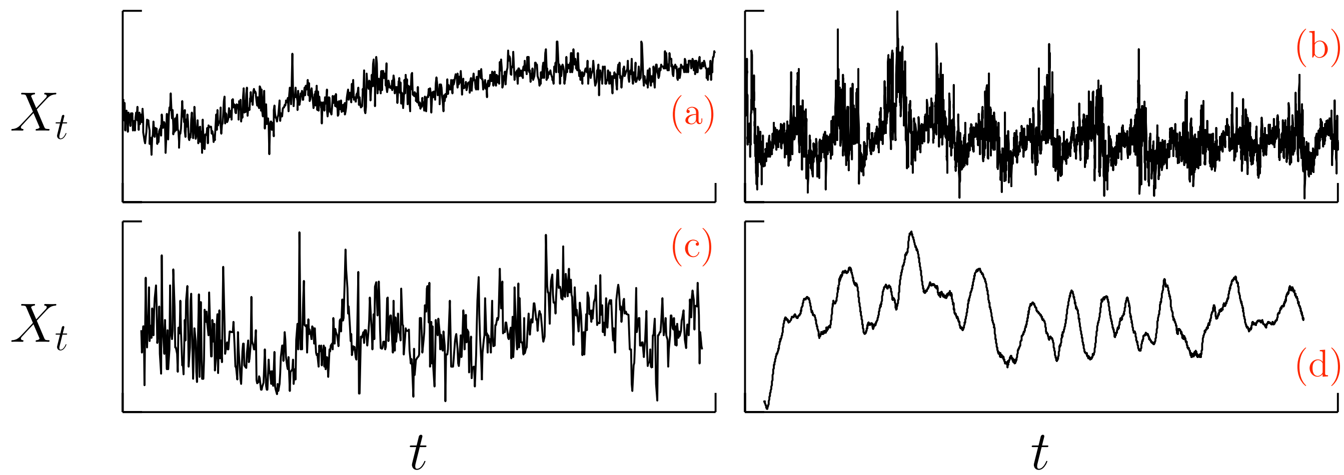
Wavelet Variance – Outline

- examples of time series to motivate discussion
- decomposition of sample variance using wavelets
- theoretical wavelet variance for stochastic processes
  - stationary processes
  - nonstationary processes with stationary differences
- sampling theory for Gaussian processes
- four examples, including use on time series with time-varying statistical properties
- summary

Examples: Time Series X_t Versus Time Index t



(a) atomic clock frequency deviates (daily observations, $N = 1025$)

(b) subtidal sea level fluctuations (twice daily, $N = 8746$)

(c) Nile River minima (annual, $N = 663$)

(d) vertical shear in the ocean (0.1 meters, $N = 4096$)

- four series are visually different
- goal of time series analysis is to quantify these differences

WMTSA: 8, 184, 192, 328

Decomposing Sample Variance of Time Series

- one approach: quantify differences by analysis of variance
- let $X_0, X_1, \ldots, X_{N-1}$ represent time series with N values
- let $\overline{X}$ denote sample mean of X_t's: $\overline{X} \equiv \frac{1}{N}\sum_{t=0}^{N-1} X_t$
- let $\hat{\sigma}_X^2$ denote sample variance of X_t's:

$$\hat{\sigma}_X^2 \equiv \frac{1}{N}\sum_{t=0}^{N-1}\left(X_t - \overline{X}\right)^2$$

- idea is to decompose (analyze, break up) $\hat{\sigma}_X^2$ into pieces that quantify how time series are different
- wavelet variance does analysis based upon differences between (possibly weighted) adjacent averages over scales

Empirical Wavelet Variance

- define empirical wavelet variance for scale $\tau_j \equiv 2^{j-1}$ as

$$\tilde{\nu}^2_X(\tau_j) \equiv \frac{1}{N}\sum_{t=0}^{N-1} \widetilde{W}^2_{j,t}, \quad \text{where } \widetilde{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l \bmod N}$$

- if $N = 2^J$, obtain analysis (decomposition) of sample variance:

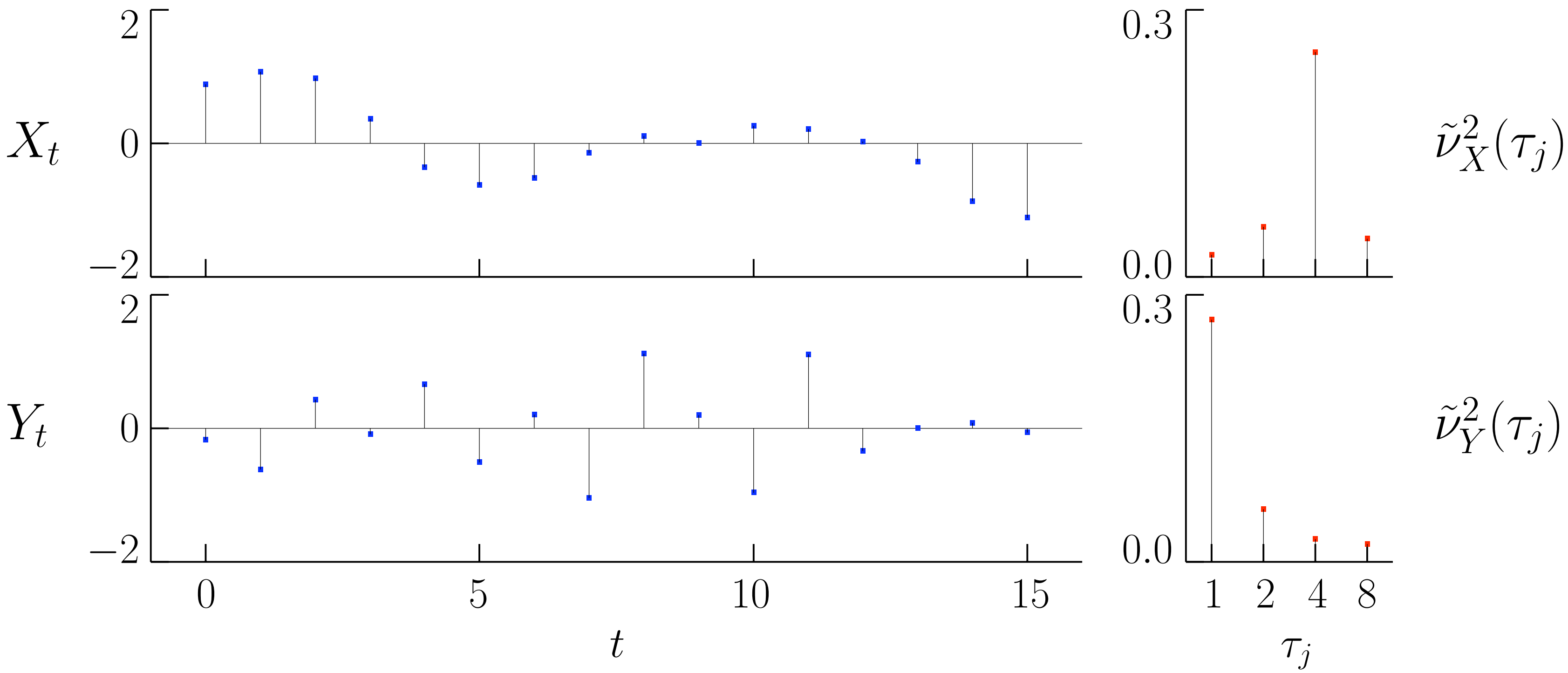
$$\hat{\sigma}^2_X = \frac{1}{N}\sum_{t=0}^{N-1}\left(X_t - \overline{X}\right)^2 = \sum_{j=1}^{J} \tilde{\nu}^2_X(\tau_j)$$

(if N not a power of 2, can analyze variance to any level J_0, but need additional component involving scaling coefficients)

- interpretation: $\tilde{\nu}^2_X(\tau_j)$ is portion of $\hat{\sigma}^2_X$ due to changes in averages over scale τ_j; i.e., 'scale by scale' analysis of variance

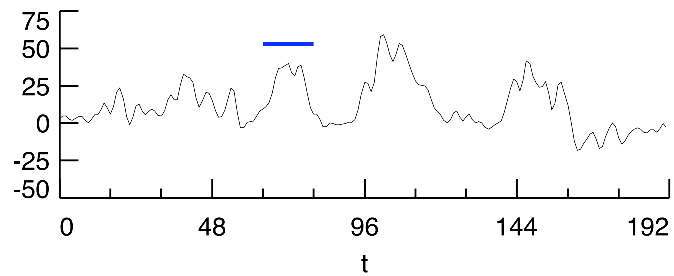
Example of Empirical Wavelet Variance

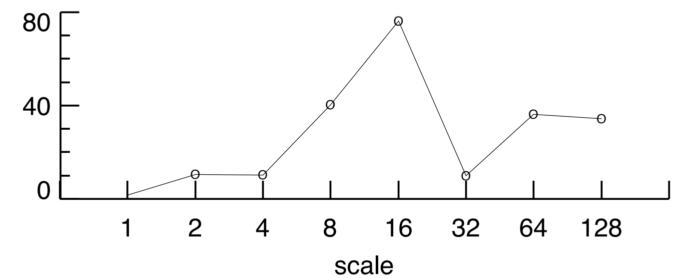
- wavelet variances for time series X_t and Y_t of length $N = 16$, each with zero sample mean and same sample variance



Second Example of Empirical Wavelet Variance

- top: part of subtidal sea level data (blue line shows scale of 16)





- bottom: empirical wavelet variances $\tilde{\nu}_X^2(\tau_j)$
- note: each $\widetilde{W}_{j,t}$ associated with a portion of X_t, so $\widetilde{W}_{j,t}^2$ versus t offers time-based decomposition of $\tilde{\nu}_X^2(\tau_j)$

Theoretical Wavelet Variance: I

- now assume X_t is a real-valued random variable (RV)
- let $\{X_t, t \in \mathbb{Z}\}$ denote a stochastic process, i.e., collection of RVs indexed by 'time' t (here $\mathbb{Z}$ denotes the set of all integers)
- use jth level equivalent MODWT filter $\{\tilde{h}_{j,l}\}$ on $\{X_t\}$ to create a new stochastic process:

$$\overline{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l}, \quad t \in \mathbb{Z},$$

which should be contrasted with

$$\widetilde{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l \bmod N}, \quad t = 0, 1, \ldots, N-1$$

Theoretical Wavelet Variance: II

- if Y is any RV, let $E\{Y\}$ denote its expectation
- let $\operatorname{var}\{Y\}$ denote its variance: $\operatorname{var}\{Y\} \equiv E\{(Y - E\{Y\})^2\}$
- definition of time dependent wavelet variance:

$$\nu_{X,t}^2(\tau_j) \equiv \operatorname{var}\{\overline{W}_{j,t}\},$$

  with conditions on X_t so that $\operatorname{var}\{\overline{W}_{j,t}\}$ exists and is finite
- $\nu_{X,t}^2(\tau_j)$ depends on τ_j and t
- will focus on time independent wavelet variance

$$\nu_X^2(\tau_j) \equiv \operatorname{var}\{\overline{W}_{j,t}\}$$

  (can adapt theory to handle time varying situation)
- $\nu_X^2(\tau_j)$ well-defined for stationary processes and certain related processes, so let's review concept of stationarity

