Daubechies Wavelet/Scaling Filters: I

- orthonormality constraints on $\{h_l\}$ yield orthonormal $\mathcal{W}$, but these alone are not sufficient to yield 'reasonable' MRA (i.e., one interpretable as a 'scale by scale' decomposition)
- 'regularity' conditions lead to Daubechies wavelet filters
- Daubechies $\{h_l\}$'s *defined* via squared gain functions:

$$\mathcal{H}^{(D)}(f) \equiv 2\sin^L(\pi f) \sum_{l=0}^{\frac{L}{2}-1} \binom{\frac{L}{2}-1+l}{l} \cos^{2l}(\pi f)$$

  - $2\sin^L(\pi f) \propto$ squared gain for difference filter of order $L/2$
  - 2nd part is squared gain for either 'all-pass' filter ($L=2$) or low-pass filter ($L=4,6,\ldots$) with width $L/2$

Daubechies Wavelet/Scaling Filters: II

- corresponding squared gain for $\{g_l\}$ given by
$$\mathcal{G}^{(D)}(f) = 2\cos^L(\pi f) \sum_{l=0}^{\frac{L}{2}-1} \binom{\frac{L}{2}-1+l}{l} \sin^{2l}(\pi f)$$
- filter $\{g_l\}$ fully defined by transfer function $G^{(D)}(\cdot)$
- specifying $\mathcal{G}^{(D)}(f) = |G^{(D)}(f)|^2$ just constrains $\{g_l\}$
- $L = 2$: 2 real-valued filters with same squared gain $\mathcal{G}^{(D)}(\cdot)$:
$$\{\, g_0 = \tfrac{1}{\sqrt{2}},\ g_1 = \tfrac{1}{\sqrt{2}} \,\} \text{ and } \{\, g_0 = -\tfrac{1}{\sqrt{2}},\ g_1 = -\tfrac{1}{\sqrt{2}}\,, \}$$
but, if we insist $\sum g_l = \sqrt{2}$ rather than $-\sqrt{2}$, only 1 filter
- $L = 4$: 4 filters with $\mathcal{G}^{(D)}(\cdot)$ (two directions paired with ± 1)
- as $L \uparrow$, get more filters with different $G^{(D)}(\cdot)$ but same $\mathcal{G}^{(D)}(\cdot)$

Daubechies Wavelet/Scaling Filters: III

- can obtain all possible $\{g_l\}$ (and hence $\{h_l\}$) systematically using a procedure called 'spectral factorization'
- Daubechies (1992) defined two classes of wavelets via criteria that select a particular scaling filter $\{g_l\}$
- one criterion leads to 'extremal phase' class
- another criterion leads to 'least asymmetric' class

Extremal Phase Scaling Filters: I

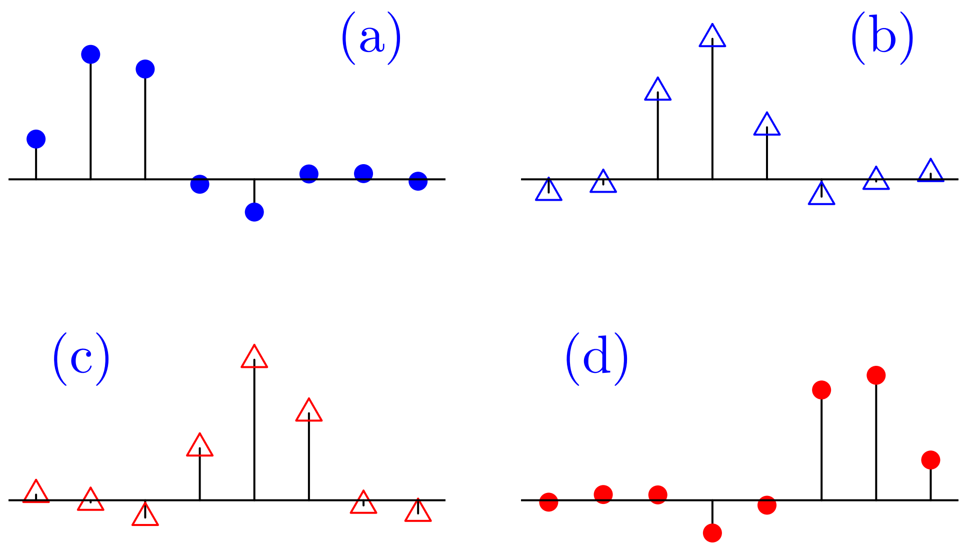
- denote these filters by $\{g_l^{(ep)}\}$
- by definition, if $\{g_l\}$ and $\{g_l^{(ep)}\}$ have same $\mathcal{G}^{(D)}(\cdot)$, then

$$\sum_{l=0}^{m} g_l^2 \leq \sum_{l=0}^{m} \left[g_l^{(ep)}\right]^2 \text{ for } m = 0, \ldots, L-1$$

- summing up to m defines mth term of partial energy sequence
- partial energy builds up fastest for $\{g_l^{(ep)}\}$ ('front loaded')
- note: above condition also called 'minimum phase'
- filter of width L called D(L) scaling filter; e.g., D(4), D(6)
- $\{g_l^{(ep)}\}$ for $L = 4, 6, \ldots, 20$ are on course Web site

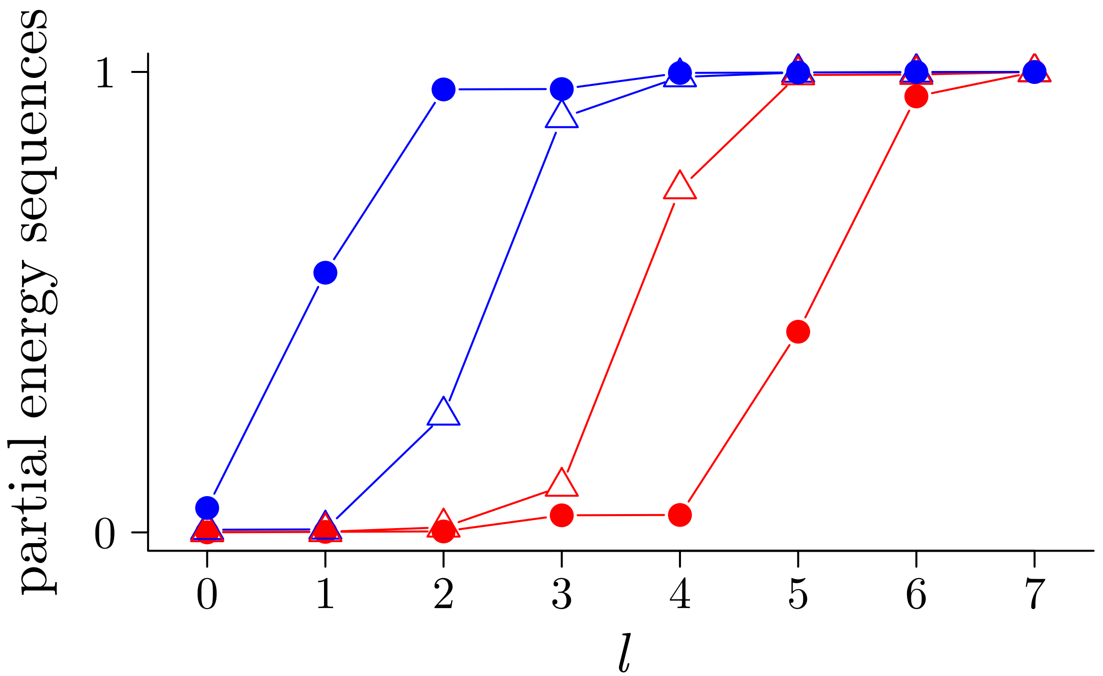
Extremal Phase Scaling Filters: II

- spectral factorization leads to four possible $\{g_l\}$ for $L = 8$



Extremal Phase Scaling Filters: III

- here are corresponding partial energy sequences



- scaling filter (a) on previous overhead is D(8) scaling filter

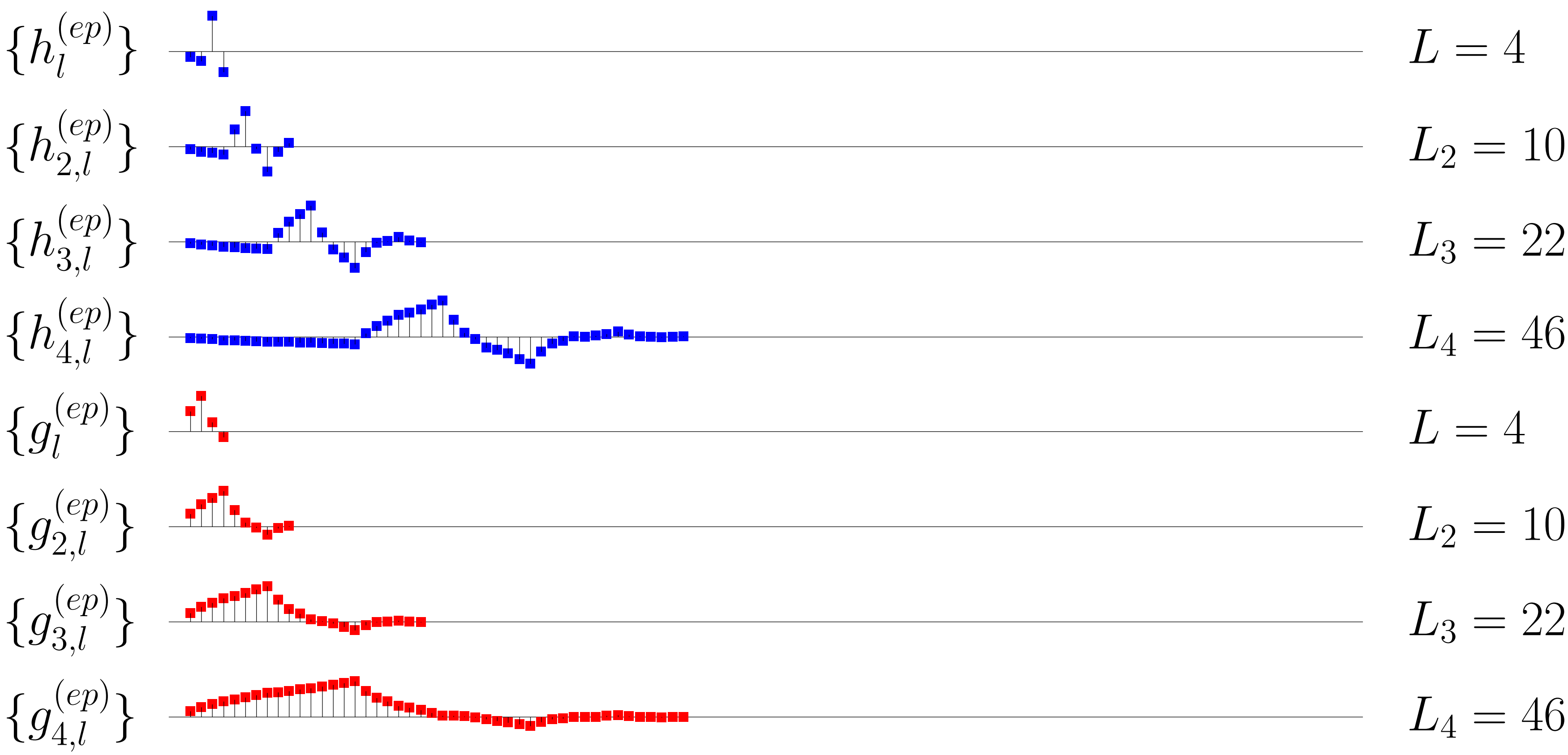
Extremal Phase Scaling Filters for $L = 4, 6, \ldots, 20$

- note that $\{g_l^{(ep)}\}$'s are front loaded

Extremal Phase Wavelet Filters for $L = 4, 6, \ldots, 20$

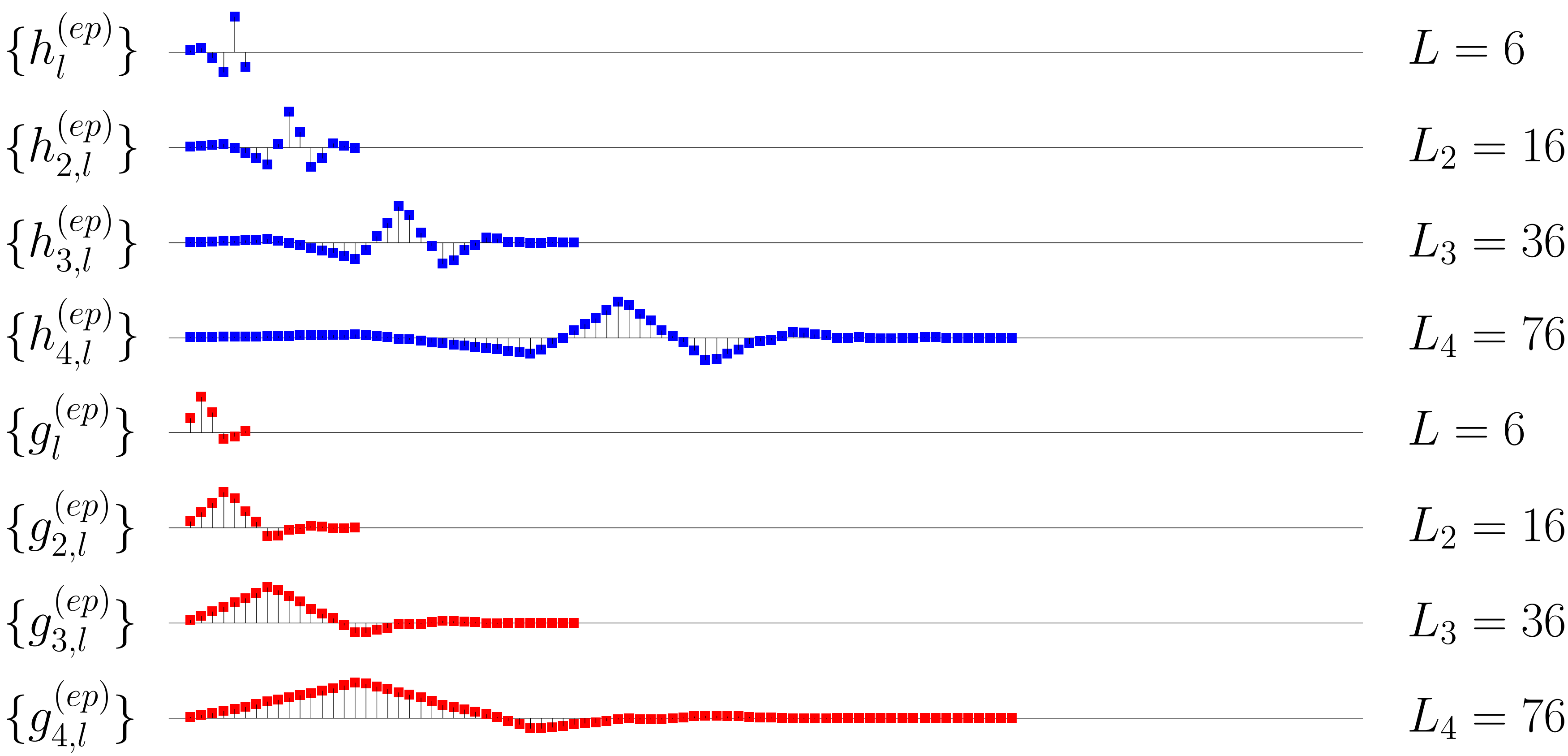
- note that $\{h_l^{(ep)}\}$'s are back loaded

