Wavelet-Based Analysis and Synthesis of Long Memory Processes

- DWT well-suited for long memory processes (LMPs)
- basic idea: DWT approximately decorrelates LMPs
- on synthesis side, leads to DWT-based simulation of LMPs
- on analysis side, leads to wavelet-based maximum likelihood and least squares estimators for LMP parameters, along with a test for homogeneity of variance

Wavelets and Long Memory Processes: I

- wavelet filters are approximate band-pass filters, with nominal pass-bands $[1/2^{j+1}, 1/2^j]$ (called jth 'octave band')
- suppose $\{X_t\}$ has $S_X(\cdot)$ as its spectral density function (SDF)
- statistical properties of $\{W_{j,t}\}$ are simple if $S_X(\cdot)$ has simple structure within jth octave band
- example: fractionally differenced (FD) process

$$(1 - B)^\delta X_t = \varepsilon_t,$$

(where B is the backward shift operator such that $(1-B)X_t = X_t - X_{t-1}$) having SDF

$$S_X(f) = \sigma_\varepsilon^2/[4\sin^2(\pi f)]^\delta$$

Wavelets and Long Memory Processes: II

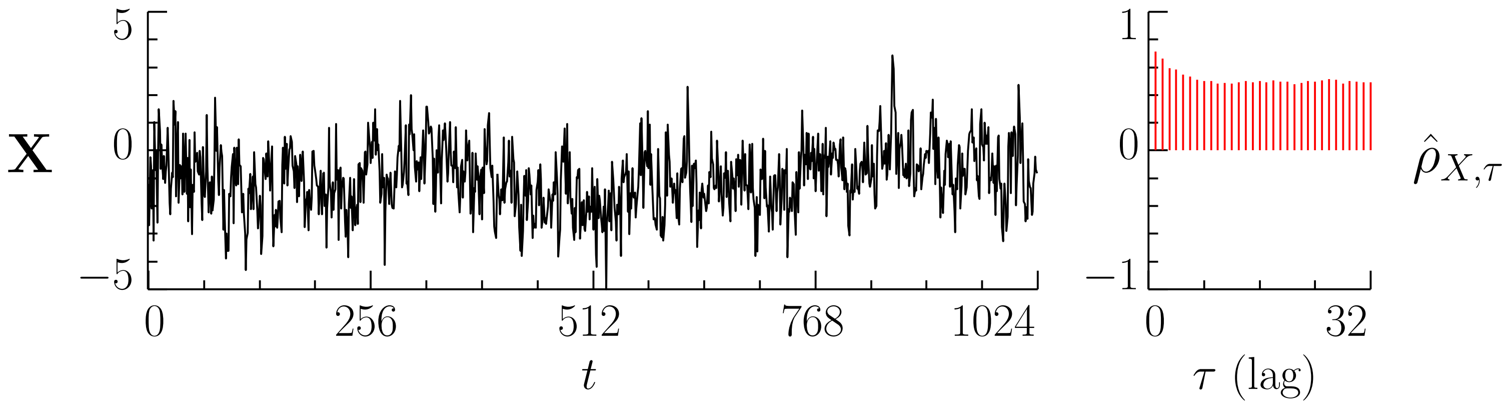
- FD process controlled by two parameters: δ and σ_ε^2
- for small f, have $S_X(f) \approx C|f|^{-2\delta}$; i.e., a power law
- $\log(S_X(f))$ vs. $\log(f)$ is approximately linear with slope -2δ
- for large τ_j, the wavelet variance at scale τ_j, namely $\nu_X^2(\tau_j)$, satisfies $\nu_X^2(\tau_j) \approx C'\tau_j^{2\delta-1}$
- $\log\left(\nu_X^2(\tau_j)\right)$ vs. $\log\left(\tau_j\right)$ is approximately linear, slope $2\delta - 1$
- approximately 'self-similar' (or 'fractal')
- stationary 'long memory' process (LMP) if $0 < \delta < 1/2$: correlation between X_t and $X_{t+\tau}$ dies down slowly as τ increases

WMTSA: 297, 284

Wavelets and Long Memory Processes: III

- power law model ubiquitous in physical sciences
  - voltage fluctuations across cell membranes
  - density fluctuations in hour glass
  - traffic fluctuations on Japanese expressway
  - impedance fluctuations in geophysical borehole
  - fluctuations in the rotation of the earth
  - X-ray time variability of galaxies
- DWT well-suited to study FD process and other LMPs
  - 'self-similar' filters used on 'self-similar' processes
  - key idea: DWT approximately decorrelates LMPs

DWT of a Long Memory Process: I



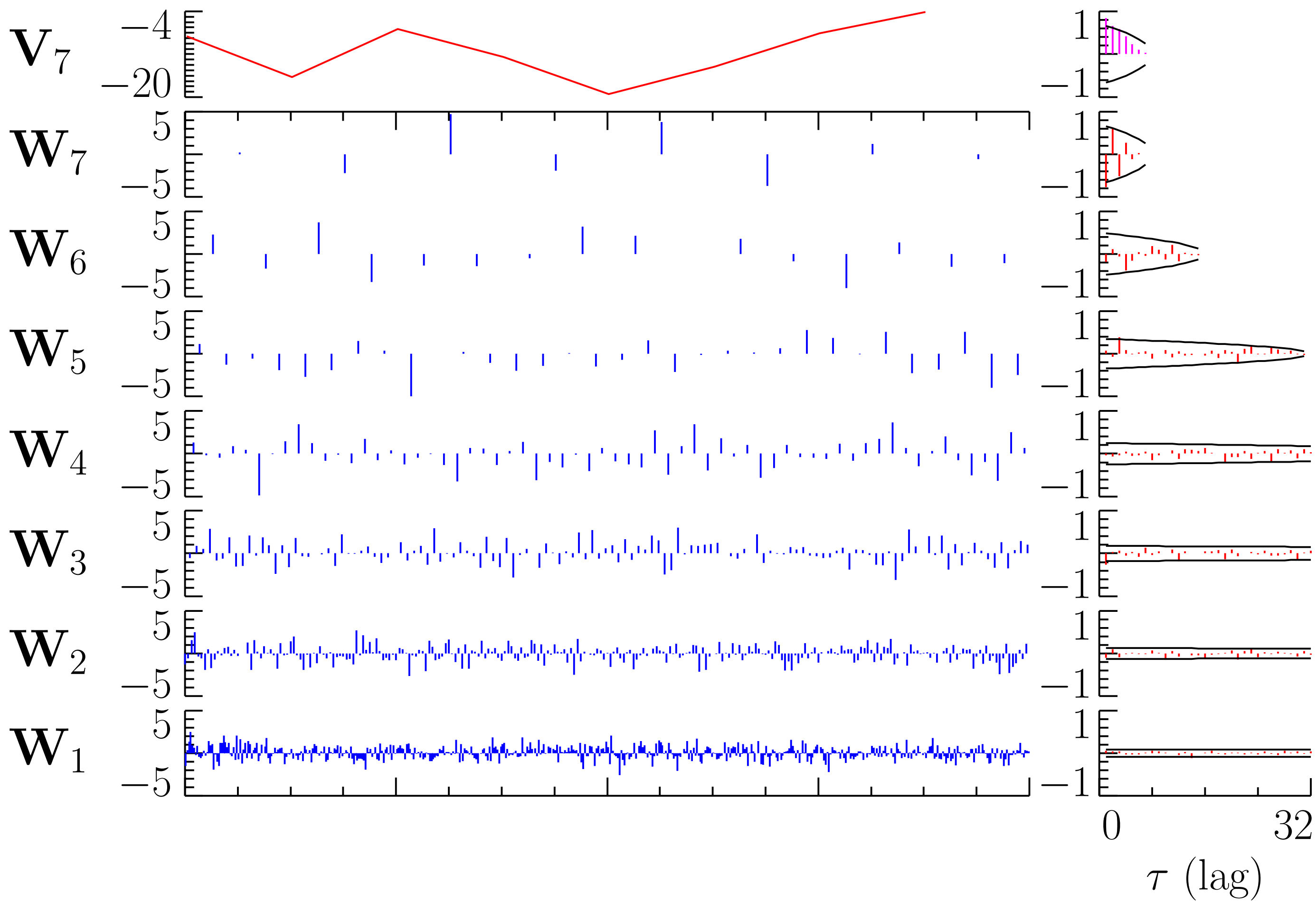
- realization of an FD(0.4) time series $\mathbf{X}$ along with its sample autocorrelation sequence (ACS): for $\tau \geq 0$,