Definition of a Stationary Process

- if U and V are two RVs, denote their covariance by

$$\text{cov}\,\{U, V\} = E\{(U - E\{U\})(V - E\{V\})\}$$

- stochastic process X_t called stationary if
  - $E\{X_t\} = \mu_X$ for all t, i.e., constant independent of t
  - $\text{cov}\{X_t, X_{t+\tau}\} = s_{X,\tau}$, i.e., depends on lag τ, but not t
- $s_{X,\tau}$, $\tau \in \mathbb{Z}$, is autocovariance sequence (ACVS)
- $s_{X,0} = \text{cov}\{X_t, X_t\} = \text{var}\{X_t\}$; i.e., variance same for all t

Spectral Density Functions: I

- spectral density function (SDF) given by

$$S_X(f) = \sum_{\tau=-\infty}^{\infty} s_{X,\tau} e^{-i2\pi f\tau}, \quad |f| \leq \frac{1}{2}$$

- above requires condition on ACVS such as

$$\sum_{\tau=-\infty}^{\infty} s_{X,\tau}^2 < \infty$$

(sufficient but not necessary)

Spectral Density Functions: II

- if square summability holds, $\{s_{X,\tau}\} \longleftrightarrow S_X(\cdot)$ says

$$\int_{-1/2}^{1/2} S_X(f) e^{i2\pi f\tau}\, df = s_{X,\tau}, \quad \tau \in \mathbb{Z}$$

- setting $\tau = 0$ yields fundamental result:

$$\int_{-1/2}^{1/2} S_X(f)\, df = s_{X,0} = \mathrm{var}\,\{X_t\};$$

  i.e., SDF decomposes $\mathrm{var}\,\{X_t\}$ across frequencies f

- interpretation: $S_X(f)\,\Delta f$ is the contribution to $\mathrm{var}\,\{X_t\}$ due to frequencies in a small interval of width Δf centered at f

White Noise Process: I

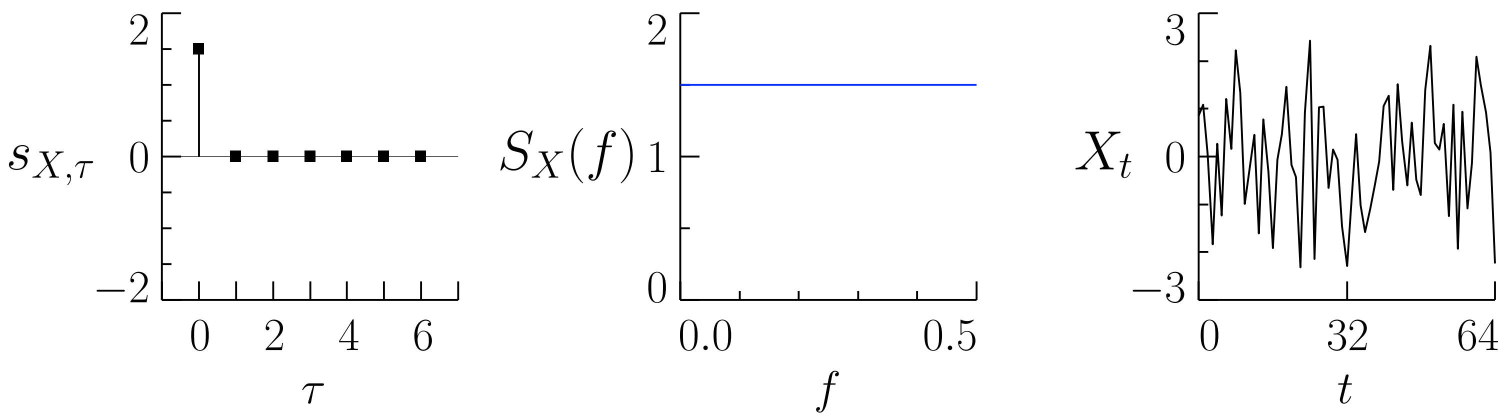
- simplest example of a stationary process is 'white noise'
- process X_t said to be white noise if
  - it has a constant mean $E\{X_t\} = \mu_X$
  - it has a constant variance $\text{var}\,\{X_t\} = \sigma_X^2$
  - $\text{cov}\,\{X_t, X_{t+\tau}\} = 0$ for all t and nonzero τ; i.e., distinct RVs in the process are uncorrelated
- ACVS and SDF for white noise take very simple forms:

$$s_{X,\tau} = \text{cov}\,\{X_t, X_{t+\tau}\} = \begin{cases} \sigma_X^2, & \tau = 0; \\ 0, & \text{otherwise.} \end{cases}$$

$$S_X(f) = \sum_{\tau=-\infty}^{\infty} s_{X,\tau} e^{-i2\pi f\tau} = s_{X,0}$$

White Noise Process: II

- ACVS (left-hand plot), SDF (middle) and a portion of length $N = 64$ of one realization (right) for a white noise process with $\mu_X = 0$ and $\sigma_X^2 = 1.5$



- since $S_X(f) = 1.5$ for all f, contribution $S_X(f)\,\Delta f$ to σ_X^2 is the same for all frequencies

Wavelet Variance for Stationary Processes

- for stationary processes, wavelet variance decomposes $\operatorname{var}\{X_t\}$:

$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \operatorname{var}\{X_t\}$$

  (above result similar to one for sample variance)

- $\nu_X^2(\tau_j)$ is thus contribution to $\operatorname{var}\{X_t\}$ due to scale τ_j
- note: $\nu_X(\tau_j)$ has same units as X_t, which is important for interpretability

Wavelet Variance for White Noise Process: I

- for a white noise process, can conclude from Exer. [8.1] that

$$\nu_X^2(\tau_j) \propto \tau_j^{-1}$$

- note that

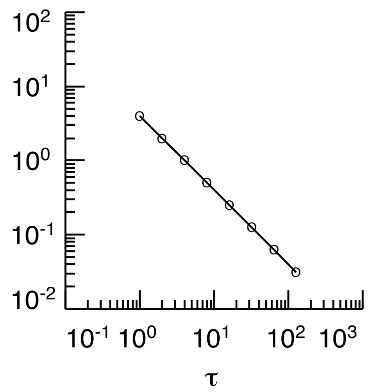
$$\log\left(\nu_X^2(\tau_j)\right) \propto -\log\left(\tau_j\right),$$

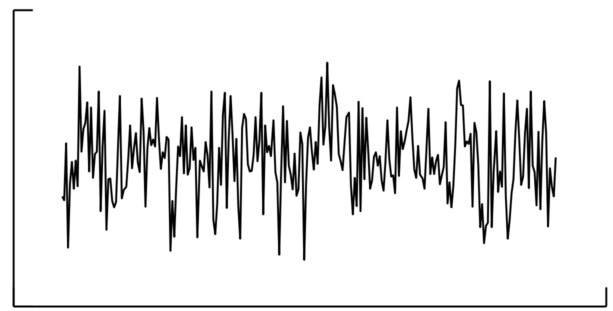
so plot of $\log\left(\nu_X^2(\tau_j)\right)$ vs. $\log\left(\tau_j\right)$ is linear with a slope of -1

Wavelet Variance for White Noise Process: II

$\delta = 0$

slope $= -1$



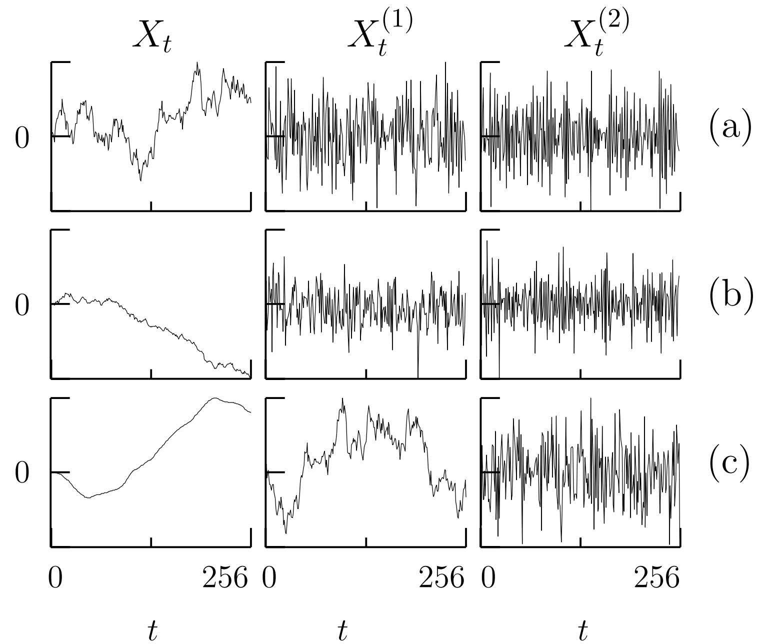


- $\nu_X^2(\tau_j)$ versus τ_j for $j = 1, \ldots, 8$ (left-hand plot), along with sample of length $N = 256$ of Gaussian white noise
- largest contribution to var $\{X_t\}$ is at smallest scale τ_1
- note: later on, we will discuss fractionally differenced (FD) processes that are characterized by a parameter δ; when $\delta = 0$, an FD process is the same as a white noise process

Generalization to Certain Nonstationary Processes

- if wavelet filter is properly chosen, $\nu_X^2(\tau_j)$ well-defined for certain processes with stationary backward differences (increments); these are also known as intrinsically stationary processes
- first order backward difference of X_t is process defined by
$$X_t^{(1)} = X_t - X_{t-1}$$
- second order backward difference of X_t is process defined by
$$X_t^{(2)} = X_t^{(1)} - X_{t-1}^{(1)} = X_t - 2X_{t-1} + X_{t-2}$$
- X_t said to have dth order stationary backward differences if
$$Y_t \equiv \sum_{k=0}^{d} \binom{d}{k} (-1)^k X_{t-k}$$
forms a stationary process (d is a nonnegative integer)

Examples of Processes with Stationary Increments



- 1st column shows, from top to bottom, realizations from

  (a) random walk: $X_t = \sum_{u=1}^{t} \epsilon_u$, & ϵ_t is zero mean white noise

  (b) like (a), but now ϵ_t has mean of -0.2

  (c) random run: $X_t = \sum_{u=1}^{t} Y_u$, where Y_t is a random walk

- 2nd & 3rd columns show 1st & 2nd differences $X_t^{(1)}$ and $X_t^{(2)}$

Wavelet Variance for Processes with Stationary Backward Differences: I

- let $\{X_t\}$ be nonstationary with dth order stationary differences
- if we use a Daubechies wavelet filter of width L satisfying $L \geq 2d$, then $\nu_X^2(\tau_j)$ is well-defined and finite for all τ_j, but now