D(4) Wavelet & Scaling Filters Revisited



- jth level D(4) wavelet filters $\{h_{j,l}^{(ep)}\}$'s are back loaded, whereas corresponding scaling filters $\{g_{j,l}^{(ep)}\}$'s are front loaded

D(6) Wavelet & Scaling Filters Revisited



- again $\{h_{j,l}^{(ep)}\}$'s are back loaded while $\{g_{j,l}^{(ep)}\}$'s are front loaded

Least Asymmetric Scaling Filters: Introduction

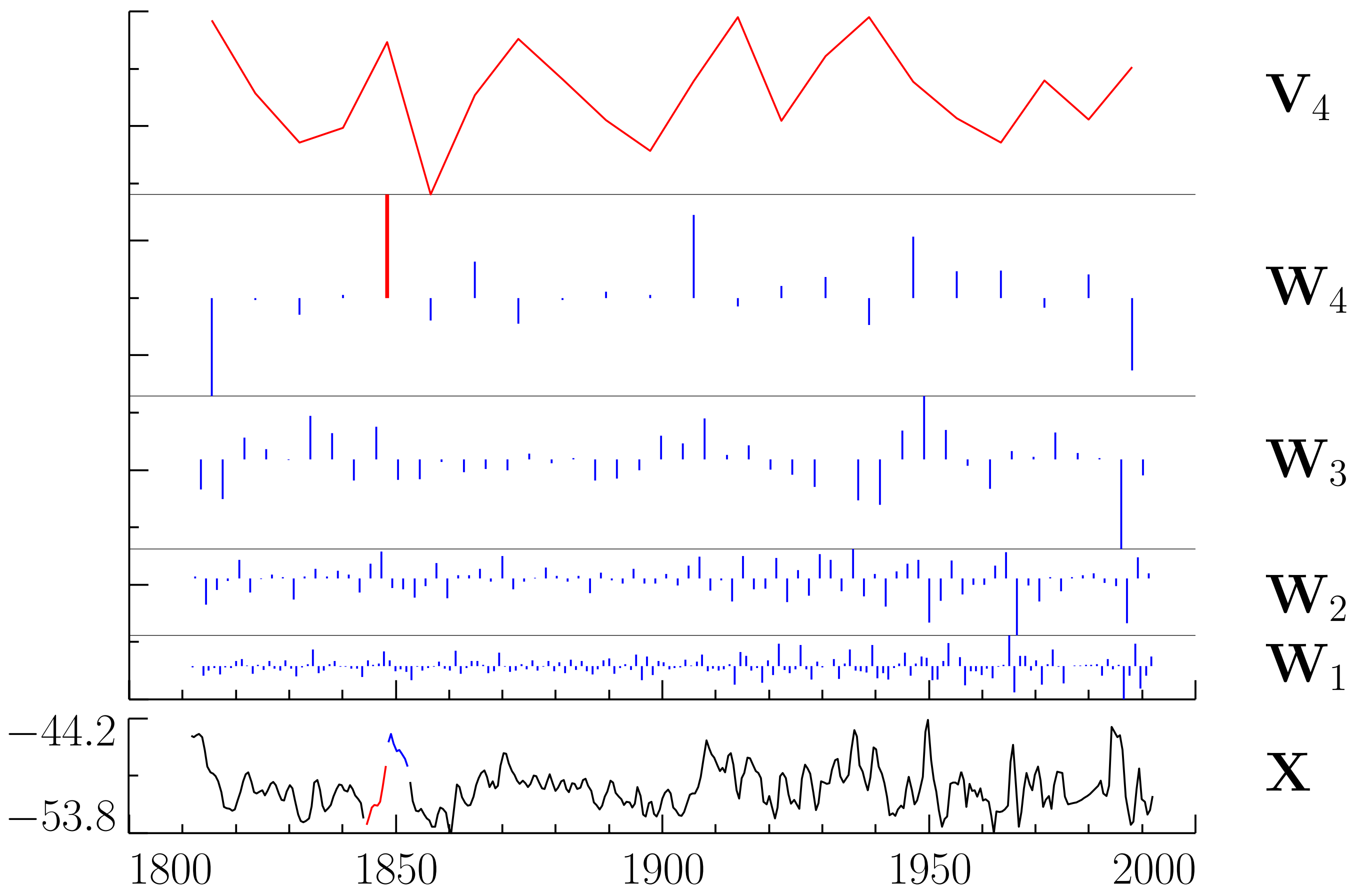
- denote these filters by $\{g_l^{(la)}\}$
- idea is to pick the filter closest to being symmetric, with symmetry being measured in terms of the phase function $\theta(\cdot)$:

$$G^{(D)}(f) = \sqrt{\mathcal{G}^{(D)}(f)}e^{i\theta(f)}$$

- filter of width L called LA(L) scaling filter; e.g., LA(8), LA(16)
- LA(2), LA(4) and LA(6) same as Haar, D(4) and D(6)
- LA(L) and D(L) scaling filters differ for $L = 8, 10, 12, \ldots$
- Q: why is symmetry of interest?

Assigning Times to Wavelet Coefficients: I

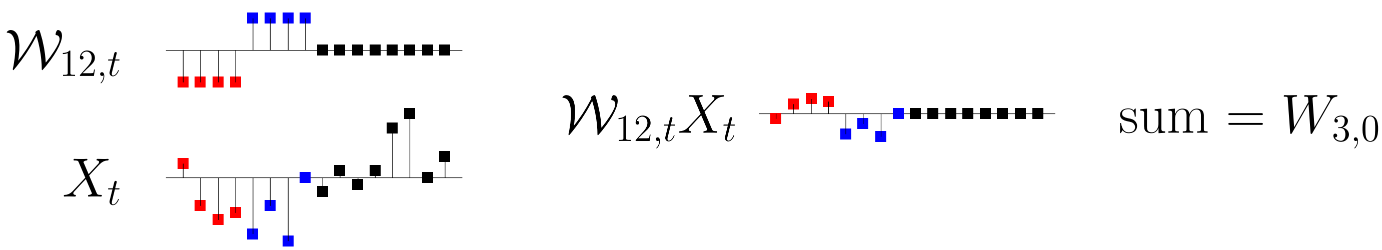
- recall example of $J_0 = 4$ partial Haar DWT:



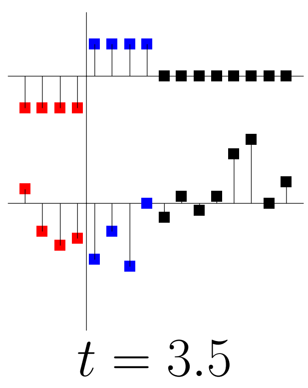
- Q: how did we decide to plot $W_{4,4}$ at 1848.3?

Assigning Times to Wavelet Coefficients: II

- symmetry in filter allows association of $W_{j,t}$ with X_t values
- recall formation of $W_{3,0}$ in $N = 16$ example:



- can associate $W_{3,0}$ with time 3.5 because Haar $\{h_{3,l}\}$ has a well-defined point of symmetry:



Zero Phase Filters: I

- LA class of wavelet and scaling filters designed to exhibit 'near symmetry' about some point in the filter
- makes it easier to align $W_{j,t}$ and $V_{J_0,t}$ with values in $\mathbf{X}$
- can quantify symmetry by considering 'zero phase' filters, so need to introduce ideas behind this type of filter
- consider filter $\{u_l\} \longleftrightarrow U(\cdot)$; i.e, $U(f) = \sum_{l=-\infty}^{\infty} u_l e^{-i2\pi fl}$
- write $U(f) = |U(f)|e^{i\theta(f)}$, where the gain function is defined by $|U(f)|$, and $\theta(\cdot)$ is the phase function

Zero Phase Filters: II

- let $\{u_l^\circ\}$ be $\{u_l\}$ periodized to length N
- Exer. [33] says that $\{u_l^\circ\} \longleftrightarrow \{U(\frac{k}{N})\}$, where both l and k take the values $0, 1, \ldots, N-1$
- let $\{X_t\}$ be time series of length N with DFT $\{\mathcal{X}_k\}$
- let $\{Y_t\}$ be $\{X_t\}$ circularly filtered with $\{u_l^\circ\}$:

$$Y_t \equiv \sum_{l=0}^{N-1} u_l^\circ X_{t-l \bmod N}, \quad t = 0, 1, \ldots, N-1$$

- hence $\{Y_t\} \longleftrightarrow \{U(\frac{k}{N})\mathcal{X}_k\}$

Zero Phase Filters: III

- since $\{Y_t\} \longleftrightarrow \{U(\frac{k}{N})\mathcal{X}_k\}$, inverse DFT says

$$Y_t = \frac{1}{N}\sum_{k=0}^{N-1} U(\tfrac{k}{N})\mathcal{X}_k e^{i2\pi kt/N}$$

- suppose $\{u_l\}$ has zero phase; i.e., $\theta(f) = 0$ for all f
- since $U(f) = |U(f)|$, have $U(\frac{k}{N}) = |U(\frac{k}{N})|$, so

$$Y_t = \frac{1}{N}\sum_{k=0}^{N-1} |U(\tfrac{k}{N})|\mathcal{X}_k e^{i2\pi kt/N}$$

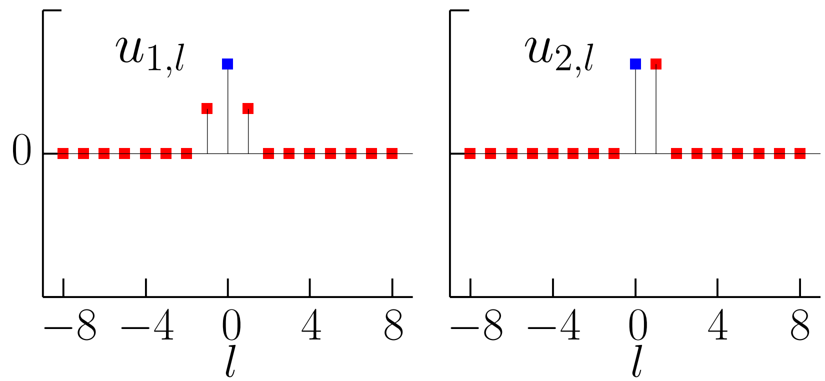
- $|U(\frac{k}{N})|\mathcal{X}_k$ & $\mathcal{X}_k$ have the same phase, but amplitudes can differ
- thus components in output $\{Y_t\}$ that are undamped by filter will line up with similar components in input $\{X_t\}$

Zero Phase Filters: IV

- examples with and without zero phase:

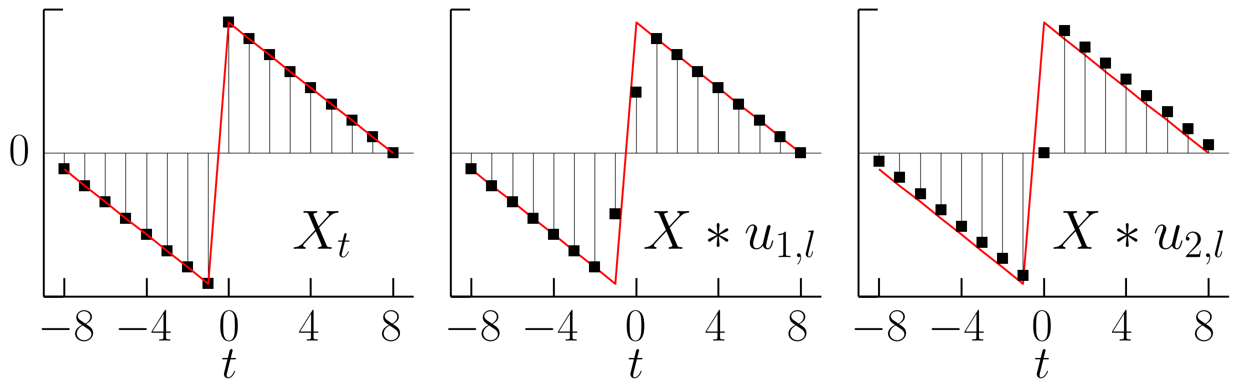
$$u_{1,l} = \begin{cases} 1/2, & l = 0; \\ 1/4, & l = \pm 1; \\ 0, & \text{otherwise}; \end{cases} \quad \text{and} \quad u_{2,l} = \begin{cases} 1/2, & l = 0, 1; \\ 0, & \text{otherwise}, \end{cases}$$

for which $\{u_{1,l}\} \longleftrightarrow \cos^2(\pi f)$ and $\{u_{2,l}\} \longleftrightarrow e^{-i\pi f}\cos(\pi f)$



Zero Phase Filters: V

- Fig. 110: example of filtering $\{X_t\}$ with low-pass filters $\{u_{1,l}\}$ and $\{u_{2,l}\}$



Linear Phase Filters: I

- LA $\{g_l\}$'s formulated in terms of linear phase filters
- to relate linear phase and zero phase ideas, consider circularly shifting $\{Y_t\}$ by ν units:

$$Y_t^{(\nu)} \equiv Y_{t+\nu \bmod N}, \quad t = 0, \ldots, N-1$$

- example: $\nu = 2$ & $N = 11$ yields $Y_8^{(2)} = Y_{8+2 \bmod 11} = Y_{10}$, with $Y_8^{(2)}$ occurring 2 time units earlier than Y_{10}
- $\{Y_t^{(\nu)}\}$ advanced version of $\{Y_t\}$ if $\nu > 0$
- $\{Y_t^{(\nu)}\}$ delayed version of $\{Y_t\}$ if $\nu < 0$