$$\hat{\rho}_{X,\tau} \equiv \frac{\frac{1}{N}\sum_{t=0}^{N-1-\tau} X_t X_{t+\tau}}{\frac{1}{N}\sum_{t=0}^{N-1} X_t^2} = \frac{\sum_{t=0}^{N-1-\tau} X_t X_{t+\tau}}{\sum_{t=0}^{N-1} X_t^2},$$

which assumes that FD(0.4) is known to have zero mean

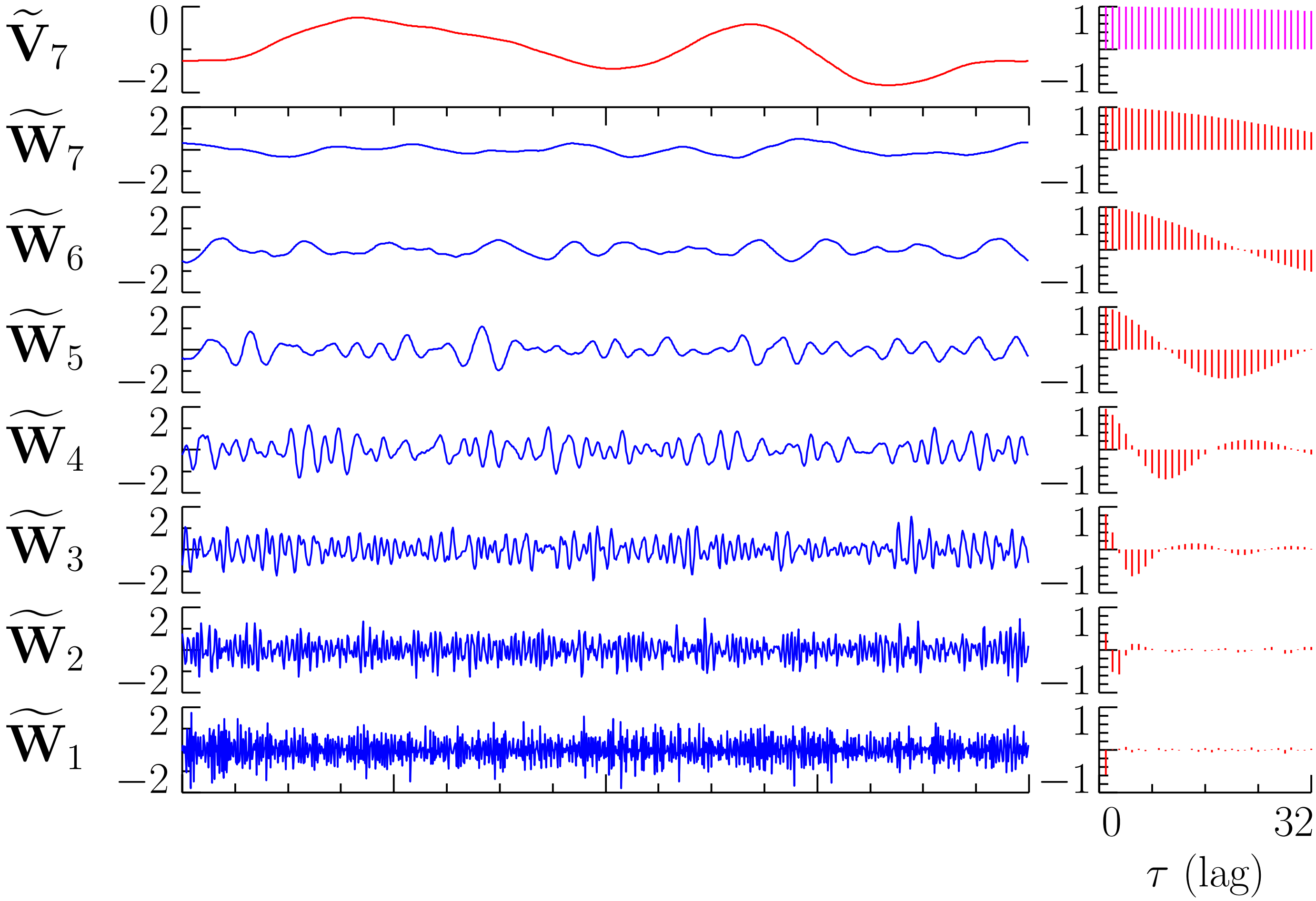
- note that ACS dies down slowly (typical for LMPs)

DWT of a Long Memory Process: II



- LA(8) DWT of FD(0.4) series and sample ACSs for each $\mathbf{W}_j$ & $\mathbf{V}_7$, along with 95% confidence intervals for white noise

MODWT of a Long Memory Process



- LA(8) MODWT of FD(0.4) series & sample ACSs for MODWT coefficients, none of which are approximately uncorrelated

DWT of a Long Memory Process: III

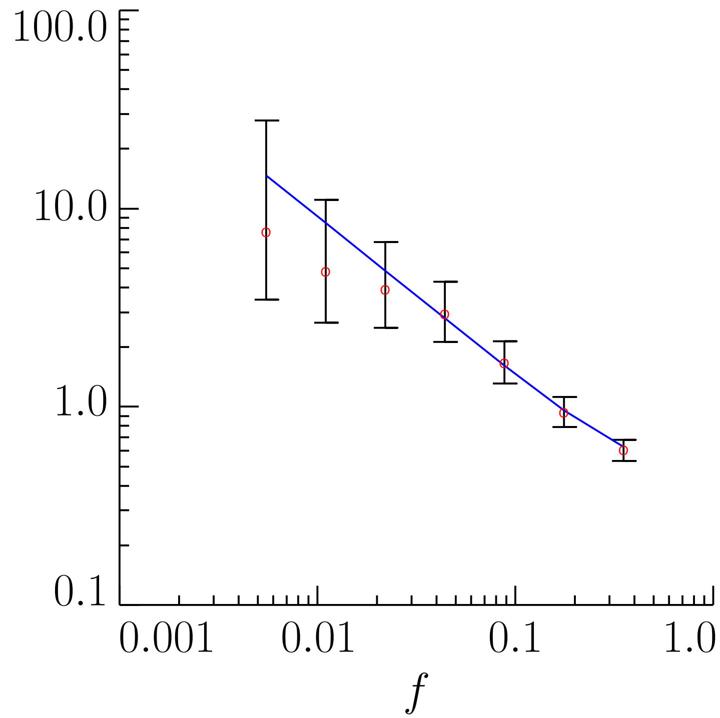
- in contrast to $\mathbf{X}$, ACSs for $\mathbf{W}_j$ consistent with white noise
- variance of $\mathbf{W}_j$ increases with j – to see why, note that