$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \infty$$

Wavelet Variance for Random Walk Process: I

- random walk process $X_t = \sum_{u=1}^{t} \epsilon_u$ has first order ($d = 1$) stationary differences since $X_t - X_{t-1} = \epsilon_t$ (i.e., white noise)
- $L \geq 2d$ holds for all wavelets when $d = 1$; for Haar ($L = 2$),

$$\nu_X^2(\tau_j) = \frac{\text{var}\{\epsilon_t\}}{6}\left(\tau_j + \frac{1}{2\tau_j}\right) \approx \frac{\text{var}\{\epsilon_t\}}{6}\tau_j,$$

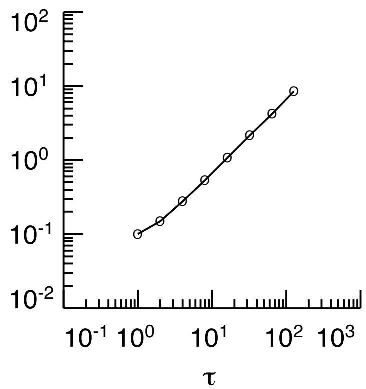
  with the approximation becoming better as τ_j increases
- note that $\nu_X^2(\tau_j)$ increases as τ_j increases
- $\log(\nu_X^2(\tau_j)) \approx \log(\text{var}\{\epsilon_t\}/6) + \log(\tau_j)$, which says that a plot of $\log(\nu_X^2(\tau_j))$ vs. $\log(\tau_j)$ is $\approx$ linear with a slope of $+1$
- as required, also have

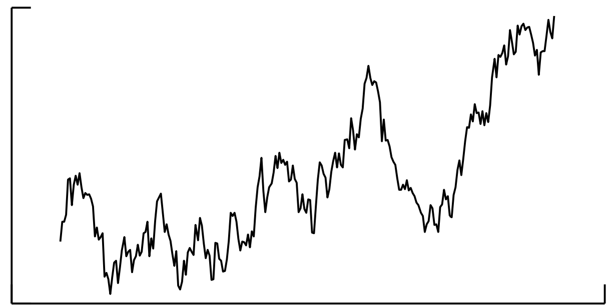
$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \frac{\text{var}\{\epsilon_t\}}{6}\left(1 + \tfrac{1}{2} + 2 + \tfrac{1}{4} + 4 + \tfrac{1}{8} + \cdots\right) = \infty$$

Wavelet Variance for Random Walk Process: II

$\delta = 1$

slope ≈ 1





- $\nu_X^2(\tau_j)$ versus τ_j for $j = 1, \ldots, 8$ (left-hand plot), along with sample of length $N = 256$ of a Gaussian random walk process
- smallest contribution to var $\{X_t\}$ is at smallest scale τ_1
- note: a fractionally differenced process with parameter $\delta = 1$ is the same as a random walk process

Wavelet Variance for Processes with Stationary Backward Differences: II

- to see why $\nu_X^2(\tau_j)$ is well-defined and finite if $L \geq 2d$, need basic result from filtering theory: if $\{Y_t\}$ stationary with SDF $S_Y(\cdot)$, then

$$Z_t \equiv \sum_{m=0}^{M-1} a_m Y_{t-m}$$

is also a stationary process, and its SDF is

$$S_Z(f) = \mathcal{A}(f) S_Y(f), \text{ where } \mathcal{A}(f) \equiv \left| \sum_{m=0}^{M-1} a_m e^{-i2\pi f m} \right|^2,$$

from which it follows that its variance is

$$\operatorname{var}\{Z_t\} = \int_{-1/2}^{1/2} S_Z(f)\, df = \int_{-1/2}^{1/2} \mathcal{A}(f) S_Y(f)\, df.$$

Wavelet Variance for Processes with Stationary Backward Differences: III

- example: first backward difference $Y_t^{(1)} = Y_t - Y_{t-1}$, i.e.,

$$\{Y_t\} \longrightarrow \boxed{\{1, -1\}} \longrightarrow \{Y_t^{(1)}\}$$

- here $a_0 = 1$, $a_1 = -1$ and $a_m = 0$ otherwise, yielding

$$\mathcal{A}(f) = 4\sin^2(\pi f) \equiv \mathcal{D}(f)$$

(proof of the above is Exer. [105b])

Wavelet Variance for Processes with Stationary Backward Differences: IV

- consider $\nu_X^2(\tau_1)$ (Exer. [304] generalizes result for τ_j, $j \geq 2$)
- by definition, $\nu_X^2(\tau_1) \equiv \text{var}\,\{\overline{W}_{1,t}\}$, with $\overline{W}_{1,t} \equiv \sum_{l=0}^{L-1} \tilde{h}_l X_{t-l}$
- because $\tilde{h}_l = h_l/\sqrt{2}$, have

$$\widetilde{\mathcal{H}}_1^{(D)}(f) = \frac{1}{2}\mathcal{H}^{(D)}(f) = \sin^L(\pi f) \sum_{l=0}^{\frac{L}{2}-1} \binom{\frac{L}{2}-1+l}{l} \cos^{2l}(\pi f)$$
$$= \mathcal{D}^{\frac{L}{2}}(f)\widetilde{\mathcal{A}}_L(f)$$

where, as before, $\mathcal{D}(f) = 4\sin^2(\pi f)$ and

$$\widetilde{\mathcal{A}}_L(f) \equiv \frac{1}{2^L} \sum_{l=0}^{\frac{L}{2}-1} \binom{\frac{L}{2}-1+l}{l} \cos^{2l}(\pi f)$$

Wavelet Variance for Processes with Stationary Backward Differences: V

- interpret $\widetilde{\mathcal{H}}_1^{(D)}(f) = \mathcal{D}^{\frac{L}{2}}(f)\widetilde{\mathcal{A}}_L(f)$ as the squared gain function for filter cascade consisting of three parts
- first part of cascade consists of a cascade of d first differences:

$$\{X_t\} \longrightarrow \underbrace{\boxed{\{1,-1\}} \longrightarrow \cdots \longrightarrow \boxed{\{1,-1\}}}_{d \text{ of these}} \longrightarrow \{Y_t\}$$

where $\{Y_t\}$ is stationary with SDF $S_Y(\cdot)$

- if $\frac{L}{2} > d$, second part uses $\frac{L}{2} - d$ first differences:

$$\{Y_t\} \longrightarrow \underbrace{\boxed{\{1,-1\}} \longrightarrow \cdots \longrightarrow \boxed{\{1,-1\}}}_{\frac{L}{2}-d \text{ of these}} \longrightarrow \{Z_t\}$$

where $\{Z_t\}$ is stationary with SDF $S_Z(f) = \mathcal{D}^{\frac{L}{2}-d}(f)S_Y(f)$

Wavelet Variance for Processes with Stationary Backward Differences: VI

- third part uses averaging filter embedded within Daubechies wavelet filter:

$$\{Z_t\} \longrightarrow \boxed{\widetilde{\mathcal{A}}_L(\cdot)} \longrightarrow \{\overline{W}_{1,t}\},$$

where $\{\overline{W}_{1,t}\}$ is stationary with SDF given by

$$\begin{aligned} S_1(f) &\equiv \widetilde{\mathcal{A}}_L(f) S_Z(f) \\ &= \mathcal{D}^{\frac{L}{2}-d}(f)\widetilde{\mathcal{A}}_L(f) S_Y(f) \\ &= \mathcal{D}^{\frac{L}{2}-d}(f)\widetilde{\mathcal{A}}_L(f)\mathcal{D}^d(f) S_X(f) = \widetilde{\mathcal{H}}_1^{(D)}(f) S_X(f) \end{aligned}$$

if we *define* an SDF for the nonstationary process $\{X_t\}$ via

$$S_X(f) \equiv \frac{S_Y(f)}{\mathcal{D}^d(f)} = \frac{S_Y(f)}{[4\sin^2(\pi f)]^d}$$

(Yaglom, 1958)

Wavelet Variance for Processes with Stationary Backward Differences: VII

- for general τ_j, can claim that, if $\{X_t\}$ has stationary increments of order d and if we use a Daubechies MODWT wavelet filter $\{\tilde{h}_l\}$ of width $L \geq 2d$, the fact that the resulting process $\{\overline{W}_{j,t}\}$ is stationary with variance $\nu_X^2(\tau_j)$ says that

$$\nu_X^2(\tau_j) = \int_{-1/2}^{1/2} \widetilde{\mathcal{H}}_j^{(D)}(f) S_X(f)\, df,$$

where $\widetilde{\mathcal{H}}_j^{(D)}(\cdot)$ is the squared gain function for the jth level equivalent filter $\{\tilde{h}_{j,l}\}$

Fractionally Differenced (FD) Processes: I

- can create a continuum of processes that 'interpolate' between white noise and random walks using notion of 'fractional differencing' (Granger and Joyeux, 1980; Hosking, 1981)
- FD(δ) process is determined by 2 parameters δ and σ_ϵ^2, where $-\infty < \delta < \infty$ and $\sigma_\epsilon^2 > 0$ (σ_ϵ^2 is less important than δ)
- if $\{X_t\}$ is an FD(δ) process, its SDF is given by

$$S_X(f) = \frac{\sigma_\epsilon^2}{\mathcal{D}^\delta(f)} = \frac{\sigma_\epsilon^2}{[4\sin^2(\pi f)]^\delta}$$

- if $\delta < 1/2$, FD process $\{X_t\}$ is stationary, and, in particular,
  - reduces to white noise if $\delta = 0$
  - has 'long memory' or 'long range dependence' if $\delta > 0$
  - is 'antipersistent' if $\delta < 0$ (i.e., $\text{cov}\,\{X_t, X_{t+1}\} < 0$)

Fractionally Differenced (FD) Processes: II

- if $\delta \geq 1/2$, FD process $\{X_t\}$ is nonstationary with dth order stationary backward differences $\{Y_t\}$
  - here $d = \lfloor \delta + 1/2 \rfloor$, where $\lfloor x \rfloor$ is integer part of x
  - $\{Y_t\}$ is stationary FD$(\delta - d)$ process
- if $\delta = 1$, FD process is the same as a random walk process
- using $\sin(x) \approx x$ for small x, can claim that, at low frequencies,

$$S_X(f) = \frac{\sigma_\epsilon^2}{[4\sin^2(\pi f)]^\delta} \approx \frac{\sigma_\epsilon^2}{(2\pi f)^{2\delta}}$$

  (approximation quite good for $f \in (0, 0.1]$)
- right-hand side describes SDF for a 'power law' process with exponent -2δ