Linear Phase Filters: II

- note following:

$$
\begin{aligned}
Y_t^{(\nu)} = Y_{t+\nu \bmod N} &= \sum_{l=0}^{N-1} u_l^\circ X_{t+\nu-l \bmod N} \\
&= \sum_{l=-\nu}^{N-1-\nu} u_{l+\nu}^\circ X_{t-l \bmod N} \\
&= \sum_{l=-\nu}^{N-1-\nu} u_{l+\nu \bmod N}^\circ X_{t-l \bmod N} \\
&= \sum_{l=0}^{N-1} u_{l+\nu \bmod N}^\circ X_{t-l \bmod N}
\end{aligned}
$$

- thus can advance filter output by advancing filter

Linear Phase Filters: III

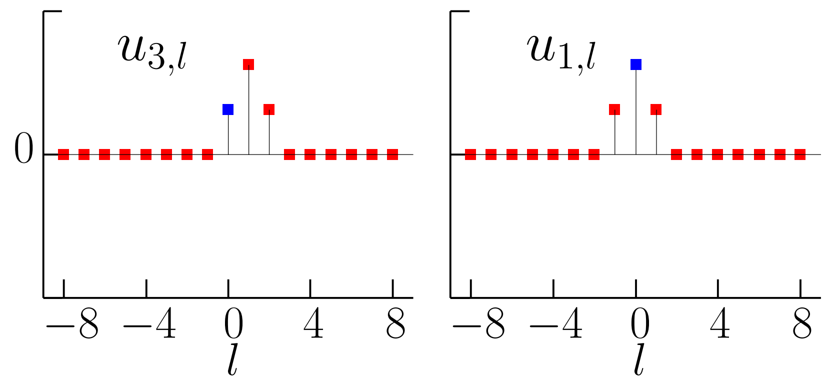
- $\{u^\circ_{l+\nu \bmod N} : l = 0, \ldots, N-1\}$ periodized version of $\{u_l^{(\nu)} \equiv u_{l+\nu} : l = \ldots, -1, 0, 1, \ldots\}$
- phase properties of $\{u^\circ_{l+\nu \bmod N}\}$ depend on transfer function $U^{(\nu)}(\cdot)$ for $\{u_l^{(\nu)}\}$
- Exer. [111]: $U^{(\nu)}(f) = e^{i2\pi f\nu}U(f)$
- suppose $\{u_l\}$ has zero phase so $U(f) = |U(f)|$
- implies $\{u_l^{(\nu)}\}$ has $\theta^{(\nu)}(f) = 2\pi f\nu$
- $\{u_l^{(\nu)}\}$ said to have linear phase
- conclusion: if ν is an integer, can convert linear phase filter to zero phase filter by appropriately advancing the filter

Linear Phase Filters: IV

- example:

$$u_{3,l} = \begin{cases} 1/2, & l = 1; \\ 1/4, & l = 0 \text{ or } 2; \\ 0, & \text{otherwise}; \end{cases} \longleftrightarrow \cos^2(\pi f) e^{-i2\pi f}$$

  - $\theta_3(f) = -2\pi f$, i.e., linear phase with $\nu = -1$
  - advancing $\{u_{3,l}\}$ by 1 unit yields zero phase filter $\{u_{1,l}\}$



Definition of Least Asymmetric Scaling Filters

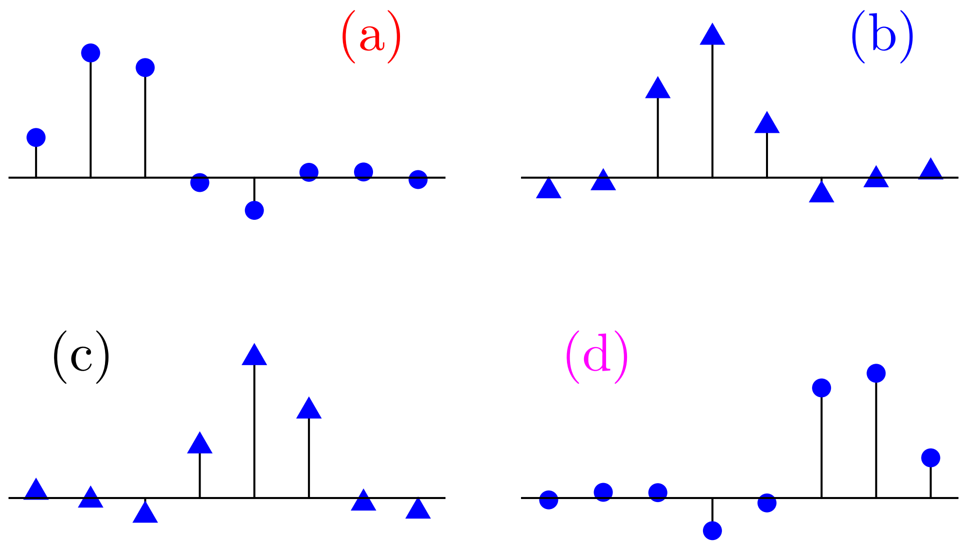
- consider the set of phase functions $\theta^{(G)}(\cdot)$ associated with all possible factorizations of $\mathcal{G}^{(D)}(\cdot)$ such that $\sum g_l = \sqrt{2}$
- definition of LA(L) scaling filter: factorization of $\mathcal{G}^{(D)}(\cdot)$ with $\theta^{(G)}(\cdot)$ such that

$$\min_{\tilde{\nu}=0,\pm 1,\ldots} \left\{ \max_{-\frac{1}{2}\leq f \leq \frac{1}{2}} \left| \theta^{(G)}(f) - 2\pi f \tilde{\nu} \right| \right\}$$

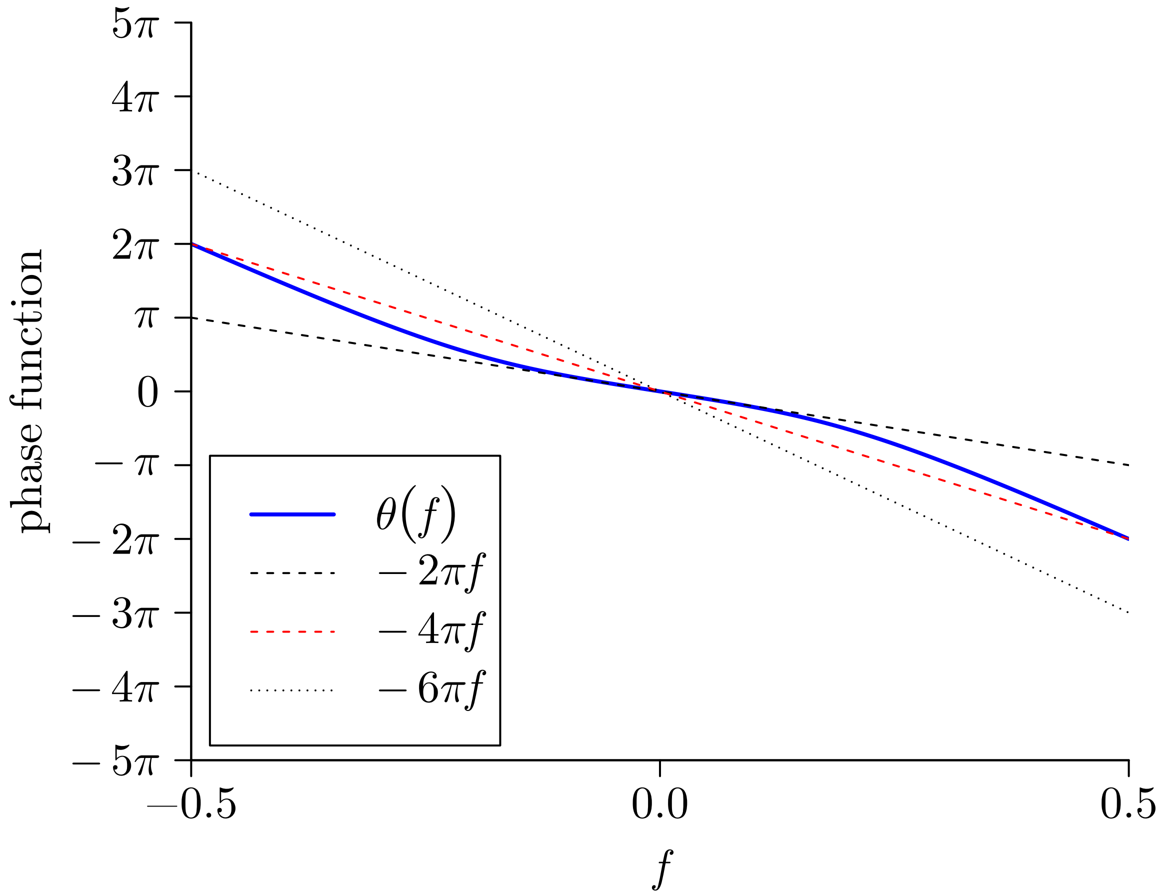
  is minimized
- let ν be the $\tilde{\nu}$ that minimizes the above; i.e., $\theta^{(G)}(f) \approx 2\pi f \nu$
- let $\{h_l^{(la)}\}$ denote wavelet filter corresponding to LA(L) scaling filter $\{g_l^{(la)}\}$

Determination of LA(8) Scaling Filter

- recall four possible $\{g_l\}$ for $L = 8$

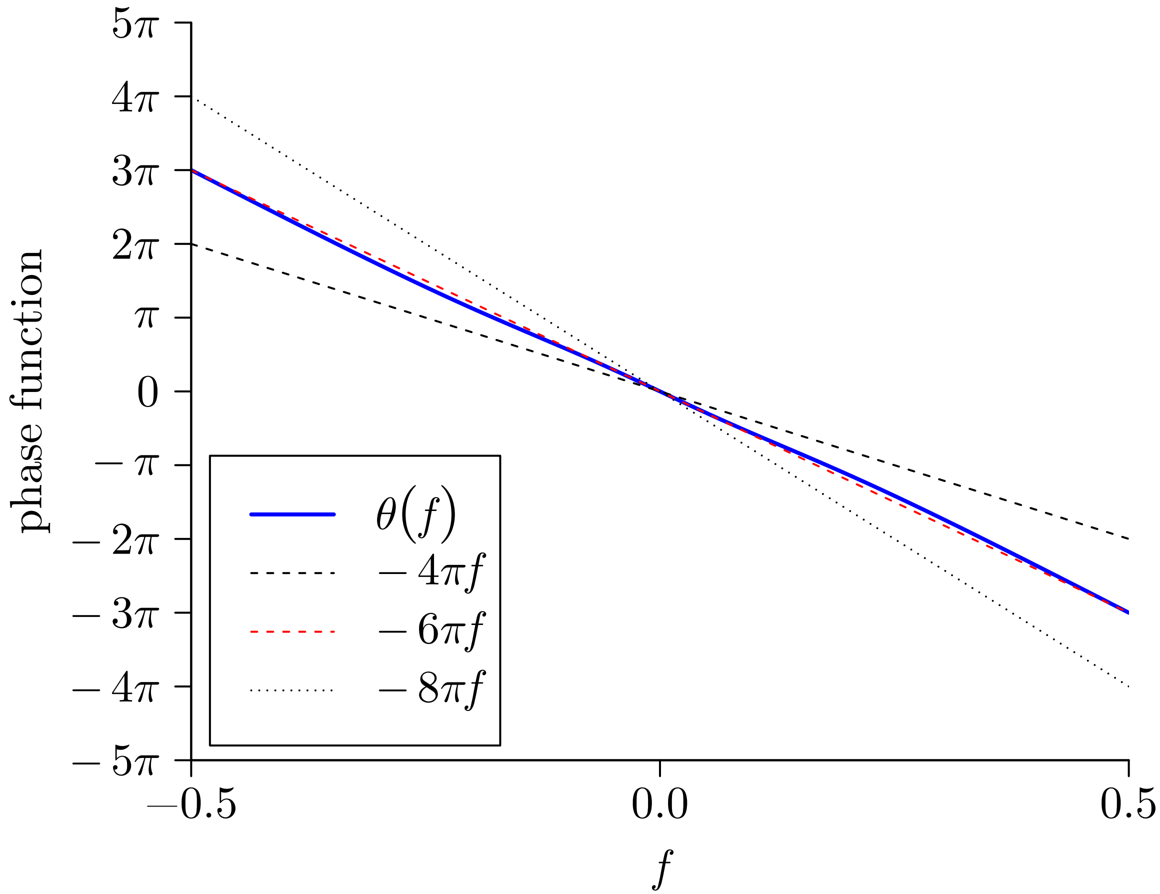


Phase Function for Filter (a)



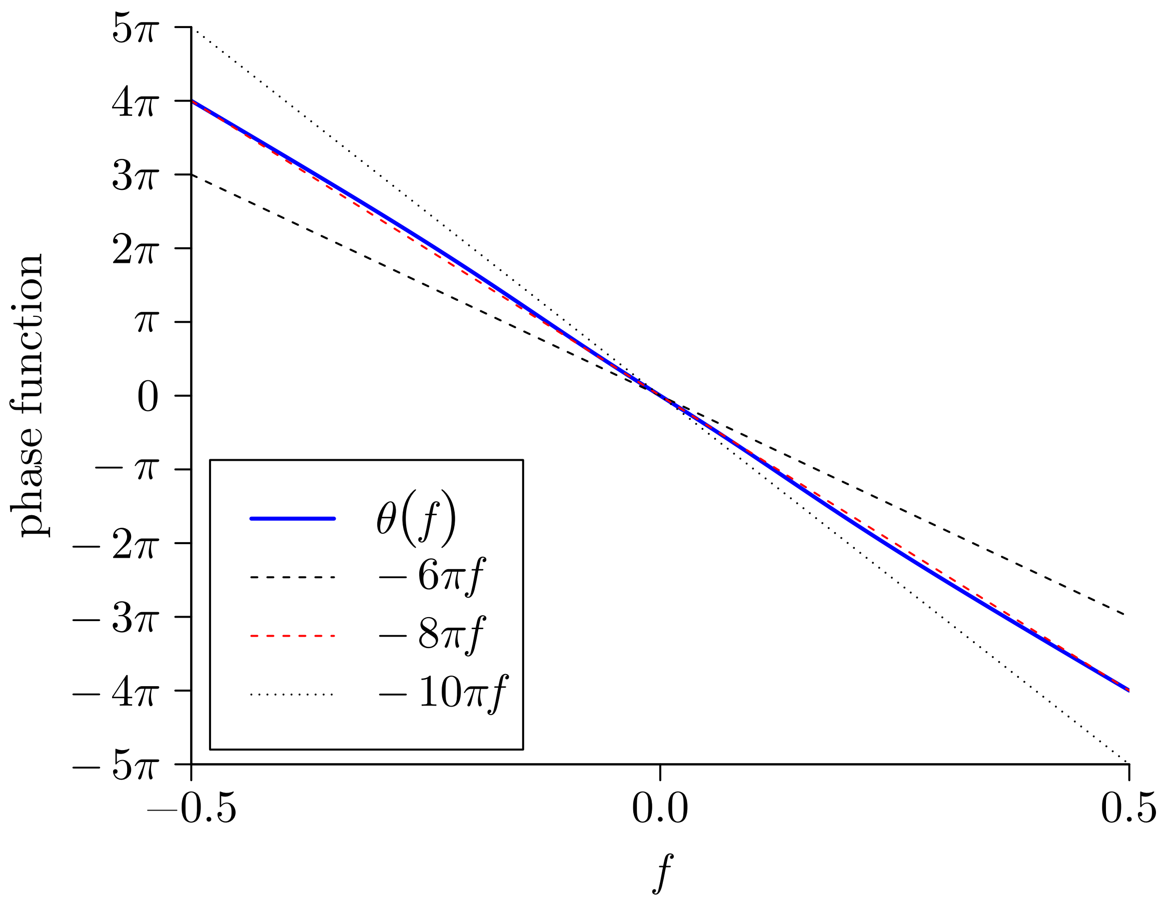
- setting $\nu = -2$ in $2\pi f \nu$ yields best approximation to $\theta^{(G)}(f)$