$$\begin{aligned}\operatorname{var}\{W_{j,t}\} &= \int_{-1/2}^{1/2} \mathcal{H}_j(f) S_X(f)\, df \\ &\approx 2 \int_{1/2^{j+1}}^{1/2^j} 2^j S_X(f)\, df \\ &= \frac{1}{\frac{1}{2^j} - \frac{1}{2^{j+1}}} \int_{1/2^{j+1}}^{1/2^j} S_X(f)\, df \equiv C_j,\end{aligned}$$

where C_j is average value of $S_X(\cdot)$ over $[1/2^{j+1}, 1/2^j]$

- for FD process, can argue that $C_j \approx S_X(1/2^{j+\frac{1}{2}})$, where $1/2^{j+\frac{1}{2}}$ is midpoint of interval $[1/2^{j+1}, 1/2^j]$

DWT of a Long Memory Process: IV



- plot shows $\widehat{\text{var}}\,\{W_{j,t}\}$ (circles) & $S_X(1/2^{j+\frac{1}{2}})$ (curve) versus $1/2^{j+\frac{1}{2}}$, along with 95% confidence intervals for $\text{var}\,\{W_{j,t}\}$
- observed $\widehat{\text{var}}\,\{W_{j,t}\}$ agrees well with theoretical $\text{var}\,\{W_{j,t}\}$

Correlations Within a Scale and Between Two Scales

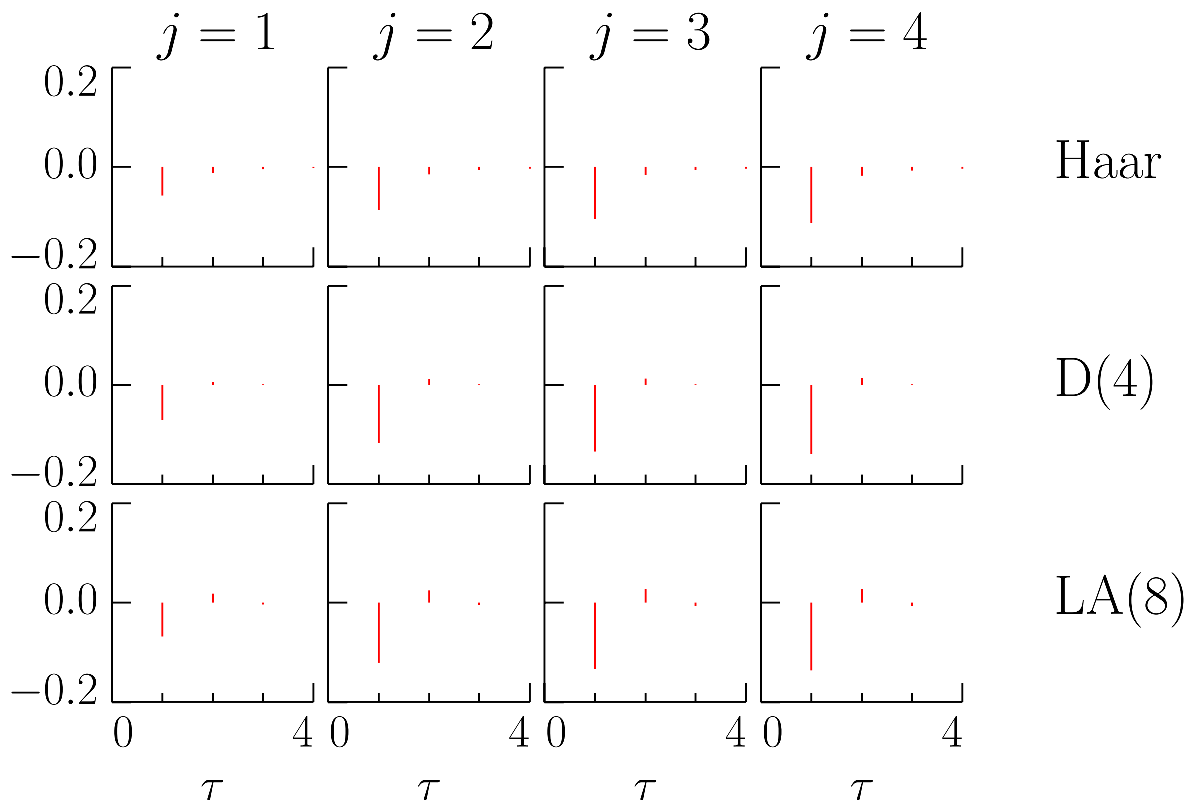
- let $\{s_{X,\tau}\}$ denote autocovariance sequence (ACVS) for $\{X_t\}$; i.e., $s_{X,\tau} = \text{cov}\{X_t, X_{t+\tau}\}$
- let $\{h_{j,l}\}$ denote equivalent wavelet filter for jth level
- to quantify decorrelation, can write

$$\text{cov}\{W_{j,t}, W_{j',t'}\} = \sum_{l=0}^{L_j-1} \sum_{l'=0}^{L_{j'}-1} h_{j,l} h_{j',l'} s_{X,2^j(t+1)-l-2^{j'}(t'+1)+l'},$$

from which we can get ACVS (and hence within-scale correlations) for $\{W_{j,t}\}$:

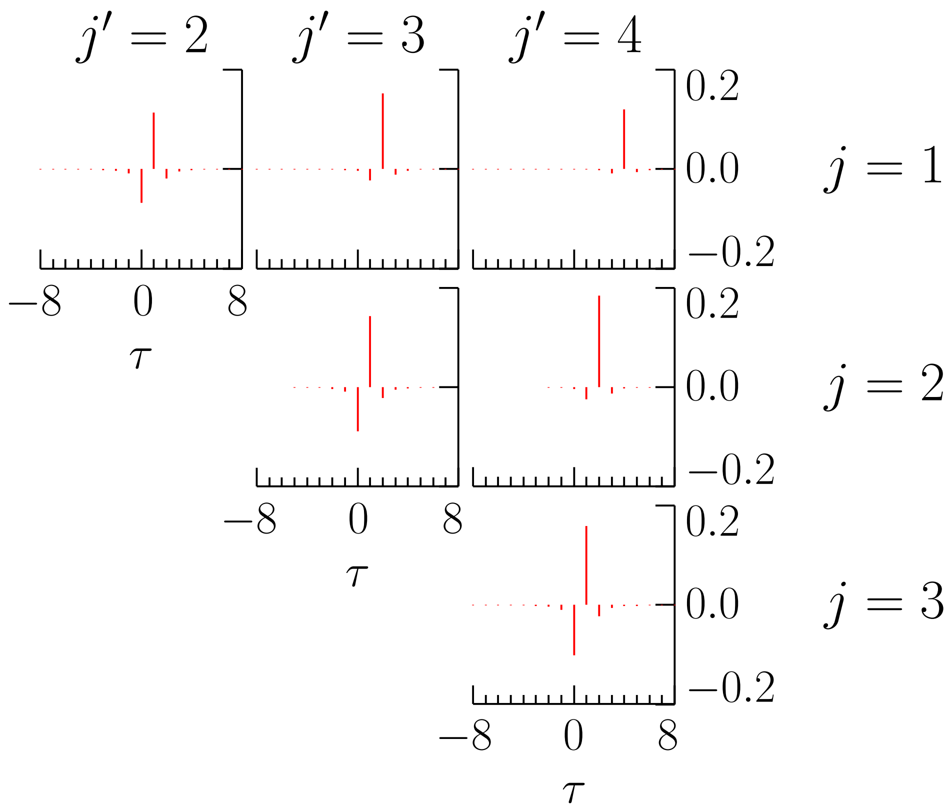
$$\text{cov}\{W_{j,t}, W_{j,t+\tau}\} = \sum_{m=-(L_j-1)}^{L_j-1} s_{X,2^j\tau+m} \sum_{l=0}^{L_j-|m|-1} h_{j,l} h_{j,l+|m|}$$

Correlations Within a Scale



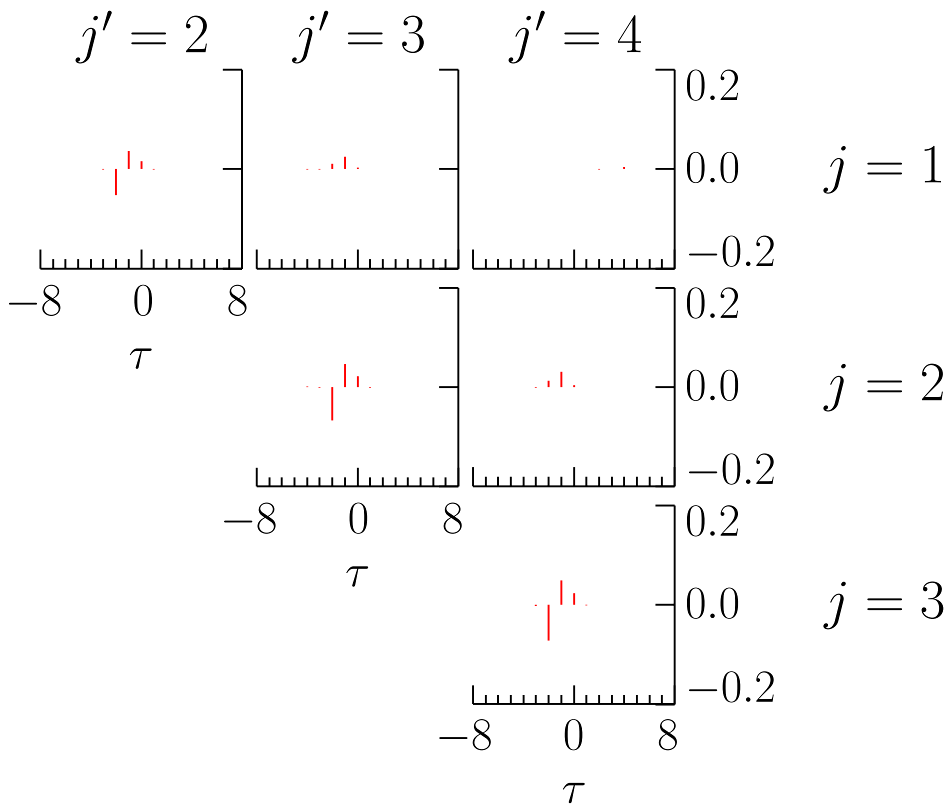
- correlations between $W_{j,t}$ and $W_{j,t+\tau}$ for an FD(0.4) process
- correlations within scale are slightly smaller for Haar
- maximum magnitude of correlation is less than 0.2

Correlations Between Two Scales: I



- correlation between Haar wavelet coefficients $W_{j,t}$ and $W_{j',t'}$ from FD(0.4) process and for levels satisfying $1 \leq j < j' \leq 4$

Correlations Between Two Scales: II



- same as before, but now for LA(8) wavelet coefficients
- correlations between scales decrease as L increases

Wavelet Domain Description of FD Process

- DWT acts as a decorrelating transform for FD process (also true for fractional Gaussian noise, pure power law etc.)
- wavelet domain description is simple
- wavelet coefficients within a given scale are approximately uncorrelated (refinement: assume 1st order autoregressive model)
- wavelet coefficients have a scale-dependent variance, but these variances are controlled by the two FD parameters (δ and σ_ε^2)
- wavelet coefficients between scales are also approximately uncorrelated (approximation improves as filter width L increases)

DWT-Based Simulation

- properties of DWT of FD processes lead to schemes for simulating time series $\mathbf{X} \equiv [X_0, \ldots, X_{N-1}]^T$ with zero mean and with a multivariate Gaussian distribution
- with $N = 2^J$, recall that $\mathbf{X} = \mathcal{W}^T \mathbf{W}$, where

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \\ \vdots \\ \mathbf{W}_j \\ \vdots \\ \mathbf{W}_J \\ \mathbf{V}_J \end{bmatrix}$$

Basic DWT-Based Simulation Scheme

- assume $\mathbf{W}$ to contain N uncorrelated Gaussian (normal) random variables (RVs) with zero mean
- assume $\mathbf{W}_j$ to have variance $C_j \approx S_X(1/2^{j+\frac{1}{2}})$
- assume single RV in $\mathbf{V}_J$ to have variance C_{J+1} (see textbook for details about how to set C_{J+1})
- approximate FD time series $\mathbf{X}$ via $\mathbf{Y} \equiv \mathcal{W}^T \Lambda^{1/2} \mathbf{Z}$, where
  - $\Lambda^{1/2}$ is $N \times N$ diagonal matrix with diagonal elements

$$\underbrace{C_1^{1/2}, \ldots, C_1^{1/2}}_{\frac{N}{2} \text{ of these}}, \underbrace{C_2^{1/2}, \ldots, C_2^{1/2}}_{\frac{N}{4} \text{ of these}}, \ldots, \underbrace{C_{J-1}^{1/2}, C_{J-1}^{1/2}}_{2 \text{ of these}}, C_J^{1/2}, C_{J+1}^{1/2}$$

  - $\mathbf{Z}$ is vector of deviations drawn from a Gaussian distribution with zero mean and unit variance

Refinements to Basic Scheme: I

- covariance matrix for approximation $\mathbf{Y}$ does not correspond to that of a stationary process
- recall $\mathcal{W}$ treats $\mathbf{X}$ as if it were circular
- let $\mathcal{T}$ be $N \times N$ 'circular shift' matrix:

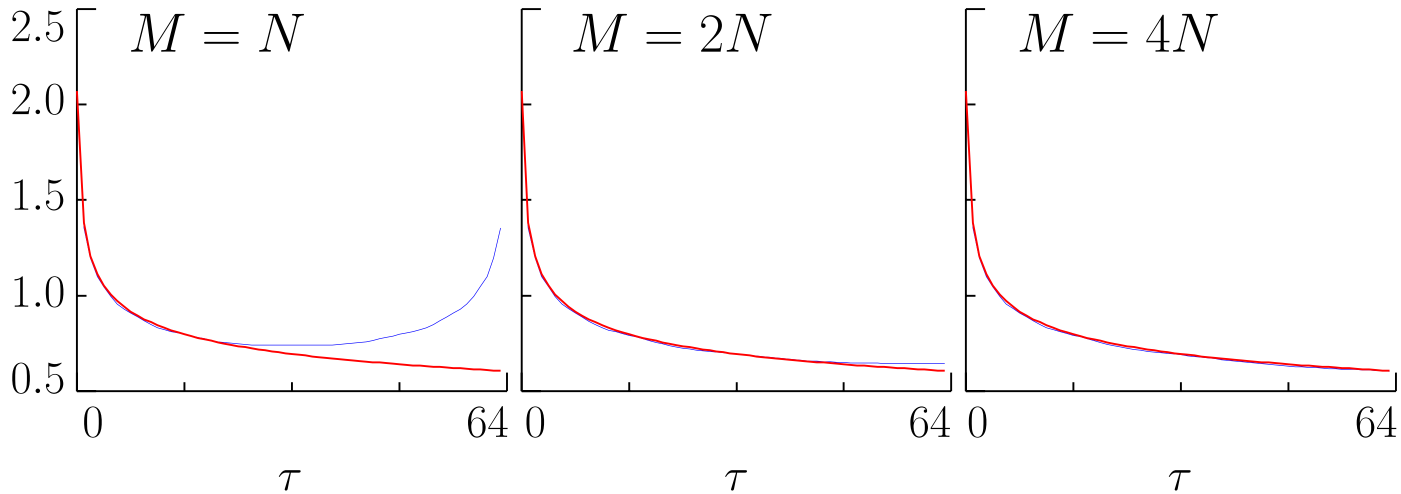
$$\mathcal{T}\begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_0 \end{bmatrix}; \quad \mathcal{T}^2\begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} Y_2 \\ Y_3 \\ Y_0 \\ Y_1 \end{bmatrix}; \quad \text{etc.}$$

- let κ be uniformily distributed over $0, \ldots, N-1$
- define $\widetilde{\mathbf{Y}} \equiv \mathcal{T}^{\kappa}\mathbf{Y}$
- $\widetilde{\mathbf{Y}}$ is stationary with ACVS given by, say, $s_{\widetilde{Y},\tau}$

Refinements to Basic Scheme: II

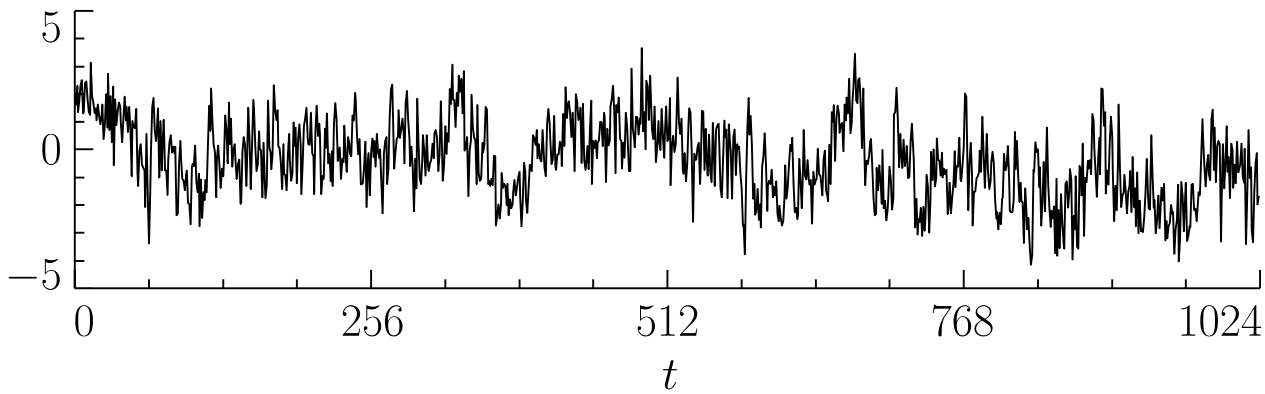
- Q: how well does $\{s_{\widetilde{Y},\tau}\}$ match $\{s_{X,\tau}\}$?
- due to circularity, find that $s_{\widetilde{Y},N-\tau} = s_{\widetilde{Y},\tau}$ for $\tau = 1, \ldots, N/2$
- implies $s_{\widetilde{Y},\tau}$ cannot approximate $s_{X,\tau}$ well for τ close to N
- can patch up by simulating $\widetilde{\mathbf{Y}}$ with $M > N$ elements and then extracting first N deviates ($M = 4N$ works well)

Refinements to Basic Scheme: III



- plot shows true ACVS $\{s_{X,\tau}\}$ (thick curves) for FD(0.4) process and wavelet-based approximate ACVSs $\{s_{\widetilde{Y},\tau}\}$ (thin curves) based on an LA(8) DWT in which an $N = 64$ series is extracted from $M = N$, $M = 2N$ and $M = 4N$ series

Example and Some Notes



- simulated FD(0.4) series (LA(8), $N = 1024$ and $M = 4N$)
- notes:
  - can form realizations faster than best exact method
  - efficient 'real-time' simulation of extremely long time series (e.g, $N = 2^{30} = 1,073,741,824$ or even longer)
  - effect of random circular shifting is to render time series non-Gaussian (a Gaussian mixture model)

MLEs of FD Parameters: I

- FD process depends on 2 parameters, namely, δ and σ_ε^2:

$$S_X(f) = \frac{\sigma_\varepsilon^2}{[4\sin^2(\pi f)]^\delta}$$

- given $\mathbf{X} = [X_0, X_1, \ldots, X_{N-1}]^T$ with $N = 2^J$, suppose we want to estimate δ and σ_ε^2
- if $\mathbf{X}$ is stationary (i.e. $\delta < 1/2$) and multivariate Gaussian, can use the maximum likelihood (ML) method

MLEs of FD Parameters: II

- definition of Gaussian likelihood function:

$$L(\delta, \sigma_\varepsilon^2 \mid \mathbf{X}) \equiv \frac{1}{(2\pi)^{N/2}|\Sigma_\mathbf{X}|^{1/2}} e^{-\mathbf{X}^T\Sigma_\mathbf{X}^{-1}\mathbf{X}/2}$$

  where $\Sigma_\mathbf{X}$ is covariance matrix for $\mathbf{X}$, with (s,t)th element given by $s_{X,s-t}$, and $|\Sigma_\mathbf{X}|$ & $\Sigma_\mathbf{X}^{-1}$ denote determinant & inverse

- ML estimators of δ and σ_ε^2 maximize $L(\delta, \sigma_\varepsilon^2 \mid \mathbf{X})$ or, equivalently, minimize

$$-2\log\left(L(\delta, \sigma_\varepsilon^2 \mid \mathbf{X})\right) = N\log(2\pi) + \log\left(|\Sigma_\mathbf{X}|\right) + \mathbf{X}^T\Sigma_\mathbf{X}^{-1}\mathbf{X}$$