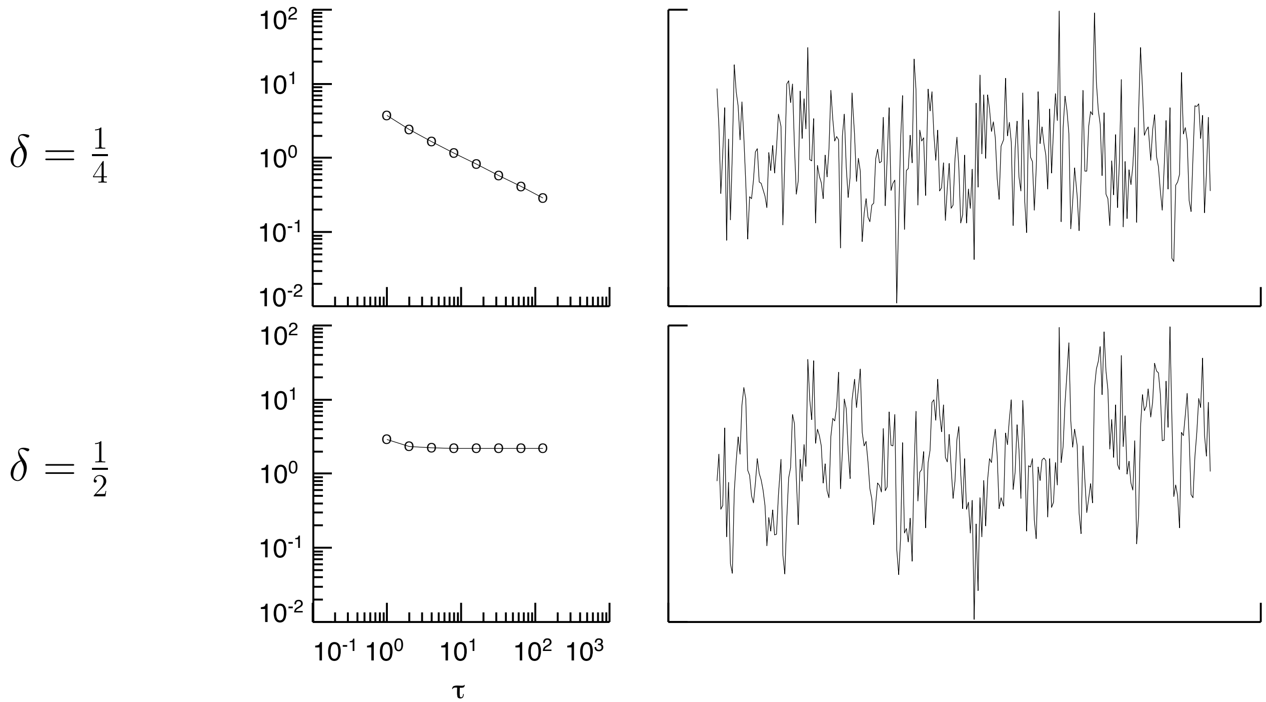
Fractionally Differenced (FD) Processes: III

- except possibly for two or three smallest scales, have

$$\begin{aligned}\nu_X^2(\tau_j) &= \int_{-1/2}^{1/2} \widetilde{\mathcal{H}}_j^{(D)}(f) S_X(f)\, df \\ &\approx 2\int_{1/2^{j+1}}^{1/2^j} \frac{\sigma_\epsilon^2}{[4\sin^2(\pi f)]^\delta}\, df \\ &\approx \frac{2\sigma_\epsilon^2}{(2\pi)^{2\delta}} \int_{1/2^{j+1}}^{1/2^j} \frac{1}{f^{2\delta}}\, df = C\tau_j^{2\delta-1}\end{aligned}$$

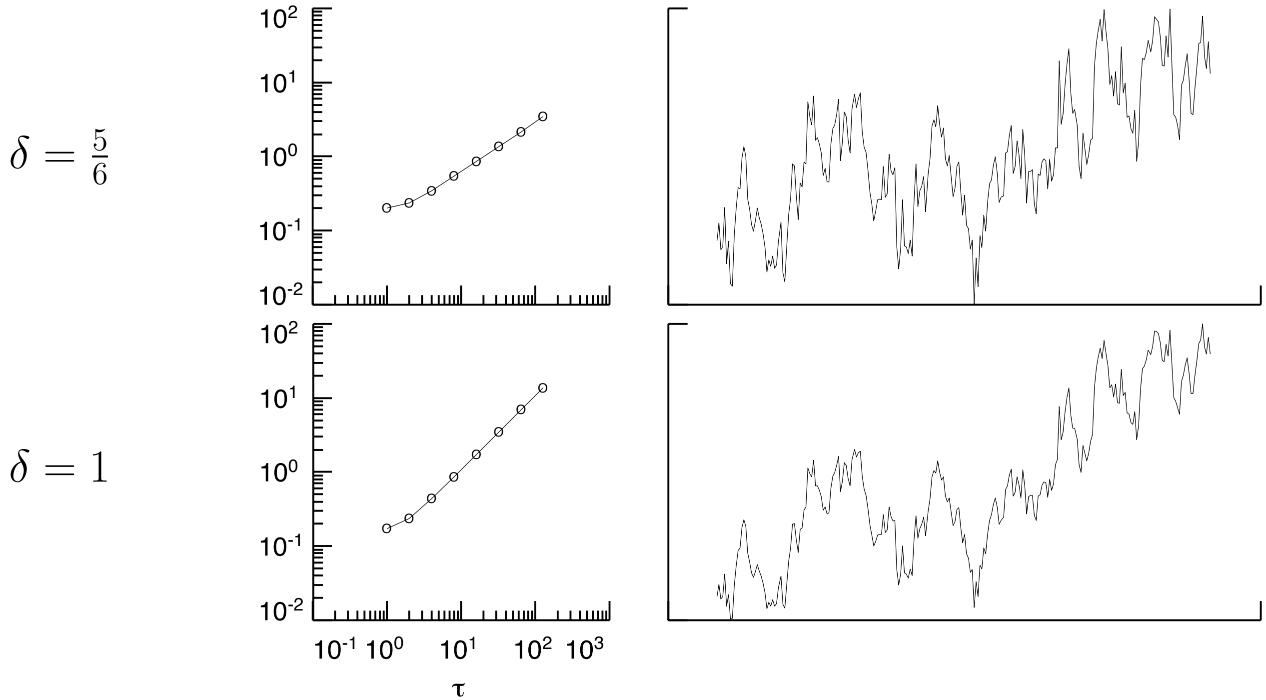
- thus $\log(\nu_X^2(\tau_j)) \approx \log(C) + (2\delta - 1)\log(\tau_j)$, so a log/log plot of $\nu_X^2(\tau_j)$ vs. τ_j looks approximately linear with slope $2\delta - 1$ for τ_j large enough

LA(8) Wavelet Variance for 2 FD Processes



- left-hand column: $\nu_X^2(\tau_j)$ versus τ_j based upon LA(8) wavelet
- right-hand: realization of length $N = 256$ from each FD process
- see overhead 16 for $\delta = 0$ (white noise), which has slope $= -1$

LA(8) Wavelet Variance for 2 More FD Processes



- $\delta = \frac{5}{6}$ is Kolmogorov turbulence; $\delta = 1$ is random walk
- note: positive slope indicates nonstationarity, while negative slope indicates stationarity

Expected Value of Wavelet Coefficients

- in preparation for considering problem of estimating $\nu_X^2(\tau_j)$ given an observed time series, let us consider $E\{\overline{W}_{j,t}\}$
- if $\{X_t\}$ is nonstationary but has dth order stationary increments, let $\{Y_t\}$ be the stationary process obtained by differencing $\{X_t\}$ a total of d times; if $\{X_t\}$ is stationary, let $Y_t = X_t$
- Exer. [305]: with $\mu_Y \equiv E\{Y_t\}$, have
  - $E\{\overline{W}_{j,t}\} = 0$ if either (i) $L > 2d$ or (ii) $L = 2d$ and $\mu_Y = 0$
  - $E\{\overline{W}_{j,t}\} \neq 0$ if $\mu_Y \neq 0$ and $L = 2d$
- thus have $E\{\overline{W}_{j,t}\} = 0$ if L is picked large enough ($L > 2d$ is sufficient, but might not be necessary)
- as the argument that follows shows, highly desirable to have $E\{\overline{W}_{j,t}\} = 0$ in order to ease the job of estimating $\nu_X^2(\tau_j)$

Estimation of a Process Variance: I

- suppose $\{U_t\}$ is a stationary process with mean $\mu_U = E\{U_t\}$ and unknown variance $\sigma_U^2 = E\{(U_t - \mu_U)^2\}$
- can be difficult to estimate σ_U^2 for a stationary process
- to understand why, assume first that μ_U is known
- when this is the case, can estimate σ_U^2 using

$$\tilde{\sigma}_U^2 \equiv \frac{1}{N} \sum_{t=0}^{N-1} (U_t - \mu_U)^2$$

- estimator above is unbiased: $E\{\tilde{\sigma}_U^2\} = \sigma_U^2$

Estimation of a Process Variance: II

- if μ_U is unknown (more common case), can estimate σ_U^2 using

$$\hat{\sigma}_U^2 \equiv \frac{1}{N}\sum_{t=0}^{N-1}(U_t - \overline{U})^2, \text{ where } \overline{U} \equiv \frac{1}{N}\sum_{t=0}^{N-1} U_t$$

- can argue that $E\{\hat{\sigma}_U^2\} = \sigma_U^2 - \text{var}\,\{\overline{U}\}$
- implies $0 \le E\{\hat{\sigma}_U^2\} \le \sigma_U^2$ because $\text{var}\,\{\overline{U}\} \ge 0$
- $E\{\hat{\sigma}_U^2\} \to \sigma_U^2$ as $N \to \infty$ if SDF exists ... but, for any

  $\epsilon > 0$ (say, $0.00\cdots01$) and sample size N (say, $N = 10^{10^{10}}$),

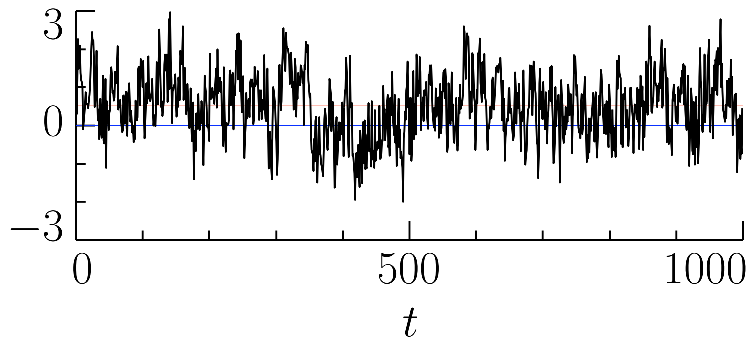
  there is some FD(δ) process $\{U_t\}$ with δ close to 1/2 such that

$$E\{\hat{\sigma}_U^2\} < \epsilon \cdot \sigma_U^2;$$

  i.e., in general, $\hat{\sigma}_U^2$ can be *badly* biased even for very large N

Estimation of a Process Variance: III

- example: realization of FD(0.4) process ($\sigma_U^2 = 1$ & $N = 1000$)