Phase Function for Filter (b)



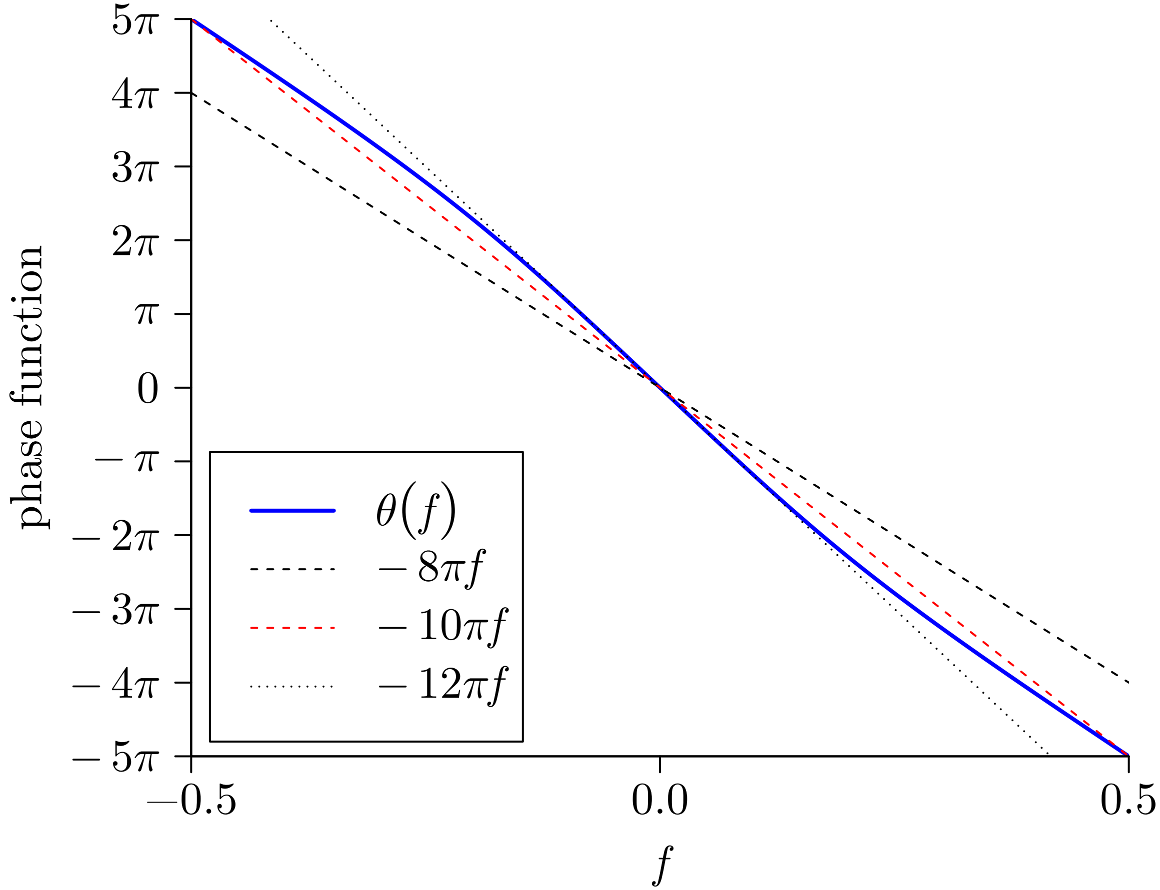
- setting $\nu = -3$ in $2\pi f\nu$ yields best approximation to $\theta^{(G)}(f)$

Phase Function for Filter (c)



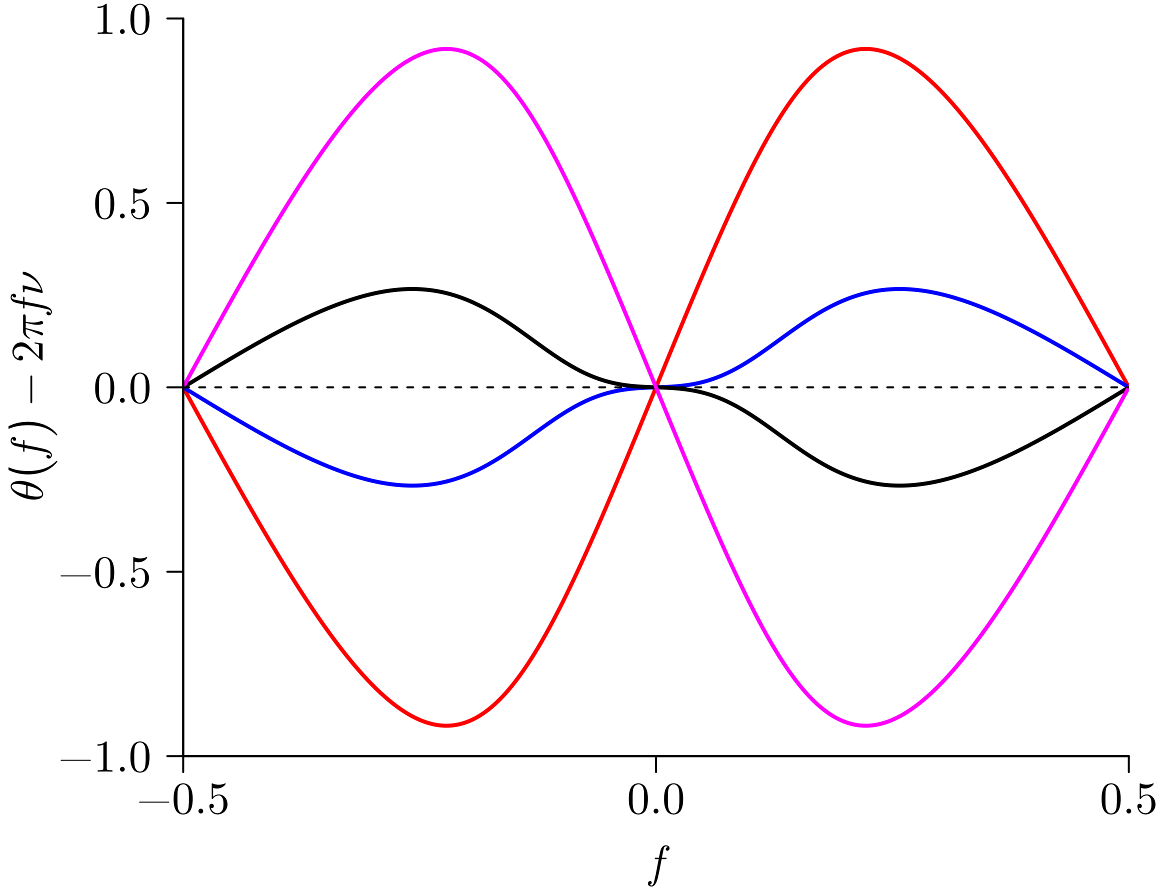
- setting $\nu = -4$ in $2\pi f \nu$ yields best approximation to $\theta^{(G)}(f)$

Phase Function for Filter (d)



- setting $\nu = -5$ in $2\pi f\nu$ yields best approximation to $\theta^{(G)}(f)$

$\theta^{(G)}(f) - 2\pi f \nu$ for Filters (a), (b), (c) and (d)

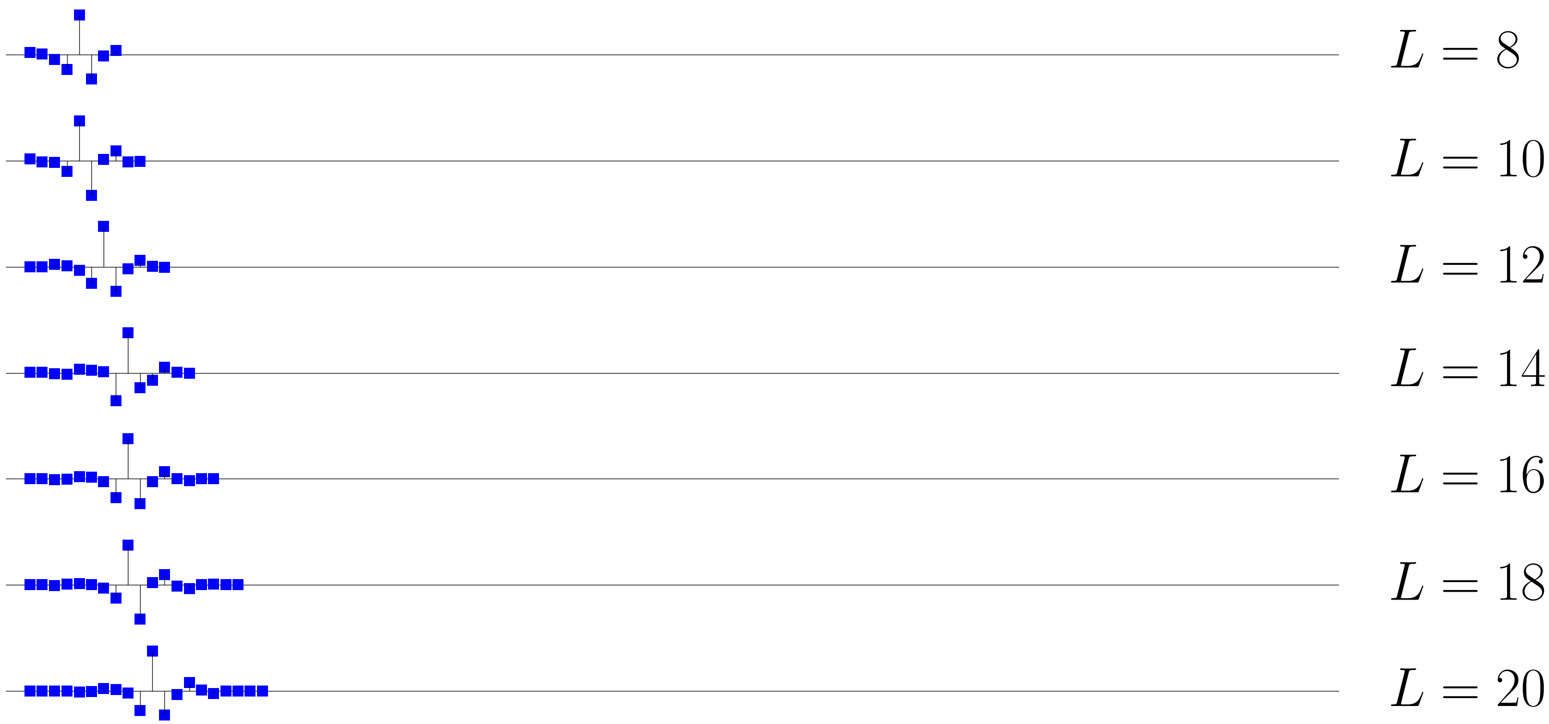


- filters (b) & (c) both qualify as least asymmetric – use (b)

Least Asymmetric Scaling Filters for $L = 8, 10, \ldots, 20$

- in contrast to D(L) scaling filters, $\{g_l^{(la)}\}$'s are not front loaded
- $\{g_l^{(la)}\}$ for $L = 8, 10, \ldots, 20$ are on course Web site

Least Asymmetric Wavelet Filters for $L = 8, 10, \ldots, 20$



- in contrast to D(L) wavelet filters, $\{h_l^{(la)}\}$'s are not back loaded

Phase Functions for LA Wavelet Filters: I

- phase function for $\{g_l^{(la)}\}$ satisfies $\theta^{(G)}(f) \approx 2\pi f \nu$
- Exer. [112]: transfer function for wavelet filter is

$$\begin{aligned} H(f) &= e^{-i2\pi f(L-1)+i\pi} G(\tfrac{1}{2} - f) \\ &= e^{-i2\pi f(L-1)+i\pi} |G(\tfrac{1}{2} - f)| e^{i\theta^{(G)}(\frac{1}{2}-f)} \end{aligned}$$

- hence phase function for wavelet filter is

$$\begin{aligned} \theta^{(H)}(f) &= -2\pi f(L-1) + \pi + \theta^{(G)}(\tfrac{1}{2} - f) \\ &\approx -2\pi f(L-1) + \pi + \pi\nu - 2\pi f\nu \\ &= -2\pi f(L-1+\nu) + \pi(\nu+1) \\ &= -2\pi f(L-1+\nu) \end{aligned}$$

  if ν is odd because $\pi(\nu+1)$ is then a multiple of 2π

- thus ν odd implies that $\{h_l^{(la)}\}$ is approximately linear phase

Phase Functions for LA Wavelet Filters: II

- for tabulated LA coefficients, have
$$\nu = \begin{cases} -\frac{L}{2}+1, & \text{if } L = 8, 12, 16, 20 \text{ (i.e., } \frac{L}{2} \text{ is even);} \\ -\frac{L}{2}, & \text{if } L = 10 \text{ or } 18; \\ -\frac{L}{2}+2, & \text{if } L = 14, \end{cases}$$
so ν is indeed odd for all 7 LA scaling filters
- conclusion: LA wavelet filters also $\approx$ linear phase
- appropriate shift to get zero phase is $-(L-1+\nu)$