- exact MLEs computationally intensive, mainly because of the need to deal with $|\Sigma_\mathbf{X}|$ and $\Sigma_\mathbf{X}^{-1}$
- good approximate MLEs of considerable interest

MLEs of FD Parameters: III

- key ideas behind first wavelet-based approximate MLEs
  - have seen that we can approximate FD time series $\mathbf{X}$ by $\mathbf{Y} = \mathcal{W}^T \Lambda^{1/2} \mathbf{Z}$, where $\Lambda^{1/2}$ is a diagonal matrix, all of whose diagonal elements are positive
  - since covariance matrix for $\mathbf{Z}$ is I_N, Equation (262c) says covariance matrix for $\mathbf{Y}$ is

    $$\mathcal{W}^T \Lambda^{1/2} I_N (\mathcal{W}^T \Lambda^{1/2})^T = \mathcal{W}^T \Lambda^{1/2} \Lambda^{1/2} \mathcal{W} = \mathcal{W}^T \Lambda \mathcal{W} \equiv \widetilde{\Sigma}_{\mathbf{X}},$$

    where $\Lambda \equiv \Lambda^{1/2} \Lambda^{1/2}$ is also diagonal
  - can consider $\widetilde{\Sigma}_{\mathbf{X}}$ to be an approximation to $\Sigma_{\mathbf{X}}$
- leads to approximation of log likelihood:

$$-2 \log \left(L(\delta, \sigma_\varepsilon^2 \mid \mathbf{X}) \right) \approx N \log (2\pi) + \log \left(|\widetilde{\Sigma}_{\mathbf{X}}| \right) + \mathbf{X}^T \widetilde{\Sigma}_{\mathbf{X}}^{-1} \mathbf{X}$$

MLEs of FD Parameters: IV

- Q: so how does this help us?
  - easy to invert $\widetilde{\Sigma}_{\mathbf{X}}$:

    $$\widetilde{\Sigma}_{\mathbf{X}}^{-1} = \left(\mathcal{W}^T \Lambda \mathcal{W}\right)^{-1} = (\mathcal{W})^{-1} \Lambda^{-1} \left(\mathcal{W}^T\right)^{-1} = \mathcal{W}^T \Lambda^{-1} \mathcal{W},$$

    where Λ^{-1} is another diagonal matrix, leading to

    $$\mathbf{X}^T \widetilde{\Sigma}_{\mathbf{X}}^{-1} \mathbf{X} = \mathbf{X}^T \mathcal{W}^T \Lambda^{-1} \mathcal{W} \mathbf{X} = \mathbf{W}^T \Lambda^{-1} \mathbf{W}$$

  - easy to compute the determinant of $\widetilde{\Sigma}_{\mathbf{X}}$:

    $$|\widetilde{\Sigma}_{\mathbf{X}}| = |\mathcal{W}^T \Lambda \mathcal{W}| = |\Lambda \mathcal{W} \mathcal{W}^T| = |\Lambda I_N| = |\Lambda|,$$

    and the determinant of a diagonal matrix is just the product of its diagonal elements

MLEs of FD Parameters: V

- define the following three functions of δ:

$$C'_j(\delta) \equiv \int_{1/2^{j+1}}^{1/2^j} \frac{2^{j+1}}{[4\sin^2(\pi f)]^\delta}\, df \approx \int_{1/2^{j+1}}^{1/2^j} \frac{2^{j+1}}{[2\pi f]^{2\delta}}\, df$$

$$C'_{J+1}(\delta) \equiv \frac{N\Gamma(1-2\delta)}{\Gamma^2(1-\delta)} - \sum_{j=1}^{J} \frac{N}{2^j} C'_j(\delta)$$

$$\sigma_\varepsilon^2(\delta) \equiv \frac{1}{N}\left(\frac{V_{J,0}^2}{C'_{J+1}(\delta)} + \sum_{j=1}^{J} \frac{1}{C'_j(\delta)} \sum_{t=0}^{\frac{N}{2^j}-1} W_{j,t}^2 \right)$$

MLEs of FD Parameters: VI

- wavelet-based approximate MLE $\tilde{\delta}$ for δ is the value that minimizes the following function of δ:

$$\tilde{l}(\delta \mid \mathbf{X}) \equiv N \log(\sigma_{\varepsilon}^2(\delta)) + \log(C'_{J+1}(\delta)) + \sum_{j=1}^{J} \frac{N}{2^j} \log(C'_j(\delta))$$

- once $\tilde{\delta}$ has been determined, MLE for σ_{ε}^2 is given by $\sigma_{\varepsilon}^2(\tilde{\delta})$
- computer experiments indicate scheme works quite well

Other Wavelet-Based Estimators of FD Parameters

- second MLE approach: formulate likelihood directly in terms of nonboundary wavelet coefficients
  - handles stationary or nonstationary FD processes (i.e., need not assume $\delta < 1/2$)
  - handles certain deterministic trends
- alternative to MLE is least square estimator (LSE)
  - recall that, for large τ and for $\beta = 2\delta - 1$, have

$$\log\left(\nu_X^2(\tau_j)\right) \approx \zeta + \beta \log\left(\tau_j\right)$$

  - suggests determining δ by regressing $\log\left(\hat{\nu}_X^2(\tau_j)\right)$ on $\log\left(\tau_j\right)$ over range of τ_j
  - weighted LSE takes into account fact that variance of $\log\left(\hat{\nu}_X^2(\tau_j)\right)$ depends upon scale τ_j (increases as τ_j increases)

Homogeneity of Variance: I

- because DWT decorrelates LMPs, nonboundary coefficients in $\mathbf{W}_j$ should resemble white noise; i.e., $\text{cov}\,\{W_{j,t}, W_{j,t'}\} \approx 0$ when $t \neq t'$, and $\text{var}\,\{W_{j,t}\}$ should not depend upon t
- can test for homogeneity of variance in $\mathbf{X}$ using $\mathbf{W}_j$ over a range of levels j
- suppose $U_0, \ldots, U_{N-1}$ are independent normal RVs with $E\{U_t\} = 0$ and $\text{var}\,\{U_t\} = \sigma_t^2$
- want to test null hypothesis

$$H_0 : \sigma_0^2 = \sigma_1^2 = \cdots = \sigma_{N-1}^2$$

- can test H_0 versus a variety of alternatives, e.g.,

$$H_1 : \sigma_0^2 = \cdots = \sigma_k^2 \neq \sigma_{k+1}^2 = \cdots = \sigma_{N-1}^2$$

  using normalized cumulative sum of squares

Homogeneity of Variance: II

- to define test statistic D, start with

$$\mathcal{P}_k \equiv \frac{\sum_{j=0}^{k} U_j^2}{\sum_{j=0}^{N-1} U_j^2}, \quad k = 0, \ldots, N-2$$

  and then compute $D \equiv \max\,(D^+, D^-)$, where

$$D^+ \equiv \max_{0 \le k \le N-2} \left(\frac{k+1}{N-1} - \mathcal{P}_k \right) \;\; \& \;\; D^- \equiv \max_{0 \le k \le N-2} \left(\mathcal{P}_k - \frac{k}{N-1} \right)$$