- using $\mu_U = 0$ (lower horizontal line), obtain $\tilde{\sigma}_U^2 \doteq 0.99$
- using $\overline{U} \doteq 0.53$ (upper line), obtain $\hat{\sigma}_U^2 \doteq 0.71$
- note that this is comparable to $E\{\hat{\sigma}_U^2\} \doteq 0.75$
- for this particular example, we would need $N \geq 10^{10}$ to get $\sigma_U^2 - E\{\hat{\sigma}_U^2\} \leq 0.01$, i.e., to reduce the bias so that it is no more than 1% of true variance $\sigma_U^2 = 1$

Estimation of a Process Variance: IV

- conclusion: $\hat{\sigma}_U^2$ can have substantial bias if μ_U is unknown (can patch up by estimating δ, but must make use of model)
- if $\{X_t\}$ stationary with mean μ_X, then, because $\sum_l \tilde{h}_{j,l} = 0$,

$$E\{\overline{W}_{j,t}\} = \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} E\{X_{t-l}\} = \mu_X \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} = 0$$

- because $E\{\overline{W}_{j,t}\}$ is known, we can form an unbiased estimator of $\text{var}\,\{\overline{W}_{j,t}\} = \nu_X^2(\tau_j)$
- more generally, if $\{X_t\}$ is nonstationary with stationary increments of order d, we can ensure $E\{\overline{W}_{j,t}\} = 0$ if we pick the filter width L such that $L > 2d$ (in some cases, we might be able to get away with just $L = 2d$)

Wavelet Variance for Processes with Stationary Backward Differences: VIII

- conclusions: $\nu_X^2(\tau_j)$ well-defined for $\{X_t\}$ that is
  - stationary: any L will do and $E\{\overline{W}_{j,t}\} = 0$
  - nonstationary with dth order stationary increments: need at least $L \geq 2d$, but might need $L > 2d$ to get $E\{\overline{W}_{j,t}\} = 0$
- if $\{X_t\}$ is stationary, then

$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \text{var}\,\{X_t\} < \infty$$

(recall that each RV in a stationary process must have the same finite variance)

Wavelet Variance for Processes with Stationary Backward Differences: IX

- if $\{X_t\}$ is nonstationary, then

$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \infty$$

- with a suitable construction, we can take the variance of a nonstationary process with dth order stationary increments to be ∞

- using this construction, we have

$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \operatorname{var}\{X_t\}$$

  for both the stationary and nonstationary cases

Background on Gaussian Random Variables

- $\mathcal{N}(\mu, \sigma^2)$ denotes a Gaussian (normal) RV with mean μ and variance σ^2
- will write
$$X \stackrel{\mathrm{d}}{=} \mathcal{N}(\mu, \sigma^2)$$
to mean 'RV X has same distribution as Gaussian RV'
- RV $\mathcal{N}(0, 1)$ often written as Z (called standard Gaussian or standard normal)
- let $\Phi(\cdot)$ be Gaussian cumulative distribution function
$$\Phi(z) \equiv \mathbf{P}[Z \leq z] = \int_{-\infty}^{z} \frac{1}{\sqrt{(2\pi)}} e^{-x^2/2}\, dx$$
- inverse $\Phi^{-1}(\cdot)$ of $\Phi(\cdot)$ is such that $\mathbf{P}[Z \leq \Phi^{-1}(p)] = p$
- $\Phi^{-1}(p)$ called $p \times 100\%$ percentage point

Background on Chi-Square Random Variables

- X said to be a chi-square RV with η degrees of freedom if its probability density function (PDF) is given by

$$f_X(x;\eta) = \frac{1}{2^{\eta/2}\Gamma(\eta/2)} x^{(\eta/2)-1} e^{-x/2}, \quad x \geq 0, \;\; \eta > 0$$

- χ^2_η denotes RV with above PDF
- 3 important facts: $E\{\chi^2_\eta\} = \eta$; $\text{var}\,\{\chi^2_\eta\} = 2\eta$; and, if η is a positive integer and if $Z_1, \ldots, Z_\eta$ are independent $\mathcal{N}(0,1)$ RVs, then

$$Z_1^2 + \cdots + Z_\eta^2 \stackrel{\mathrm{d}}{=} \chi^2_\eta$$

- let $Q_\eta(p)$ denote the pth percentage point for the RV χ^2_η:

$$\mathbf{P}[\chi^2_\eta \leq Q_\eta(p)] = p$$

Unbiased Estimator of Wavelet Variance: I

- given a realization of $X_0, X_1, \ldots, X_{N-1}$ from a process with dth order stationary differences, want to estimate $\nu_X^2(\tau_j)$
- for wavelet filter such that $L \geq 2d$ and $E\{\overline{W}_{j,t}\} = 0$, have

$$\nu_X^2(\tau_j) = \operatorname{var}\{\overline{W}_{j,t}\} = E\{\overline{W}_{j,t}^2\}$$

- can base estimator on squares of

$$\widetilde{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l \bmod N}, \quad t = 0, 1, \ldots, N-1$$

- recall that

$$\overline{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l}, \qquad t \in \mathbb{Z}$$

Unbiased Estimator of Wavelet Variance: II

- comparing

$$\widetilde{W}_{j,t} = \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l \bmod N} \text{ with } \overline{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l}$$

  says that $\widetilde{W}_{j,t} = \overline{W}_{j,t}$ if 'mod N' not needed; this happens when $L_j - 1 \leq t < N$ (recall that $L_j = (2^j - 1)(L-1) + 1)$

- if $N - L_j \geq 0$, unbiased estimator of $\nu_X^2(\tau_j)$ is

$$\hat{\nu}_X^2(\tau_j) \equiv \frac{1}{N - L_j + 1} \sum_{t=L_j-1}^{N-1} \widetilde{W}_{j,t}^2 = \frac{1}{M_j} \sum_{t=L_j-1}^{N-1} \overline{W}_{j,t}^2,$$

  where $M_j \equiv N - L_j + 1$

Statistical Properties of $\hat{\nu}^2_X(\tau_j)$

- assume that $\{\overline{W}_{j,t}\}$ is Gaussian stationary process with mean zero and ACVS $\{s_{j,\tau}\}$
- suppose $\{s_{j,\tau}\}$ is such that

$$A_j \equiv \sum_{\tau=-\infty}^{\infty} s^2_{j,\tau} < \infty$$

(if $A_j = \infty$, can make it finite usually by just increasing L)

- can show that $\hat{\nu}^2_X(\tau_j)$ is asymptotically Gaussian with mean $\nu^2_X(\tau_j)$ and large sample variance $2A_j/M_j$; i.e.,

$$\frac{\hat{\nu}^2_X(\tau_j) - \nu^2_X(\tau_j)}{(2A_j/M_j)^{1/2}} = \frac{M_j^{1/2}(\hat{\nu}^2_X(\tau_j) - \nu^2_X(\tau_j))}{(2A_j)^{1/2}} \stackrel{\text{d}}{=} \mathcal{N}(0,1)$$

approximately for large $M_j \equiv N - L_j + 1$

Estimation of A_j

- in practical applications, need to estimate $A_j = \sum_\tau s^2_{j,\tau}$
- can argue that, for large M_j, the estimator

$$\hat{A}_j \equiv \frac{\left(\hat{s}^{(p)}_{j,0}\right)^2}{2} + \sum_{\tau=1}^{M_j-1} \left(\hat{s}^{(p)}_{j,\tau}\right)^2,$$

is approximately unbiased, where

$$\hat{s}^{(p)}_{j,\tau} \equiv \frac{1}{M_j} \sum_{t=L_j-1}^{N-1-|\tau|} \widetilde{W}_{j,t}\widetilde{W}_{j,t+|\tau|}, \quad 0 \le |\tau| \le M_j - 1$$

- Monte Carlo results: $\hat{A}_j$ reasonably good for $M_j \ge 128$

Confidence Intervals for $\nu_X^2(\tau_j)$: I

- based upon large sample theory, can form a $100(1-2p)\%$ confidence interval (CI) for $\nu_X^2(\tau_j)$:

$$\left[\hat{\nu}_X^2(\tau_j) - \Phi^{-1}(1-p)\frac{\sqrt{2A_j}}{\sqrt{M_j}}, \hat{\nu}_X^2(\tau_j) + \Phi^{-1}(1-p)\frac{\sqrt{2A_j}}{\sqrt{M_j}}\right];$$

  i.e., random interval traps unknown $\nu_X^2(\tau_j)$ with probability $1-2p$
- if A_j replaced by $\hat{A}_j$, approximate $100(1-2p)\%$ CI
- critique: lower limit of CI can very well be negative even though $\nu_X^2(\tau_j) \geq 0$ always
- can avoid this problem by using a χ^2 approximation

Confidence Intervals for $\nu_X^2(\tau_j)$: II

- χ_η^2 useful for approximating distribution of linear combinations of squared Gaussians
- let $U_1, U_2, \ldots, U_K$ be K independent Gaussian RVs with mean 0 & variance σ^2; then, since $\text{var}\,\{U_k^2\} = 2\sigma^4$,

$$Q \equiv \sum_{k=1}^{K} \lambda_k U_k^2 \text{ has } E\{Q\} = \sigma^2 \sum_{k=1}^{K} \lambda_k \;\&\; \text{var}\,\{Q\} = 2\sigma^4 \sum_{k=1}^{K} \lambda_k^2$$

- take distribution of Q to be that of the RV $a\chi_\eta^2$, where a and equivalent degrees of freedom (EDOF) η are to be determined
- because $E\{\chi_\eta^2\} = \eta$ and $\text{var}\,\{\chi_\eta^2\} = 2\eta$, we have $E\{a\chi_\eta^2\} = a\eta$ and $\text{var}\,\{a\chi_\eta^2\} = 2a^2\eta$
- can equate $E\{Q\}$ & $\text{var}\,\{Q\}$ to $a\eta$ & $2a^2\eta$ to determine a & η

Confidence Intervals for $\nu_X^2(\tau_j)$: III

- obtain

$$E\{Q\} = a\eta = \sigma^2 \sum_{k=1}^{K} \lambda_k \text{ and } \operatorname{var}\{Q\} = 2a^2\eta = 2\sigma^4 \sum_{k=1}^{K} \lambda_k^2,$$

which, when combined, yield

$$\eta = \frac{2(E\{Q\})^2}{\operatorname{var}\{Q\}} = \frac{(\sum_{k=1}^{K} \lambda_k)^2}{\sum_{k=1}^{K} \lambda_k^2} \text{ and } a = \sigma^2 \frac{\sum_{k=1}^{K} \lambda_k^2}{\sum_{k=1}^{K} \lambda_k}$$

- can also use to approximate sums of correlated squared Gaussians with zero means, e.g., $\hat{\nu}_X^2(\tau_j) = \frac{1}{M_j} \sum_{t=L_j-1}^{N-1} \overline{W}_{j,t}^2$
- can determine η based upon $E\{\hat{\nu}_X^2(\tau_j)\} = \nu_X^2(\tau_j)$ and an approximation for $\operatorname{var}\{\hat{\nu}_X^2(\tau_j)\}$