Shifts for Higher Level Filters: I

- since

$$\{g_{j,l}\} \longleftrightarrow G_j(f) = \prod_{l=0}^{j-1} G(2^l f)$$

$$\{h_{j,l}\} \longleftrightarrow H_j(f) = H(2^{j-1} f) G_{j-1}(f),$$

phase functions for $\{g_{j,l}\}$ and $\{h_{j,l}\}$ are given by

$$\theta_j^{(G)}(f) = \sum_{l=0}^{j-1} \theta^{(G)}(2^l f) \;\&\; \theta_j^{(H)}(f) = \theta^{(H)}(2^{j-1} f) + \sum_{l=0}^{j-2} \theta^{(G)}(2^l f),$$

so $\{g_{j,l}\}$ & $\{h_{j,l}\}$ are approximately linear phase also

Shifts for Higher Level Filters: II

- Exer. [114]:

$$\theta_j^{(G)}(f) \approx 2\pi f \nu_j^{(G)} \quad \text{with} \quad \nu_j^{(G)} \equiv (2^j - 1)\nu$$
$$\theta_j^{(H)}(f) \approx 2\pi f \nu_j^{(H)} \quad \text{with} \quad \nu_j^{(H)} \equiv -(2^{j-1}[L-1] + \nu)$$

- in terms of widths $L_j = (2^j - 1)(L - 1) + 1$ of $\{g_{j,l}\}$ & $\{h_{j,l}\}$, have $\nu_j^{(G)} \approx \nu_j^{(H)} \approx -\frac{L_j}{2}$ in all cases
- note: $\frac{L}{2}$ odd poorer approximation to linear phase than $\frac{L}{2}$ even (for details, see discussion concerning Fig. 115 in textbook)

Aligning Filter Outputs

- can use $\nu_j^{(H)}$ & $\nu_{J_0}^{(G)}$ to align elements of $\mathbf{W}_j$ & $\mathbf{V}_{J_0}$ with $\mathbf{X}$
- working through some details (see pp. 114–5 of text), find that, if X_t is associated with actual time $t_0 + t\,\Delta t$, LA wavelet coefficient $W_{j,t}$ can be associated with an interval of width $2\tau_j\,\Delta t$ centered at

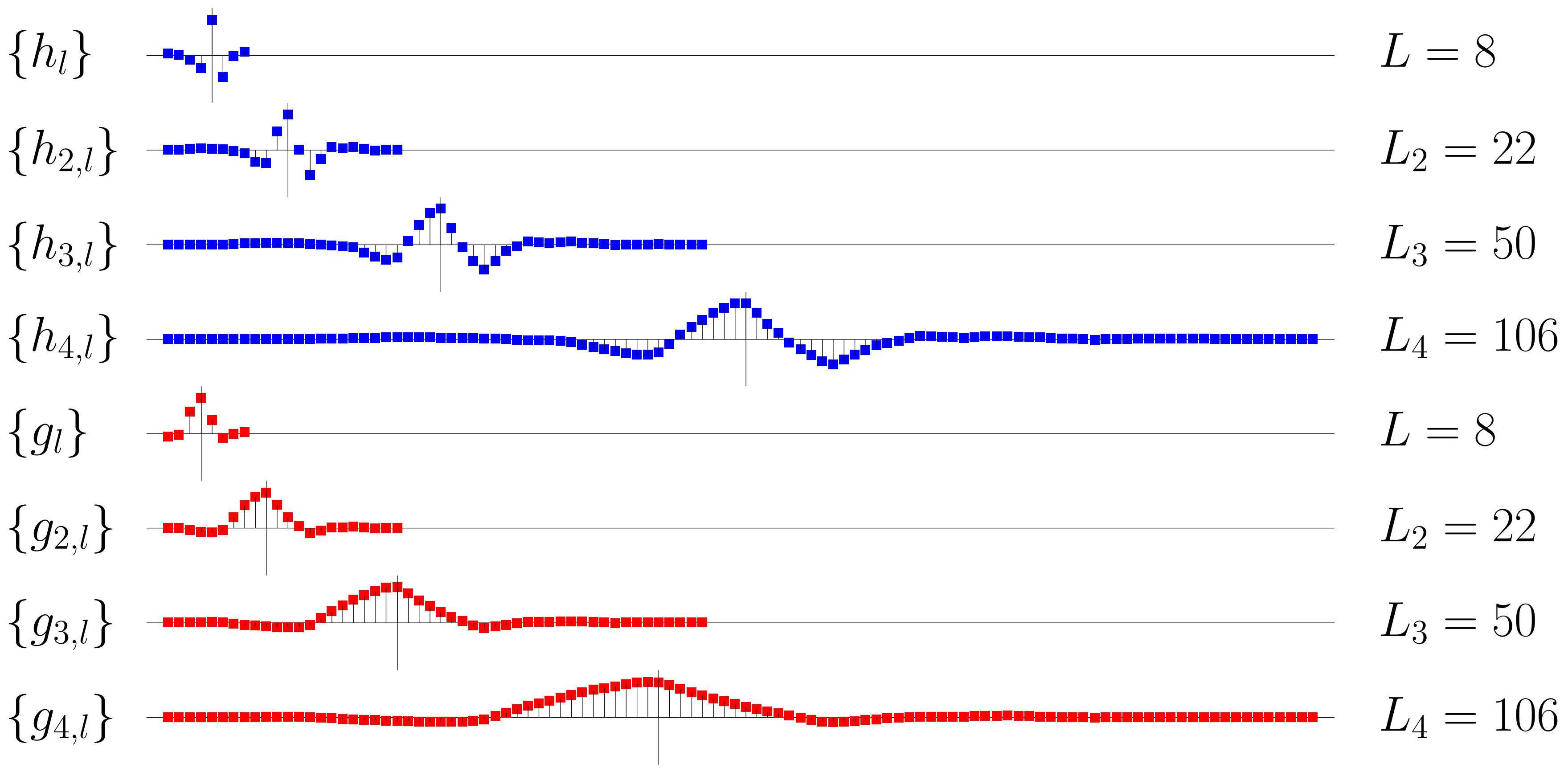
$$t_0 + (2^j(t+1) - 1 - |\nu_j^{(H)}| \bmod N)\,\Delta t,$$

where, e.g., $|\nu_j^{(H)}| = [7(2^j - 1) + 1]/2$ for LA(8) wavelet

- similarly, LA scaling coefficient $V_{J_0,t}$ can be associated with an interval of width $\lambda_{J_0}\,\Delta t$ centered at

$$t_0 + (2^{J_0}(t+1) - 1 - |\nu_{J_0}^{(G)}| \bmod N)\,\Delta t$$

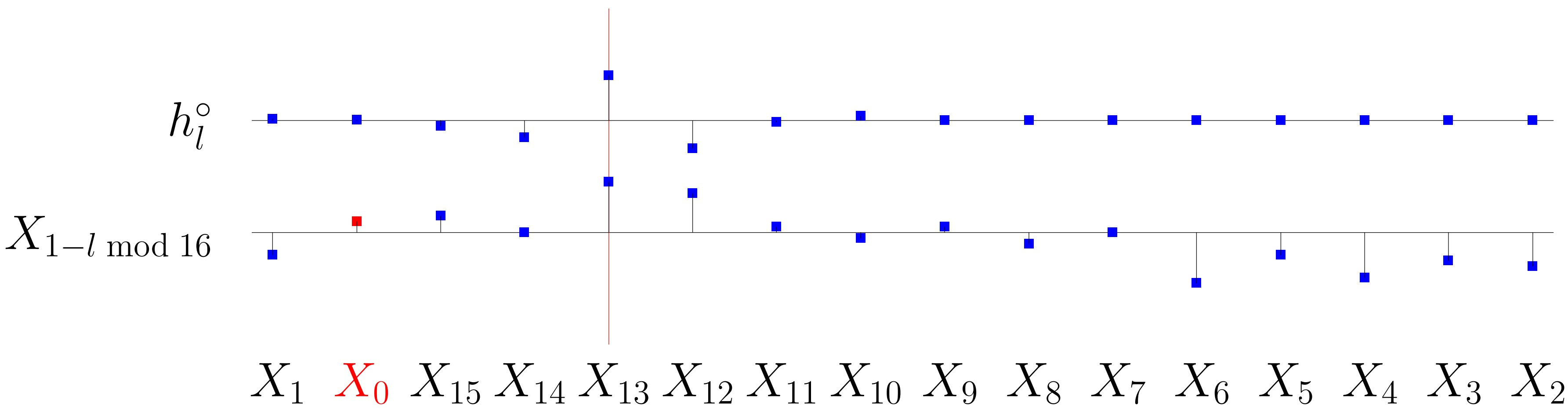
LA(8) Wavelet & Scaling Filters Revisited



- vertical lines indicate point of approximate symmetry

Aligning Wavelet Coefficients with Time Series: I

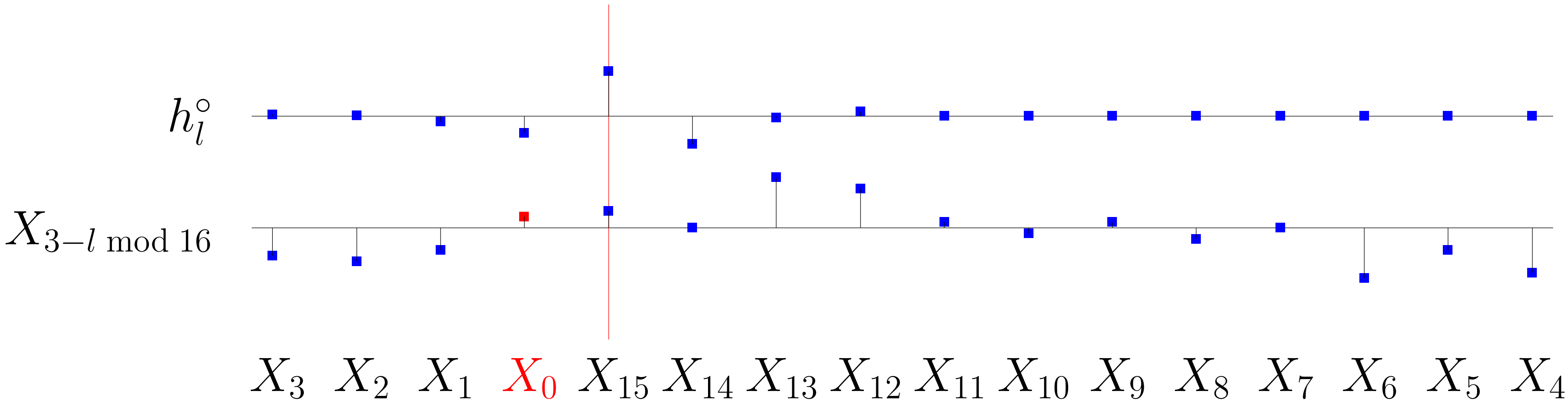
- $W_{1,0} = \sum_{l=0}^{15} h_l^\circ X_{1-l \bmod 16}$, i.e., inner product of vectors:



coefficient	$W_{1,0}$	$W_{1,1}$	$W_{1,2}$	$W_{1,3}$	$W_{1,4}$	$W_{1,5}$	$W_{1,6}$	$W_{1,7}$
associated time	13							

Aligning Wavelet Coefficients with Time Series: I

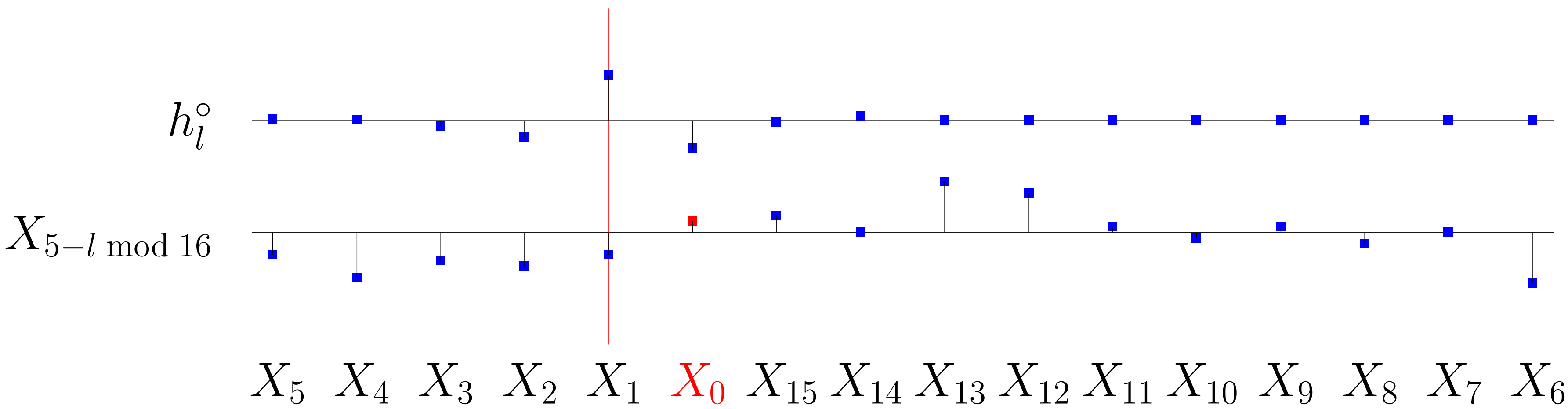
- $W_{1,1} = \sum_{l=0}^{15} h_l^\circ X_{3-l \bmod 16}$, i.e., inner product of vectors:



coefficient	$W_{1,0}$	$W_{1,1}$	$W_{1,2}$	$W_{1,3}$	$W_{1,4}$	$W_{1,5}$	$W_{1,6}$	$W_{1,7}$
associated time	13	15						