- can reject H_0 if observed D is 'too large,' where 'too large' is quantified by considering distribution of D under H_0
- need to find critical value x_α such that $\mathbf{P}[D \ge x_\alpha] = \alpha$ for, e.g., $\alpha = 0.01$, 0.05 or 0.1

Homogeneity of Variance: III

- once determined, can perform α level test of H_0:
  - compute D statistic from data $U_0, \ldots, U_{N-1}$
  - reject H_0 at level α if $D \geq x_\alpha$
  - fail to reject H_0 at level α if $D < x_\alpha$
- can determine critical values x_α in two ways
  - Monte Carlo simulations
  - large sample approximation to distribution of D:

$$\mathbf{P}[(N/2)^{1/2} D \geq x] \approx 1 + 2 \sum_{l=1}^{\infty} (-1)^l e^{-2l^2x^2}$$

(reasonable approximation for $N \geq 128$)

Homogeneity of Variance: IV

- idea: given time series $\{X_t\}$, compute D using nonboundary wavelet coefficients $W_{j,t}$ (there are $M'_j \equiv N_j - L'_j$ of these):

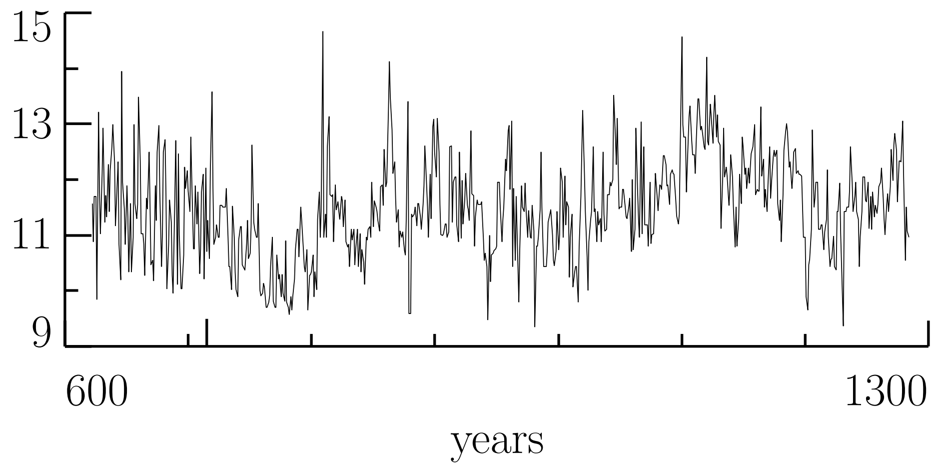
$$\mathcal{P}_k \equiv \frac{\sum_{t=L'_j}^{k} W_{j,t}^2}{\sum_{t=L'_j}^{N_j-1} W_{j,t}^2}, \quad k = L'_j, \ldots, N_j - 2$$

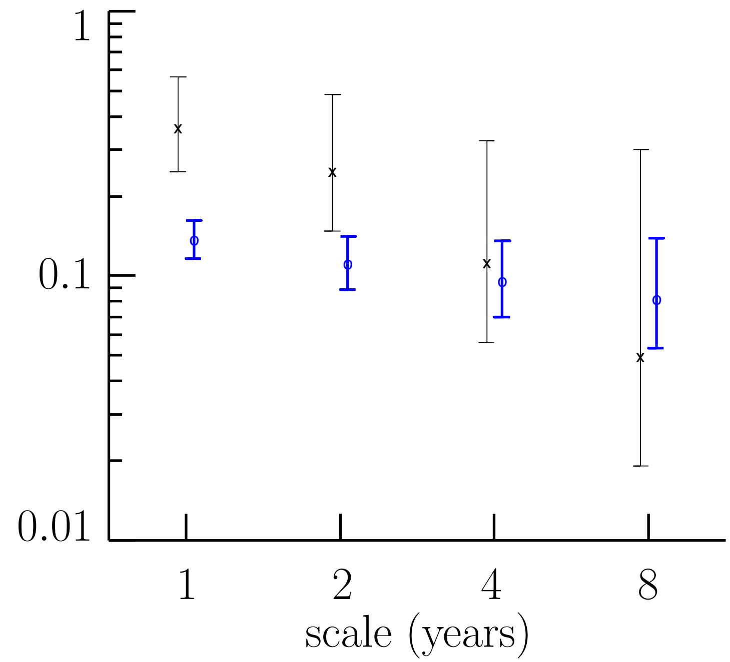
- if null hypothesis rejected at level j, can use nonboundary MODWT coefficients to locate change point based on

$$\widetilde{\mathcal{P}}_k \equiv \frac{\sum_{t=L_j-1}^{k} \widetilde{W}_{j,t}^2}{\sum_{t=L_j-1}^{N-1} \widetilde{W}_{j,t}^2}, \quad k = L_j - 1, \ldots, N - 2$$

along with analogs $\widetilde{D}_k^+$ and $\widetilde{D}_k^-$ of D_k^+ and D_k^-