Three Ways to Set η: I

1. use large sample theory with appropriate estimates:
$$\eta = \frac{2(E\{\hat{\nu}_X^2(\tau_j)\})^2}{\operatorname{var}\{\hat{\nu}_X^2(\tau_j)\}} \approx \frac{2\nu_X^4(\tau_j)}{2A_j/M_j} \text{ suggests } \hat{\eta}_1 = \frac{M_j \hat{\nu}_X^4(\tau_j)}{\hat{A}_j}$$

2. assume nominal shape for SDF of $\{X_t\}$: $S_X(f) = hC(f)$, where $C(\cdot)$ is known, but h is not; though questionable, get acceptable CIs using
$$\eta_2 = \frac{2\left(\sum_{k=1}^{\lfloor (M_j-1)/2 \rfloor} C_j(f_k)\right)^2}{\sum_{k=1}^{\lfloor (M_j-1)/2 \rfloor} C_j^2(f_k)} \text{ \& } C_j(f) \equiv \int_{-1/2}^{1/2} \widetilde{\mathcal{H}}_j^{(D)}(f) C(f)\, df$$

3. make an assumption about the effect of wavelet filter on $\{X_t\}$ to obtain simple (but effective!) approximation
$$\eta_3 = \max\{M_j/2^j, 1\}$$

Three Ways to Set η: II

- comments on three approaches
  1. $\hat{\eta}_1$ requires estimation of A_j
     - works well for $M_j \geq 128$ (5% to 10% errors on average)
     - can yield optimistic CIs for smaller M_j
  2. η_2 requires specification of shape of $S_X(\cdot)$
     - common practice in, e.g., atomic clock literature
  3. η_3 assumes band-pass approximation
     - default method if M_j small and there is no reasonable guess at shape of $S_X(\cdot)$

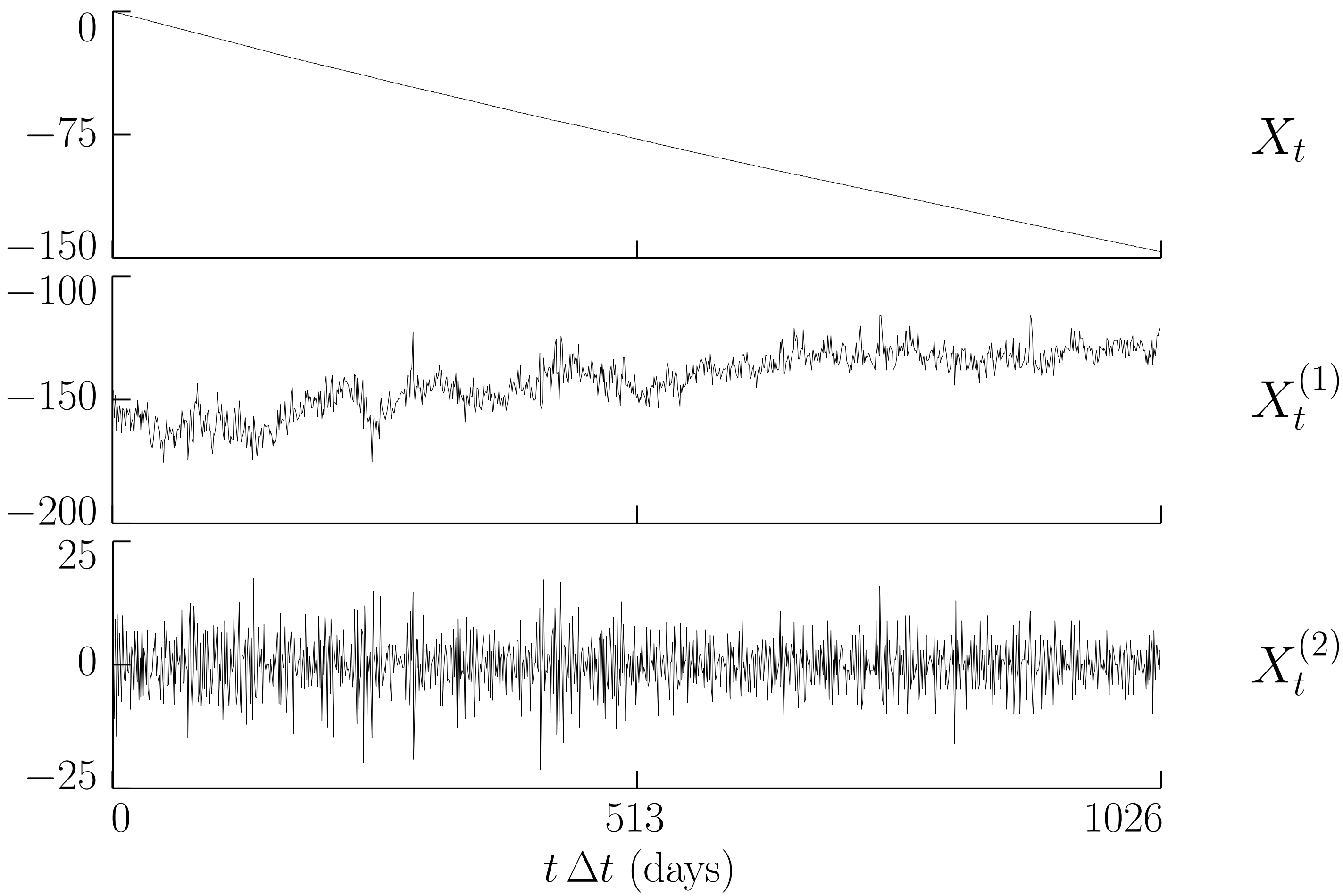
Confidence Intervals for $\nu_X^2(\tau_j)$: IV

- after η has been determined, can obtain a CI for $\nu_X^2(\tau_j)$
- Exer. [313b]: with prob. $1-2p$, the random interval

$$\left[\frac{\eta\hat{\nu}_X^2(\tau_j)}{Q_\eta(1-p)}, \frac{\eta\hat{\nu}_X^2(\tau_j)}{Q_\eta(p)}\right]$$

  traps the true unknown $\nu_X^2(\tau_j)$
- lower limit is now nonnegative
- get approximate $100(1-2p)\%$ CI for $\nu_X^2(\tau_j)$, with approximation improving as $N \to \infty$, if we use $\hat{\eta}_1$ to estimate η
- as $N \to \infty$, above CI and Gaussian-based CI converge

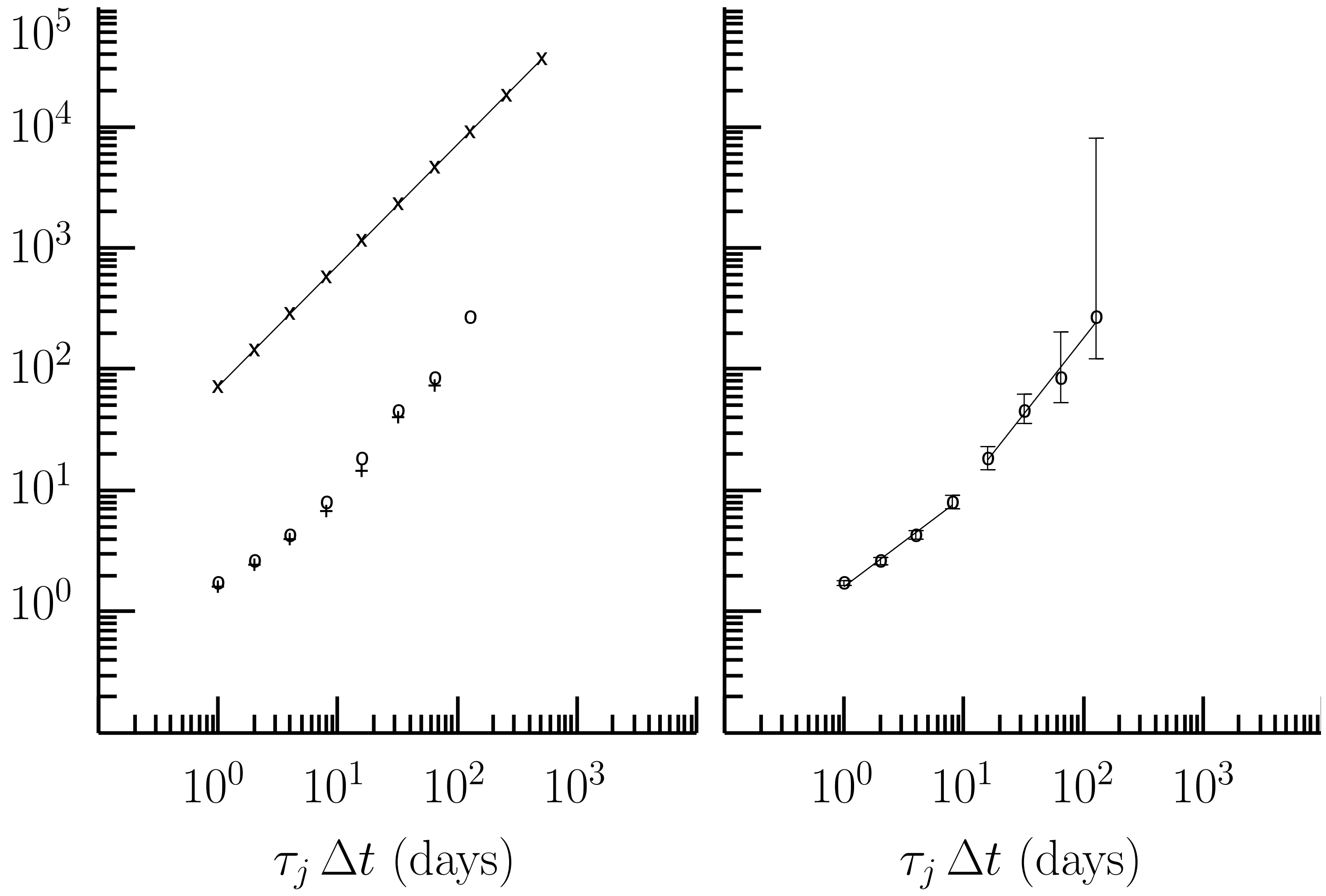
Atomic Clock Deviates: I



Atomic Clock Deviates: II

- top plot: errors $\{X_t\}$ in time kept by atomic clock 571 as compared to time kept at Naval Observatory (measured in microseconds, where 1,000,000 microseconds = 1 second)
- middle: first backward differences $\{X_t^{(1)}\}$ in nanoseconds (1000 nanoseconds = 1 microsecond)
- bottom: second backward differences $\{X_t^{(2)}\}$, also in nanoseconds
- if $\{X_t\}$ nonstationary with dth order stationary increments, need $L \geq 2d$, but might need $L > 2d$ to get $E\{\overline{W}_{j,t}\} = 0$
- Q: what is an appropriate L here?