Aligning Wavelet Coefficients with Time Series: I

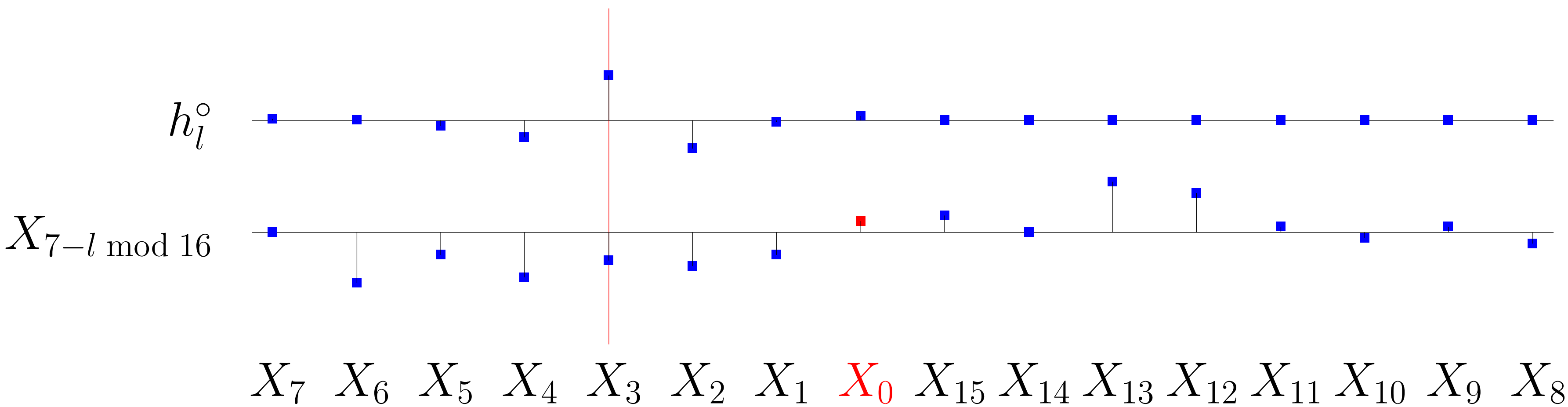
- $W_{1,2} = \sum_{l=0}^{15} h_l^\circ X_{5-l \bmod 16}$, i.e., inner product of vectors:



coefficient	$W_{1,0}$	$W_{1,1}$	$W_{1,2}$	$W_{1,3}$	$W_{1,4}$	$W_{1,5}$	$W_{1,6}$	$W_{1,7}$
associated time	13	15	1					

Aligning Wavelet Coefficients with Time Series: I

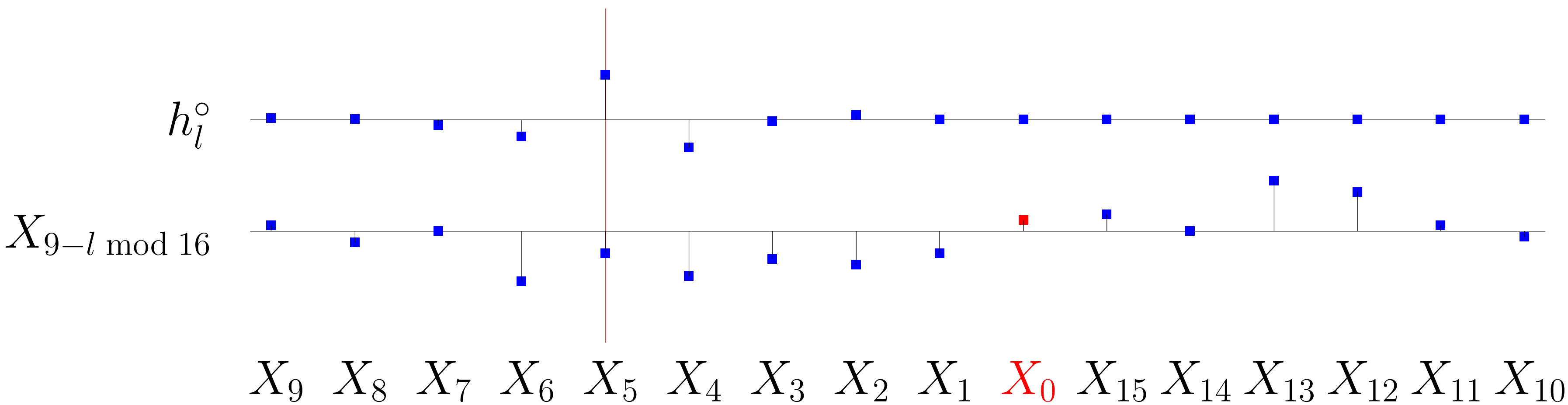
- $W_{1,3} = \sum_{l=0}^{15} h_l^\circ X_{7-l \bmod 16}$, i.e., inner product of vectors:



coefficient	$W_{1,0}$	$W_{1,1}$	$W_{1,2}$	$W_{1,3}$	$W_{1,4}$	$W_{1,5}$	$W_{1,6}$	$W_{1,7}$
associated time	13	15	1	3				

Aligning Wavelet Coefficients with Time Series: I

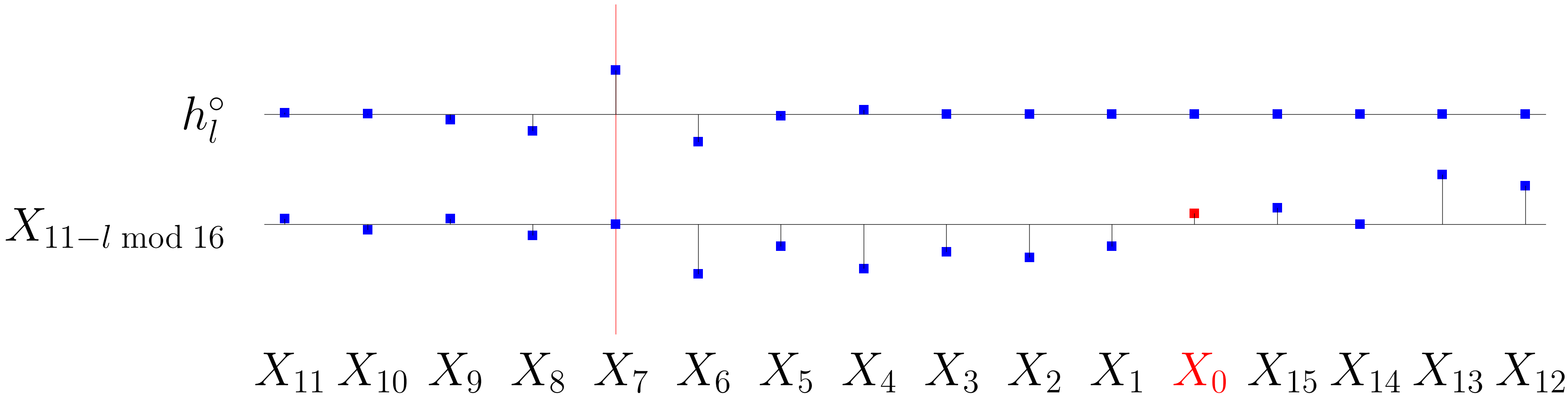
- $W_{1,4} = \sum_{l=0}^{15} h_l^\circ X_{9-l \bmod 16}$, i.e., inner product of vectors:



coefficient	$W_{1,0}$	$W_{1,1}$	$W_{1,2}$	$W_{1,3}$	$W_{1,4}$	$W_{1,5}$	$W_{1,6}$	$W_{1,7}$
associated time	13	15	1	3	5			

Aligning Wavelet Coefficients with Time Series: I

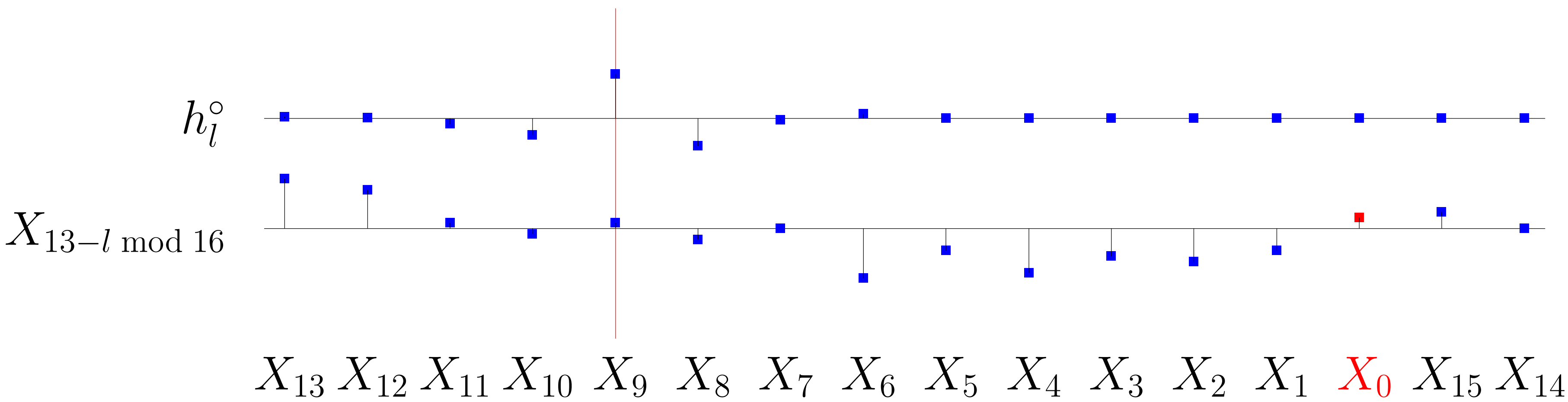
- $W_{1,5} = \sum_{l=0}^{15} h_l^\circ X_{11-l \bmod 16}$, i.e., inner product of vectors:



coefficient	$W_{1,0}$	$W_{1,1}$	$W_{1,2}$	$W_{1,3}$	$W_{1,4}$	$W_{1,5}$	$W_{1,6}$	$W_{1,7}$
associated time	13	15	1	3	5	7		

Aligning Wavelet Coefficients with Time Series: I

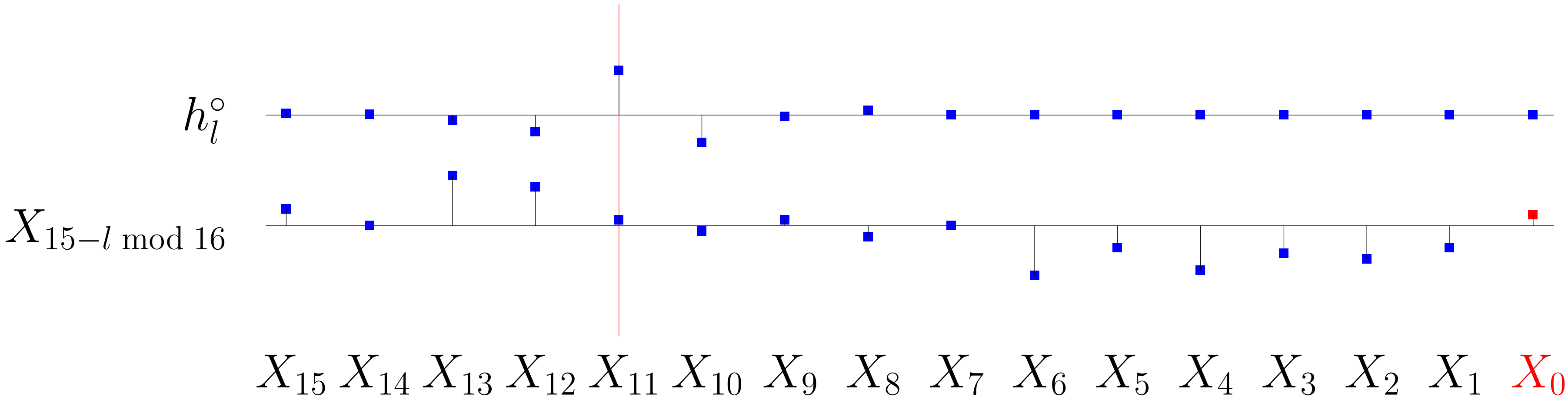
- $W_{1,6} = \sum_{l=0}^{15} h_l^\circ X_{13-l \bmod 16}$, i.e., inner product of vectors:



coefficient	$W_{1,0}$	$W_{1,1}$	$W_{1,2}$	$W_{1,3}$	$W_{1,4}$	$W_{1,5}$	$W_{1,6}$	$W_{1,7}$
associated time	13	15	1	3	5	7	9	

Aligning Wavelet Coefficients with Time Series: I

- $W_{1,7} = \sum_{l=0}^{15} h_l^\circ X_{15-l \bmod 16}$, i.e., inner product of vectors:



coefficient	$W_{1,0}$	$W_{1,1}$	$W_{1,2}$	$W_{1,3}$	$W_{1,4}$	$W_{1,5}$	$W_{1,6}$	$W_{1,7}$
associated time	13	15	1	3	5	7	9	11

- order in which elements of $\mathbf{W}_1$ should be displayed is thus

$$W_{1,2}, W_{1,3}, W_{1,4}, W_{1,5}, W_{1,6}, W_{1,7}, W_{1,0}, W_{1,1}$$

Aligning Wavelet Coefficients with Time Series: II

- recall that we can use $N \times N$ matrix $\mathcal{T}^k$ to circularly shift $\mathbf{W}_1$ by k units
  - shift is to the right if k is positive
  - shift is to the left if k is negative
- can express reordering elements of