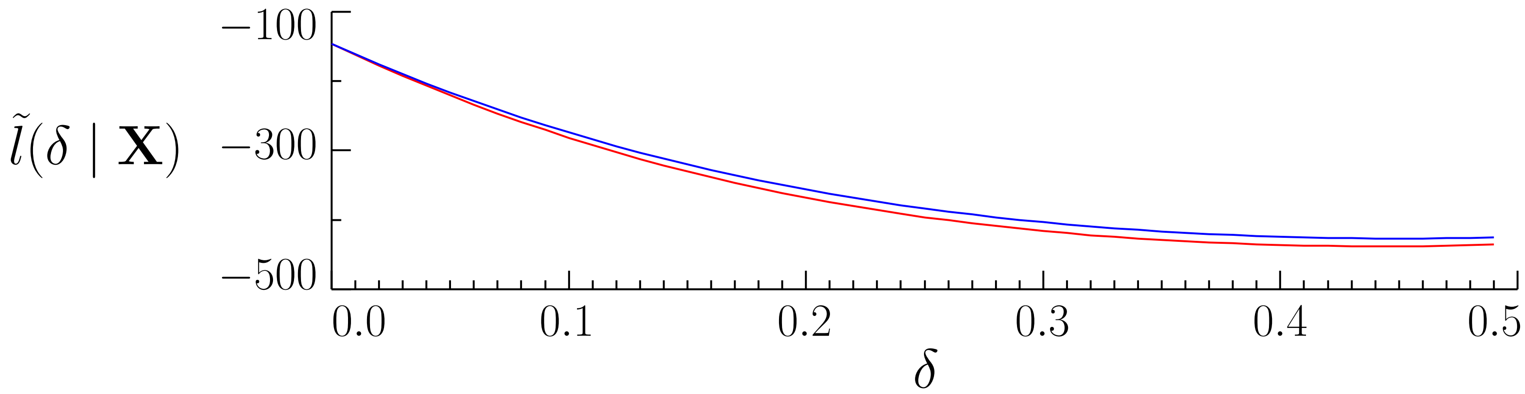
Annual Minima of Nile River





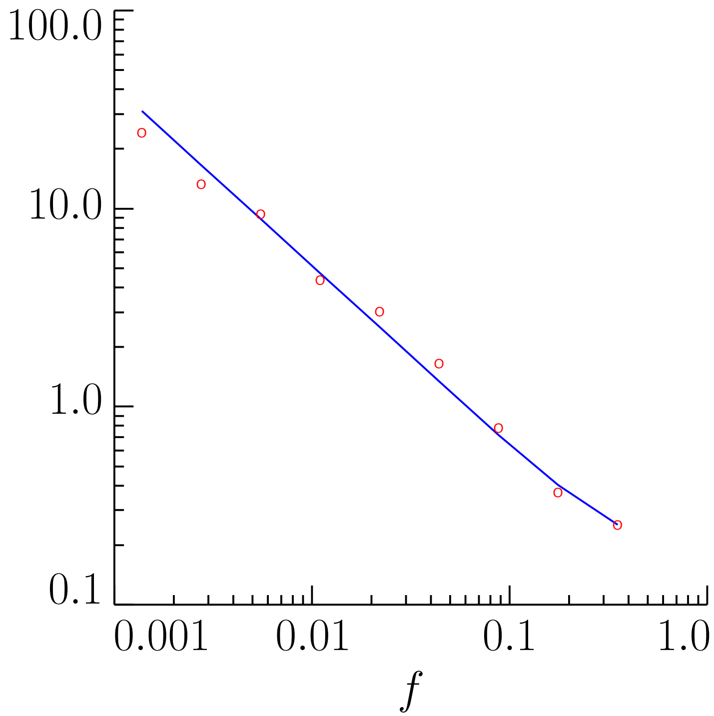
- left-hand plot: annual minima of Nile River
- new measuring device introduced around year 715
- right: Haar $\hat{\nu}^2_X(\tau_j)$ before ($\mathtt{x}$'s) and after ($\mathtt{o}$'s) year 715.5, with 95% confidence intervals based upon $\chi^2_{\eta_3}$ approximation

Example – Annual Minima of Nile River: II



- based upon last 512 values (years 773 to 1284), plot shows $\tilde{l}(\delta \mid \mathbf{X})$ versus δ for the first wavelet-based approximate MLE using the LA(8) wavelet (upper curve) and corresponding curve for exact MLE (lower)
  - wavelet-based approximate MLE is value minimizing upper curve: $\tilde{\delta} \doteq 0.4532$
  - exact MLE is value minimizing lower curve: $\hat{\delta} \doteq 0.4452$

Example – Annual Minima of Nile River: III



- using last 512 values again, variance of wavelet coefficients computed via LA(8) MLEs $\tilde{\delta}$ and $\sigma^2_\varepsilon(\tilde{\delta})$ (solid curve) as compared to sample variances of LA(8) wavelet coefficients (circles)
- agreement is almost too good to be true!

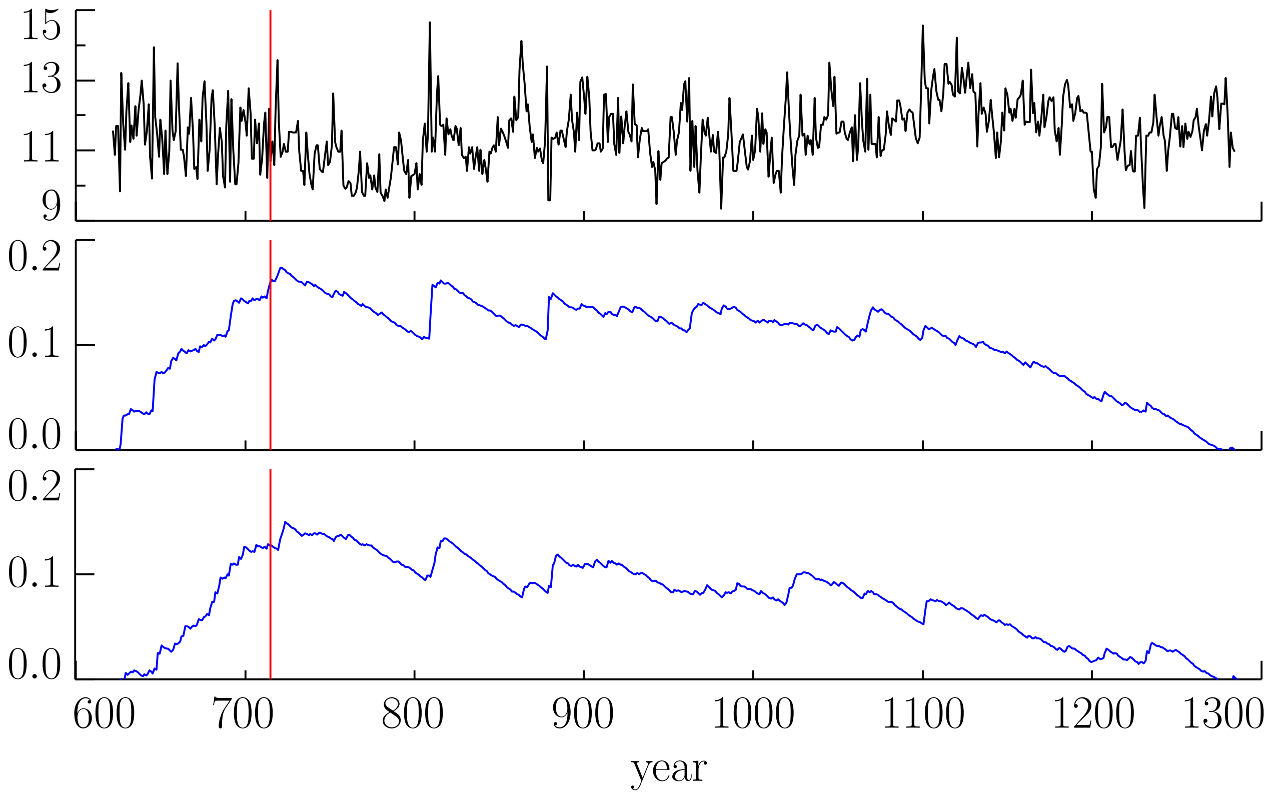
Example – Annual Minima of Nile River: IV

- results of testing all Nile River minima for homogeneity of variance using the Haar wavelet filter with critical values determined by computer simulations

			critical levels		
τ_j	M'_j	D	10%	5%	1%
1 year	331	0.1559	0.0945	0.1051	0.1262
2 years	165	0.1754	0.1320	0.1469	0.1765
4 years	82	0.1000	0.1855	0.2068	0.2474
8 years	41	0.2313	0.2572	0.2864	0.3436

- can reject null hypothesis of homogeneity of variance at level of significance 0.05 for scales τ_1 & τ_2, but not at larger scales

Example – Annual Minima of Nile River: V



- Nile River minima (top plot) along with curves (constructed per Equation (382)) for scales τ_1 & τ_2 (middle & bottom) to identify change point via time of maximum deviation (vertical lines denote year 715)

Summary

- wavelets approximately decorrelate LMPs
- leads to practical and flexible schemes for simulating LMPs
- also leads to schemes for estimating parameters of LMPs
  - approximate maximum likelihood estimators (two varieties)
  - weighted least squares estimator
- can also devise wavelet-based tests for
  - homogeneity of variance
  - trends (see Section 9.4 & Craigmile *et al.*, *Environmetrics*, 15, 313–35, 2004, for details)