Atomic Clock Deviates: III



Atomic Clock Deviates: IV

- square roots of wavelet variance estimates for atomic clock time errors $\{X_t\}$ based upon unbiased MODWT estimator with
  - Haar wavelet ($\mathbf{x}$'s in left-hand plot, with linear fit)
  - D(4) wavelet (circles in left- and right-hand plots)
  - D(6) wavelet (pluses in left-hand plot).
- Haar wavelet inappropriate
  - need $\{X_t^{(1)}\}$ to be a realization of a stationary process with mean 0 (stationarity might be OK, but mean 0 is way off)
  - see Exer. [320b] for explanation of linear appearance
- 95% confidence intervals in the right-hand plot are the square roots of intervals computed using the chi-square approximation with η given by $\hat{\eta}_1$ for $j = 1, \ldots, 6$ and by η_3 for $j = 7$ & 8

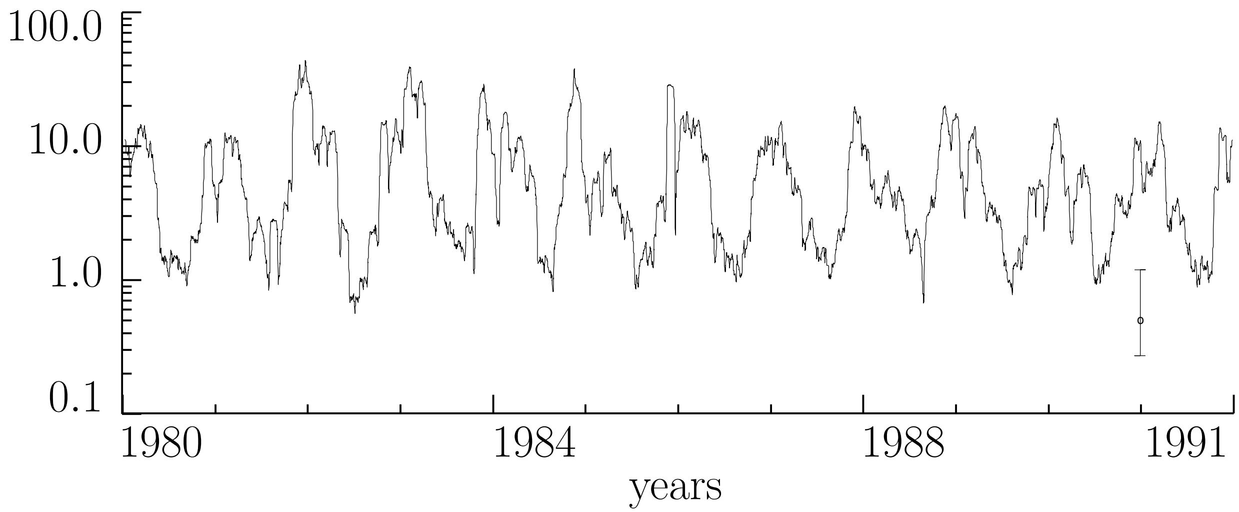
Wavelet Variance Analysis of Time Series with Time-Varying Statistical Properties

- each wavelet coefficient $\widetilde{W}_{j,t}$ formed using portion of X_t
- suppose X_t associated with actual time $t_0 + t\,\Delta t$
  - t_0 is actual time of first observation X_0
  - Δt is spacing between adjacent observations
- suppose $\tilde{h}_{j,l}$ is least asymmetric Daubechies wavelet
- can associate $\widetilde{W}_{j,t}$ with an interval of width $2\tau_j\,\Delta t$ centered at

$$t_0 + (2^j(t+1) - 1 - |\nu_j^{(H)}| \bmod N)\,\Delta t,$$

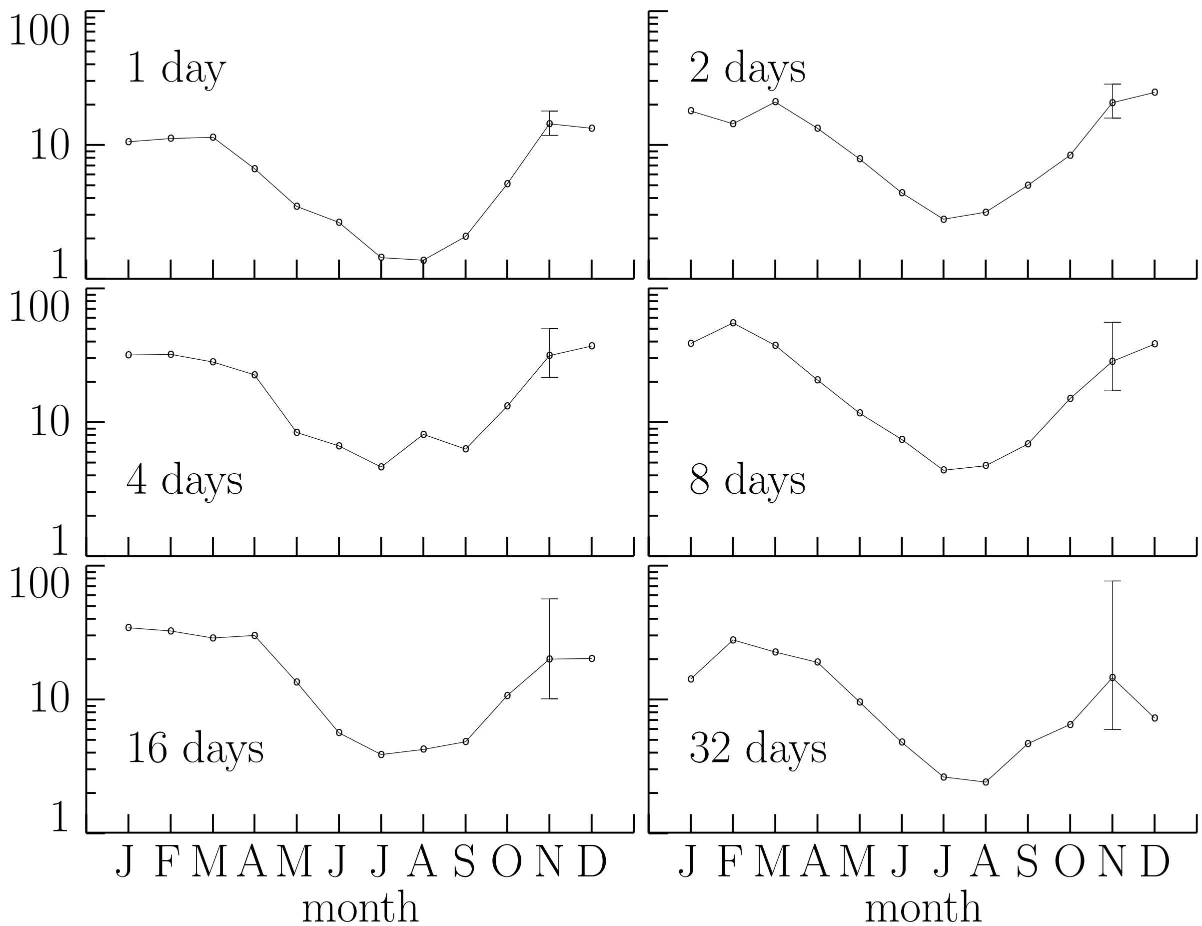
  where, e.g., $|\nu_j^{(H)}| = [7(2^j - 1) + 1]/2$ for LA(8) wavelet
- can thus form 'localized' wavelet variance analysis (implicitly assumes stationarity or stationary increments locally)

Subtidal Sea Level Fluctuations: I



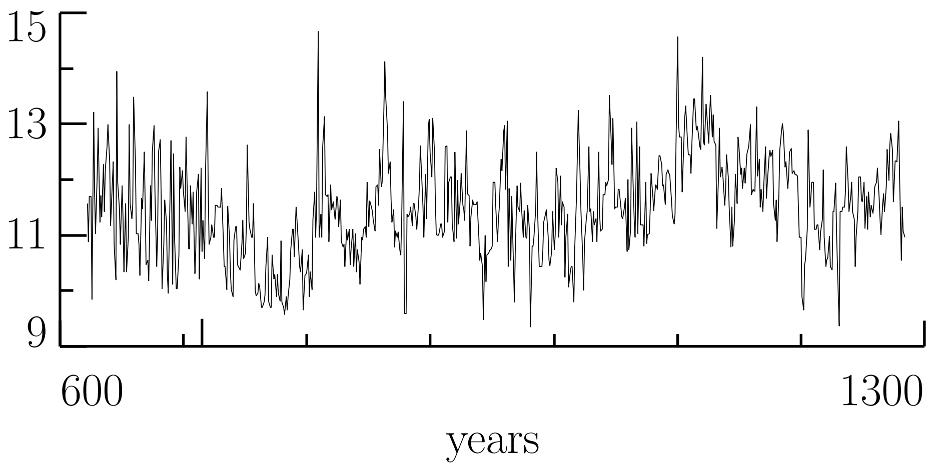
- estimated time-dependent LA(8) wavelet variances for physical scale $\tau_2 \Delta t = 1$ day based upon averages over monthly blocks (30.5 days, i.e., 61 data points)
- plot also shows a representative 95% confidence interval based upon a hypothetical wavelet variance estimate of $1/2$ and a chi-square distribution with $\eta = 15.25$

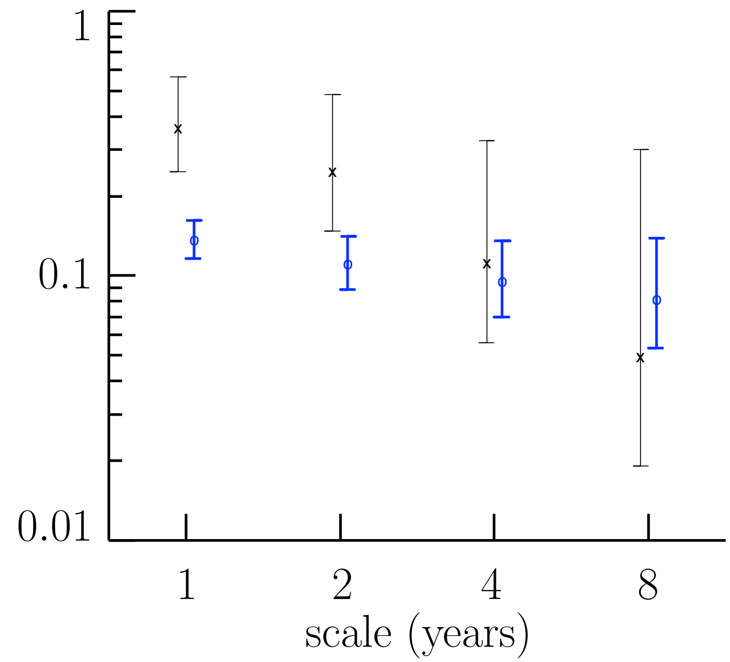
Subtidal Sea Level Fluctuations: II



- estimated LA(8) wavelet variances for physical scales $\tau_j\,\Delta t = 2^{j-2}$ days, $j = 2, \ldots, 7$, grouped by calendar month