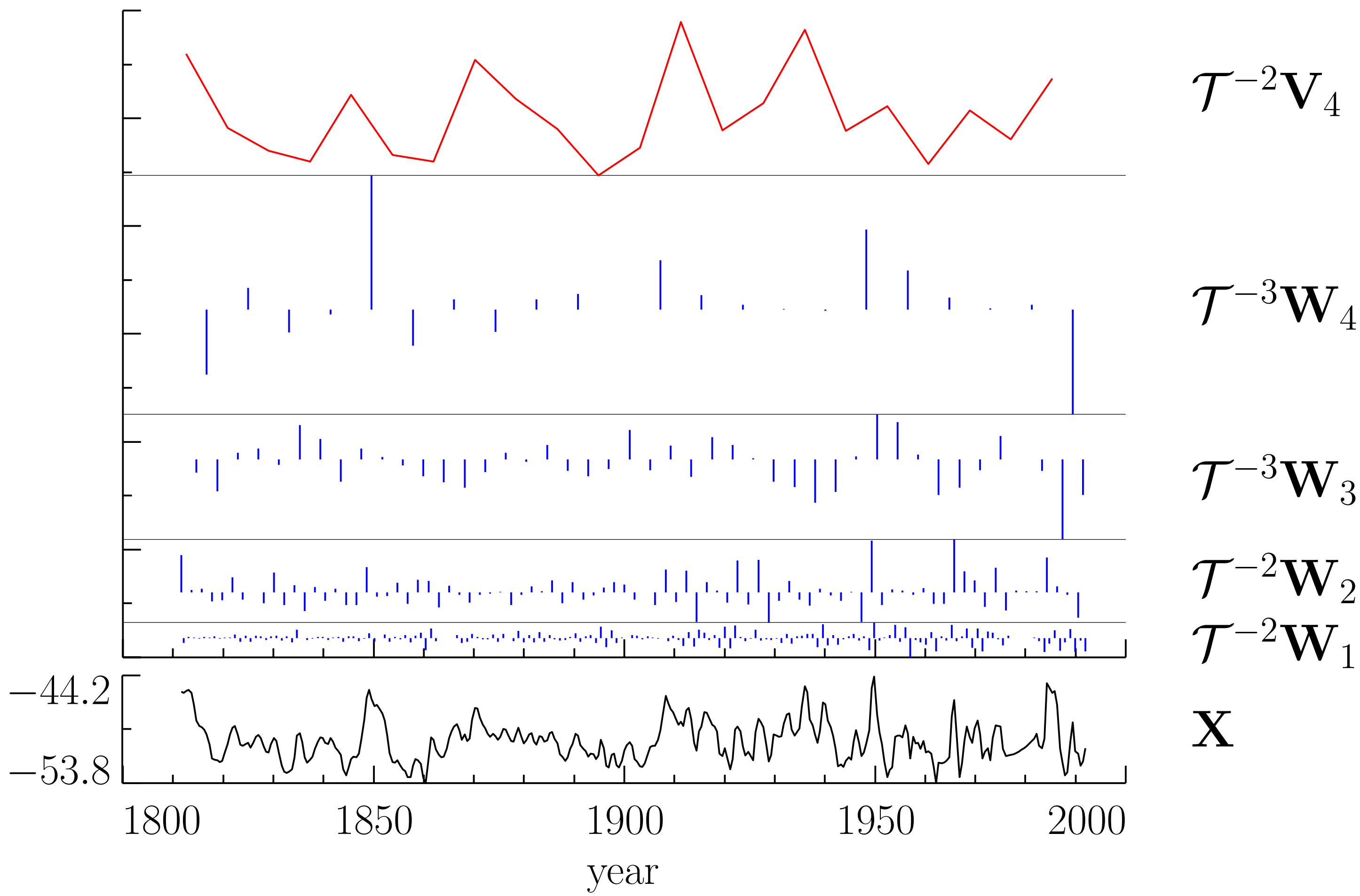
$$\mathbf{W}_1 = [W_{1,0}, W_{1,1}, W_{1,2}, W_{1,3}, W_{1,4}, W_{1,5}, W_{1,6}, W_{1,7}]^T$$

as they occur in time using

$$\mathcal{T}^{-2}\mathbf{W}_1 = [W_{1,2}, W_{1,3}, W_{1,4}, W_{1,5}, W_{1,6}, W_{1,7}, W_{1,0}, W_{1,1}]^T$$

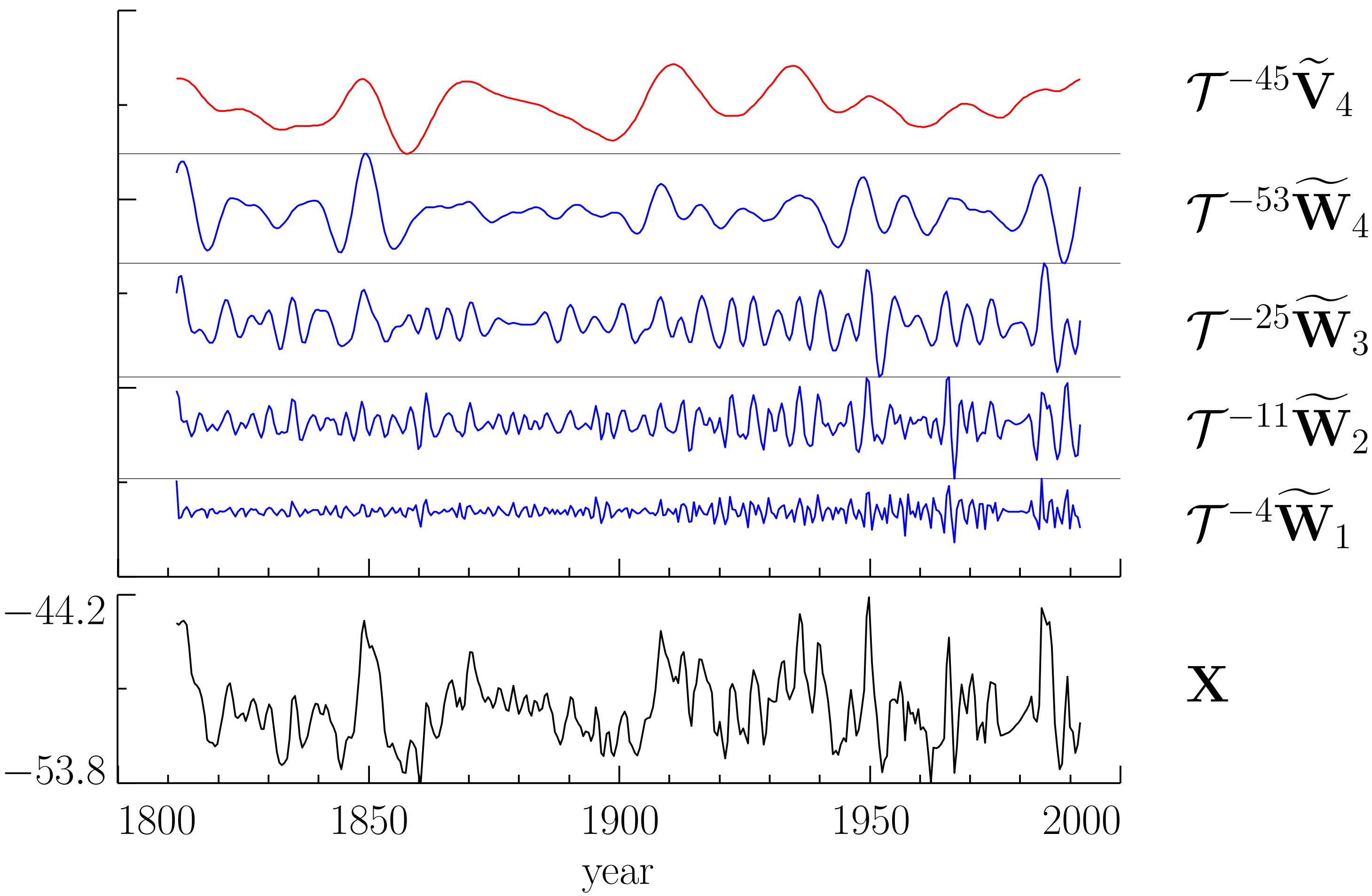
Example of $J_0 = 4$ LA(8) Partial DWT

- oxygen isotope records $\mathbf{X}$ from Antarctic ice core



Example of $J_0 = 4$ LA(8) MODWT

- oxygen isotope records $\mathbf{X}$ from Antarctic ice core



Summary of Daubechies Filters: I

- by definition, scaling filters $\{g_l\}$ of the Daubechies class have a squared gain function given by

$$\mathcal{G}^{(D)}(f) = 2\cos^L(\pi f) \sum_{l=0}^{\frac{L}{2}-1} \binom{\frac{L}{2}-1+l}{l} \sin^{2l}(\pi f)$$

- for given width L, there are several filters with the same $\mathcal{G}^{(D)}(\cdot)$ (these differ only in their phase functions)
- need to impose additional constraints to pick unique filter

Summary of Daubechies Filters: II

- extremal (or minimum) phase constraint leads to the D(L) scaling filters, denoted as $\{g_l^{(ep)}\}$ (these maximize the increase in the partial energy sequence)
- least asymmetric constraint leads to the LA(L) scaling filters, denoted as $\{g_l^{(la)}\}$
  - approximately zero phase after shifting by ν
  - zero phase helps align filter output with input
  - shift ν depends on L in a simple manner
  - corresponding wavelet filters $\{h_l^{(la)}\}$ are also approximately zero phase after shifting by $\nu_1^{(H)} \equiv -(L-1-\nu)$

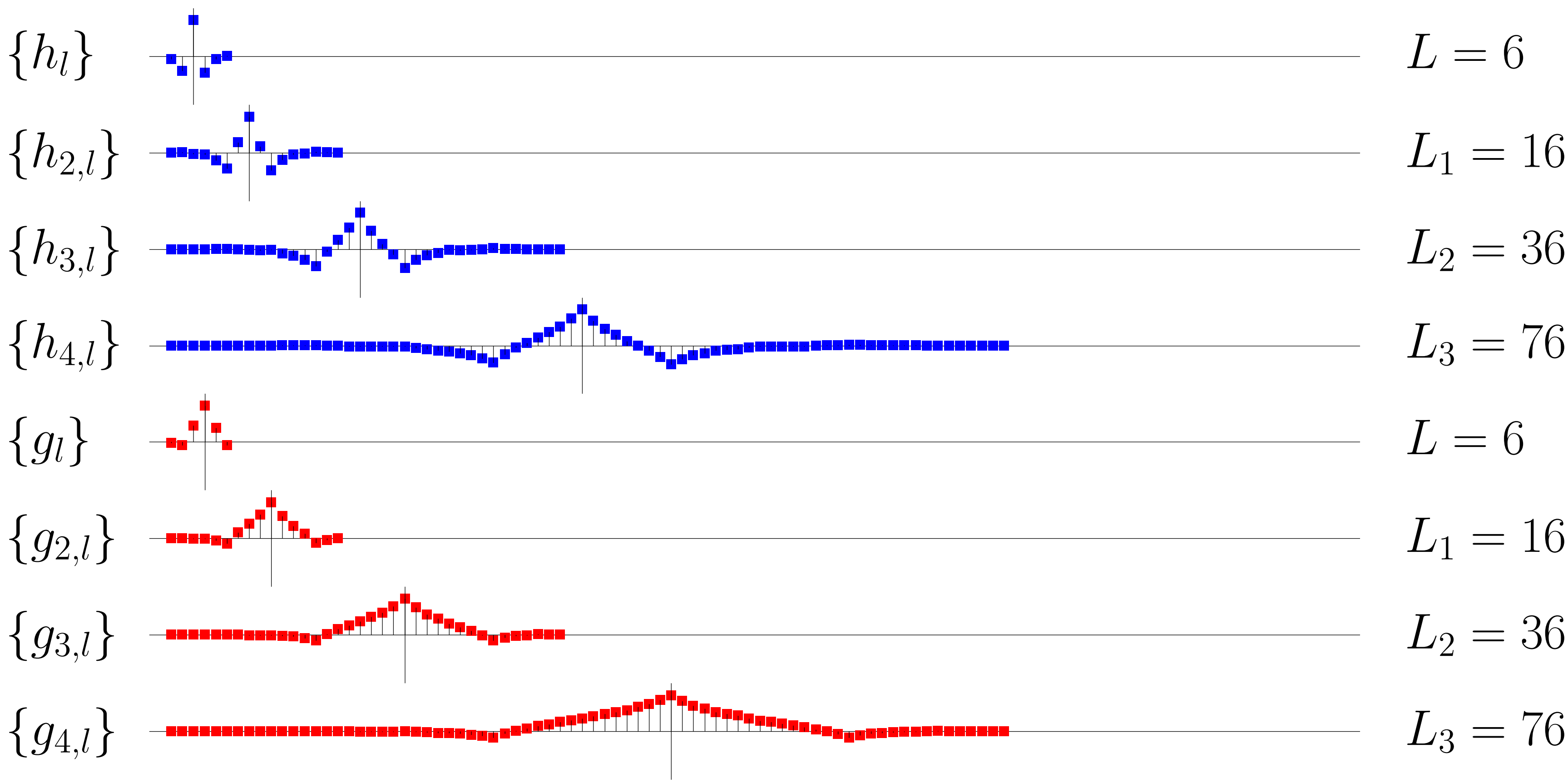
Coiflets

- another class of filters yielding differences of weighted averages (due to Daubechies, but suggested by R. Coifman)
- C(L) filters defined for widths $L = 6, 12, 18, 24$ and 30
- has $L/3$ embedded differencing operations rather than $L/2$
- can express squared gain function $\mathcal{H}^{(c)}(f)$ as

$$(2\sin(\pi f))^{\frac{2L}{3}} \left(\sum_{l=0}^{\frac{L}{6}-1} \binom{\frac{L}{6}-1+l}{l} \cos^{2l}(\pi f) + \cos^{\frac{L}{3}}(\pi f) F(f) \right)^2,$$

  where $F(\cdot)$ is chosen so that $\mathcal{H}^{(c)}(f) + \mathcal{H}^{(c)}(f + \frac{1}{2}) = 2$ (however, $F(\cdot)$ cannot be expressed in closed form)
- by some measures, coiflets are more symmetric than LA filters, but their triangular shapes can be problematic

C(6) Wavelet & Scaling Filters Revisited



- vertical lines indicate point of approximate symmetry

Zero-Phase Wavelet (Zephlet) Transform: I

- possible to construct orthonormal DWT based on filters whose squared gain functions are consistent with those of Daubechies, but with *exact* zero phase, as follows
- with N being a positive even integer, let $\mathcal{H}(\cdot)$ be a squared gain function satisfying
$$\mathcal{H}(\tfrac{k}{N}) + \mathcal{H}(\tfrac{k}{N} + \tfrac{1}{2}) = 2 \text{ for all } \tfrac{k}{N}$$
- let $\{\bar{h}_l\}$ be the inverse DFT of the sequence $\{\mathcal{H}^{1/2}(\tfrac{k}{N})\}$:
$$\bar{h}_l \equiv \frac{1}{N} \sum_{k=0}^{N-1} \mathcal{H}^{1/2}(\tfrac{k}{N}) e^{i2\pi kl/N}, \quad l = 0, 1, \ldots, N-1$$
- define $\bar{g}_l = (-1)^l \bar{h}_l$, and let $\{\overline{G}(\tfrac{k}{N})\}$ denote its DFT
- with $\mathcal{G}(\tfrac{k}{N}) \equiv |\overline{G}(\tfrac{k}{N})|^2$, can argue that $\mathcal{H}(\tfrac{k}{N}) + \mathcal{G}(\tfrac{k}{N}) = 2$