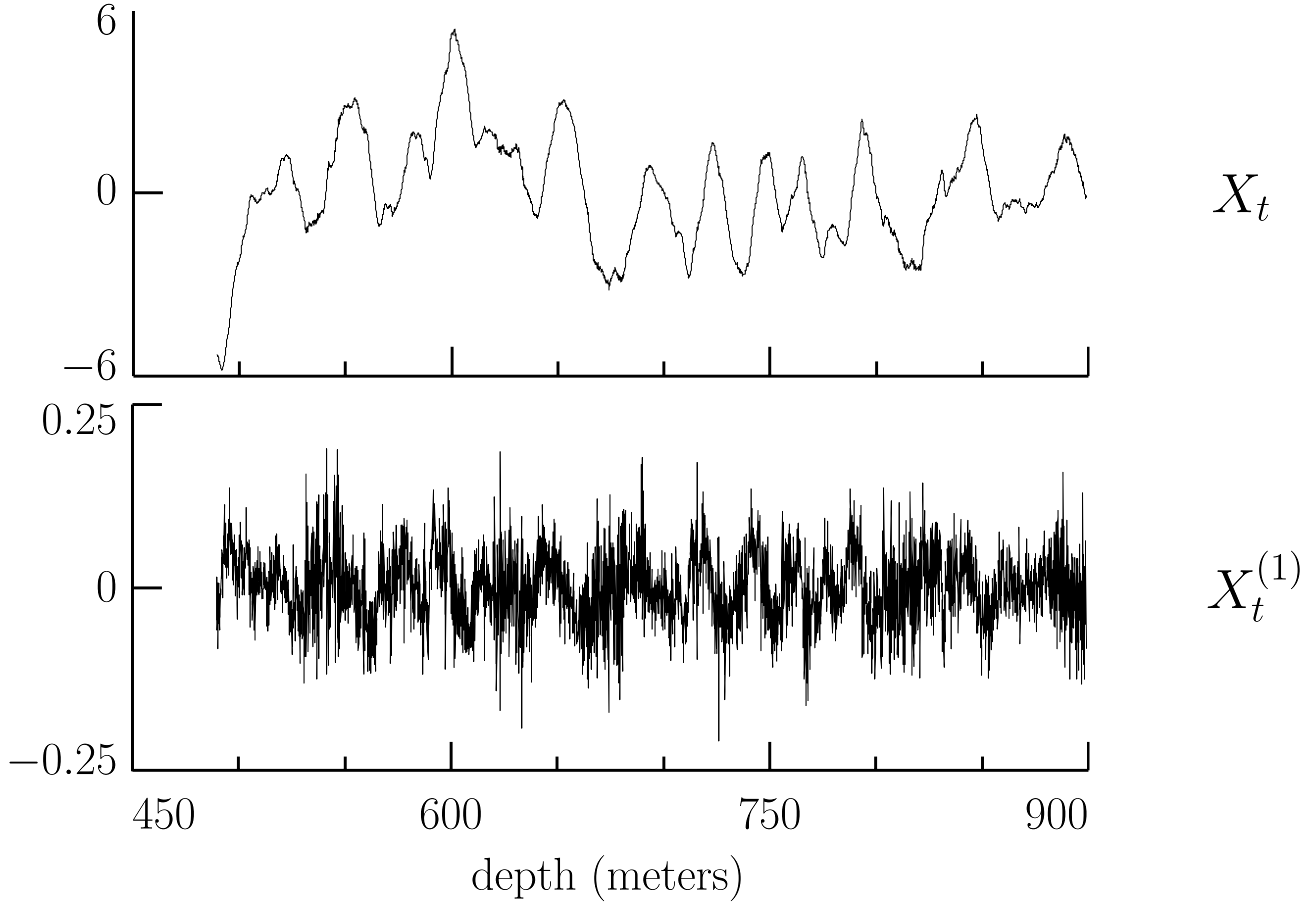
Annual Minima of Nile River





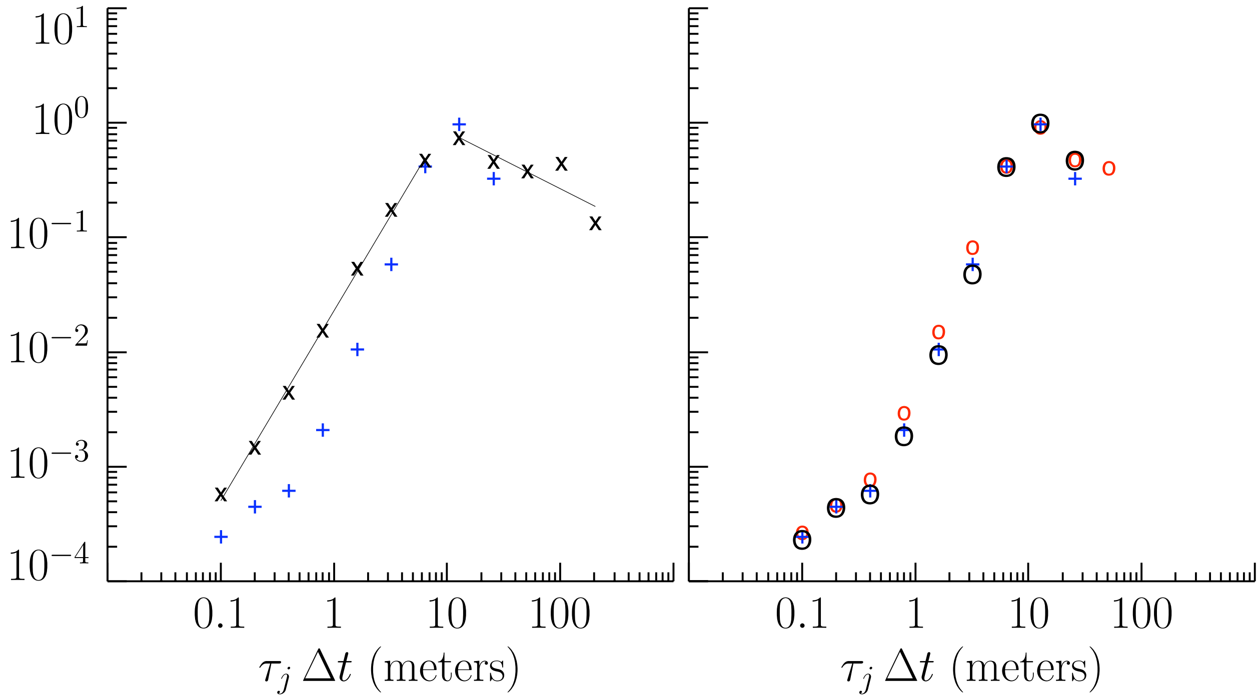
- left-hand plot: annual minima of Nile River
- right: Haar $\hat{\nu}_X^2(\tau_j)$ before ($\mathtt{x}$'s) and after ($\mathtt{o}$'s) year 715.5, with 95% confidence intervals based upon $\chi^2_{\eta_3}$ approximation

Vertical Shear in the Ocean: I



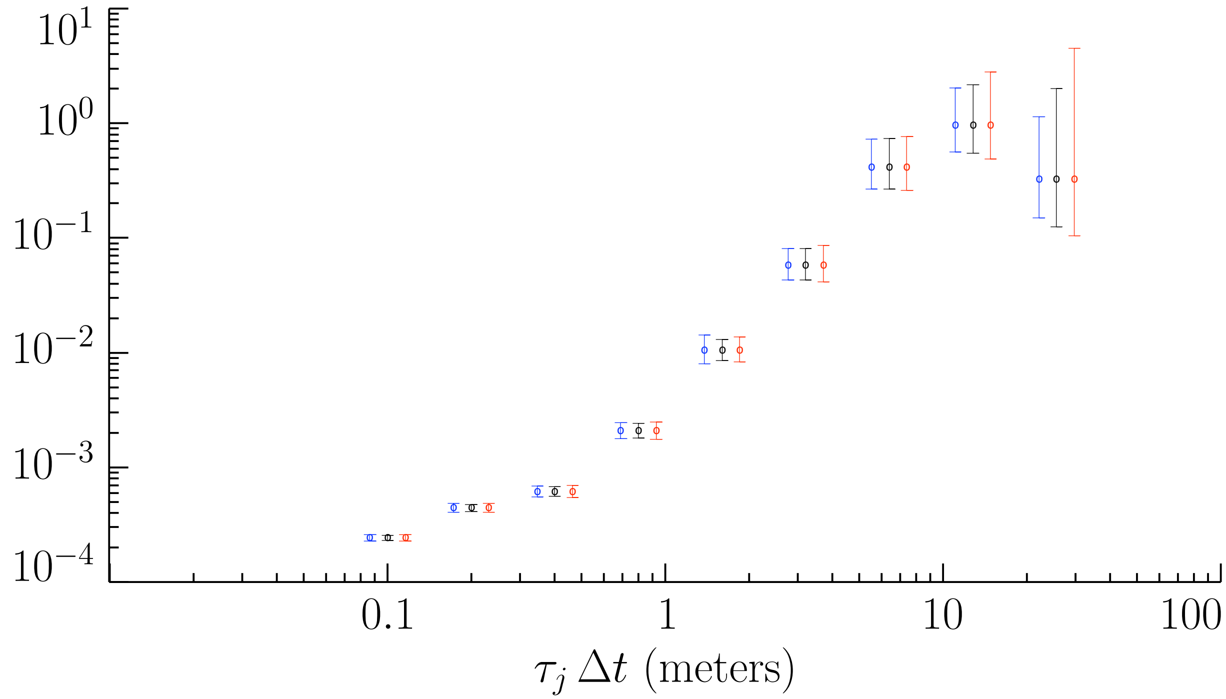
- selected 'stationary' portion of vertical shear measurements $\{X_t\}$ (top plot) and their first backward differences $\{X_t^{(1)}\}$

Vertical Shear in the Ocean: II



- unbiased MODWT wavelet variance estimates using the following wavelet filters: Haar (**x**'s in left-hand plot, through which two regression lines have been fit); D(4) (small circles, right-hand plot); D(6) (pluses, both plots); and LA(8) (big circles, right-hand plot).

Vertical Shear in the Ocean: III



- D(6) wavelet variance estimates, along with 95% confidence intervals for true wavelet variance with EDOFs determined by, from left to right within each group of 3, $\hat{\eta}_1$ (estimated from data), η_2 (using a nominal model for $S_X(\cdot)$) and $\eta_3 = \max\{M_j/2^j, 1\}$

Some Extensions and Ongoing Work

- biased estimators of wavelet variance
- unbiased estimator of wavelet variance for ‘gappy’ time series
- asymptotic theory for non-Gaussian processes satisfying a certain ‘mixing’ condition
- wavelet cross-covariance and cross-correlation
- extension of notion and estimators to random fields

Summary

- wavelet variance gives scale-based analysis of variance
- presented statistical theory for Gaussian processes with stationary increments
- in addition to the applications we have considered, the wavelet variance has been used to analyze
  - genome sequences
  - changes in variance of soil properties
  - canopy gaps in forests
  - accumulation of snow fields in polar regions
  - boundary layer atmospheric turbulence
  - regular and semiregular variable stars