Zero-Phase Wavelet (Zephlet) Transform: II

- define the $\frac{N}{2} \times N$ matrices

$$\mathcal{D}_1 = \begin{bmatrix} \bar{h}_1 & \bar{h}_0 & \bar{h}_{N-1} & \bar{h}_{N-2} & \bar{h}_{N-3} & \cdots & \bar{h}_5 & \bar{h}_4 & \bar{h}_3 & \bar{h}_2 \\ \bar{h}_3 & \bar{h}_2 & \bar{h}_1 & \bar{h}_0 & \bar{h}_{N-1} & \cdots & \bar{h}_7 & \bar{h}_6 & \bar{h}_5 & \bar{h}_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \bar{h}_{N-1} & \bar{h}_{N-2} & \bar{h}_{N-3} & \bar{h}_{N-4} & \bar{h}_{N-5} & \cdots & \bar{h}_3 & \bar{h}_2 & \bar{h}_1 & \bar{h}_0 \end{bmatrix}$$

and

$$\mathcal{C}_1 = \begin{bmatrix} \bar{g}_0 & \bar{g}_{N-1} & \bar{g}_{N-2} & \bar{g}_{N-3} & \bar{g}_{N-4} & \cdots & \bar{g}_4 & \bar{g}_3 & \bar{g}_2 & \bar{g}_1 \\ \bar{g}_2 & \bar{g}_1 & \bar{g}_0 & \bar{g}_{N-1} & \bar{g}_{N-2} & \cdots & \bar{g}_6 & \bar{g}_5 & \bar{g}_4 & \bar{g}_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \bar{g}_{N-2} & \bar{g}_{N-3} & \bar{g}_{N-4} & \bar{g}_{N-5} & \bar{g}_{N-6} & \cdots & \bar{g}_2 & \bar{g}_1 & \bar{g}_0 & \bar{g}_{N-1} \end{bmatrix}$$

(note that, while $\mathcal{D}_1$ has a form analogous to $\mathcal{W}_1$ & $\mathcal{V}_1$, corresponding rows in $\mathcal{C}_1$ and $\mathcal{D}_1$ differ by a circular shift of one)

Zero-Phase Wavelet (Zephlet) Transform: III

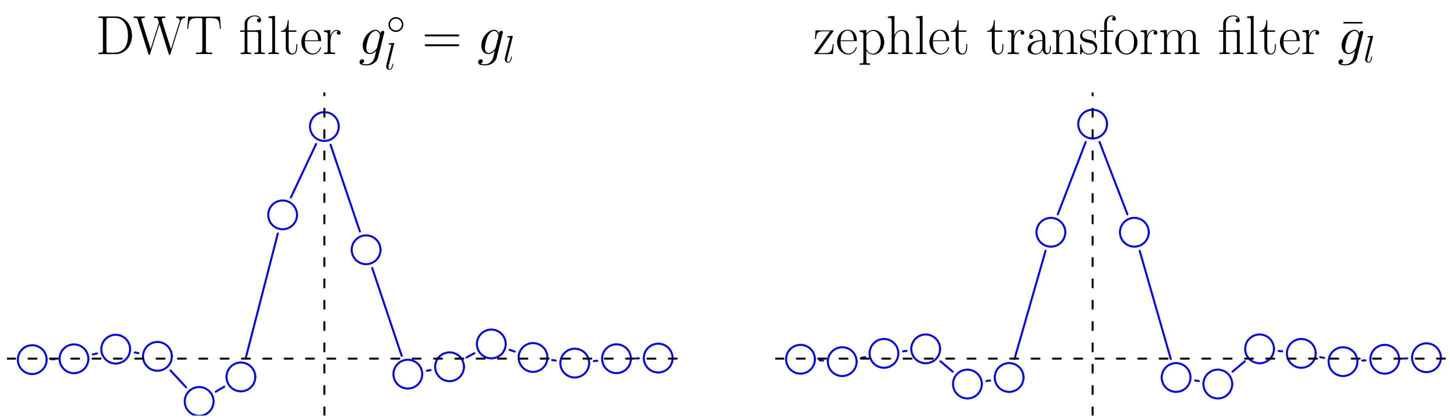
- can show that the $N \times N$ matrix formed by stacking $\mathcal{D}_1$ on top of $\mathcal{C}_1$ is a real-valued orthonormal matrix; i.e,

$$\mathcal{D} \equiv \begin{bmatrix} \mathcal{D}_1 \\ \mathcal{C}_1 \end{bmatrix} \text{ is such that } \mathcal{D}^T \mathcal{D} = I_N$$

- proof of above result (subject of forthcoming exercise!) is similar in spirit to proof that $\mathcal{W}$ is orthonormal, but details differ
- algorithms for computing DWT and zephlet transform are, respectively, $\mathcal{O}(N)$ and $\mathcal{O}(N \cdot \log_2(N))$

Zero-Phase Wavelet (Zephlet) Transform: IV

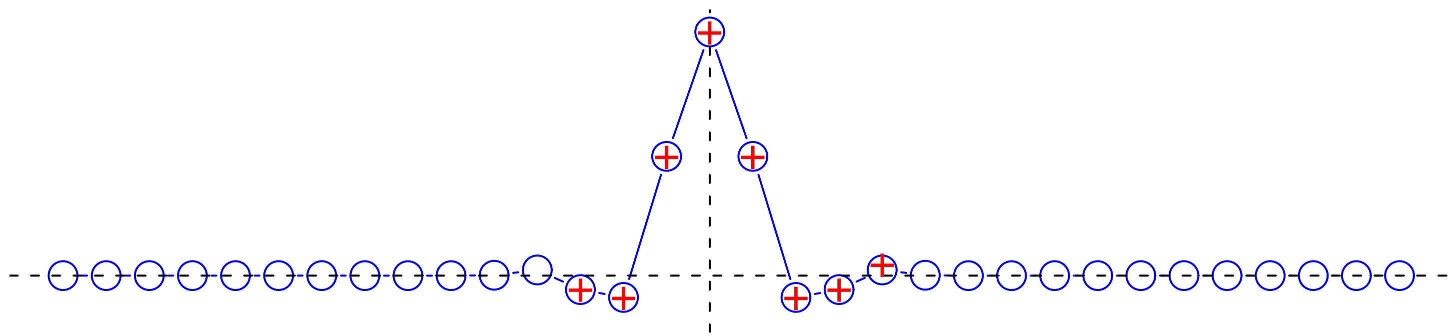
- for case $N = L = 16$, let's compare values in rows of $\mathcal{V}_1$ based on Daubechies' least asymmetric filter and corresponding $\mathcal{C}_1$ (after alignments for easier comparison)



- for given N & L, squared magnitudes of DFTs of $\{g_l^{\circ}\}$ & $\{\bar{g}_l\}$ at $f_k = k/N$ are *exactly* the same, but phase functions differ, with that for $\{\bar{g}_l\}$ given by $\theta(f_k) = 0$

Zero-Phase Wavelet (Zephlet) Transform: V

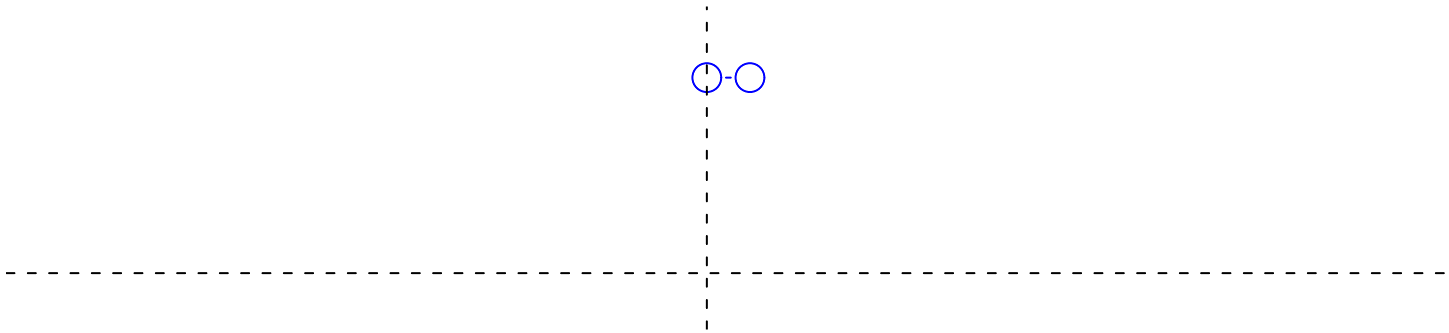
- for fixed $L \geq 8$, values in rows of zephlet transform change as N increases (DWT rows just add more 0's for all $N \geq L$)
- consider zephlet transform based on least asymmetric filter for $L = 8$ and cases $N = 8$ (pluses) and $N = 32$ (circles)



Zero-Phase Wavelet (Zephlet) Transform: VI

- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):
$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$
for large $N = 2M$, where
$$S_{l,\pm} \equiv \sin([2l \pm 1]\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm 1}{4})}{\sin(\pi\frac{2l\pm 1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 2$:



Zero-Phase Wavelet (Zephlet) Transform: VI

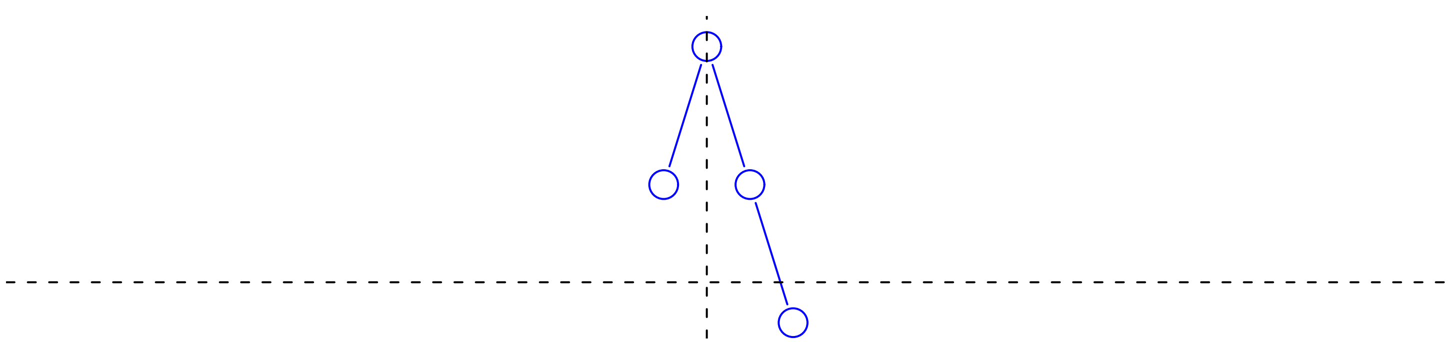
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large $N = 2M$, where

$$S_{l,\pm} \equiv \sin([2l \pm 1]\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm1}{4})}{\sin(\pi\frac{2l\pm1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 4$:



Zero-Phase Wavelet (Zephlet) Transform: VI

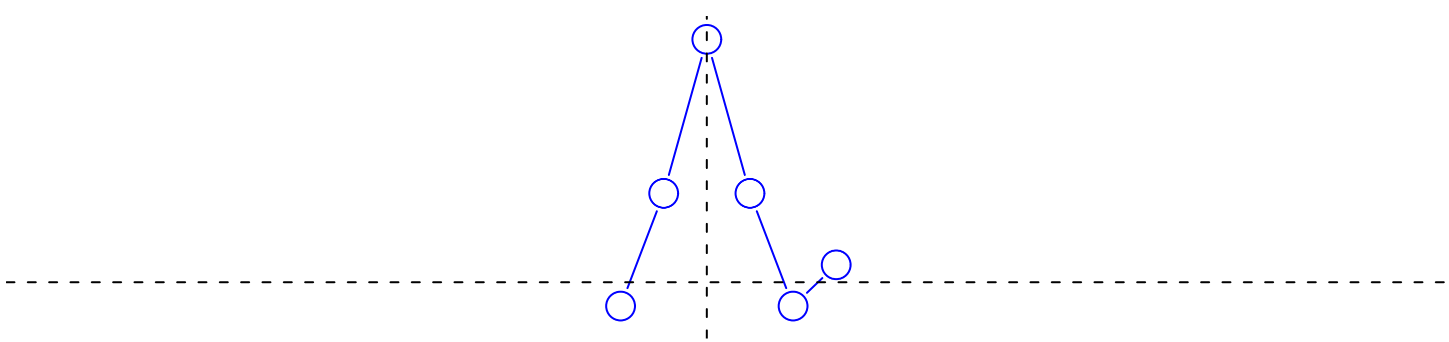
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large $N = 2M$, where

$$S_{l,\pm} \equiv \sin([2l \pm 1]\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm1}{4})}{\sin(\pi\frac{2l\pm1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 6$:



Zero-Phase Wavelet (Zephlet) Transform: VI

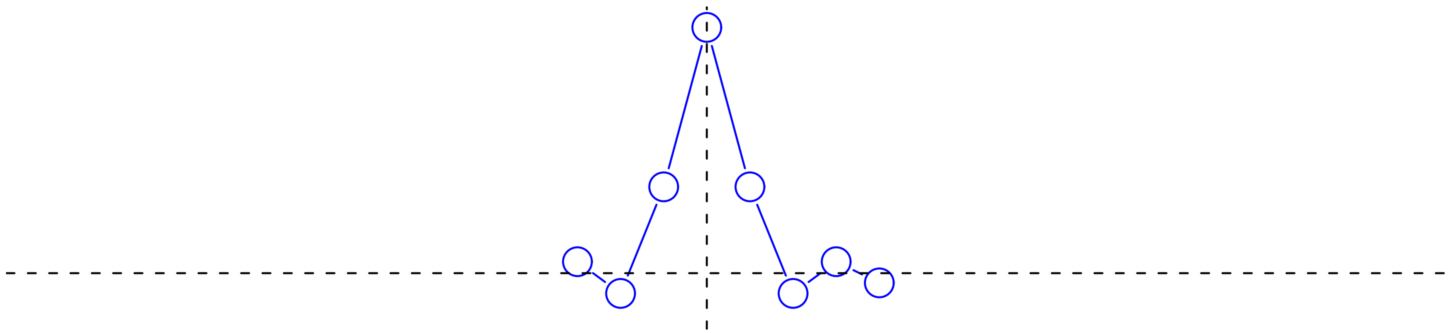
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large $N = 2M$, where

$$S_{l,\pm} \equiv \sin([2l \pm 1]\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm1}{4})}{\sin(\pi\frac{2l\pm1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 8$:



Zero-Phase Wavelet (Zephlet) Transform: VI

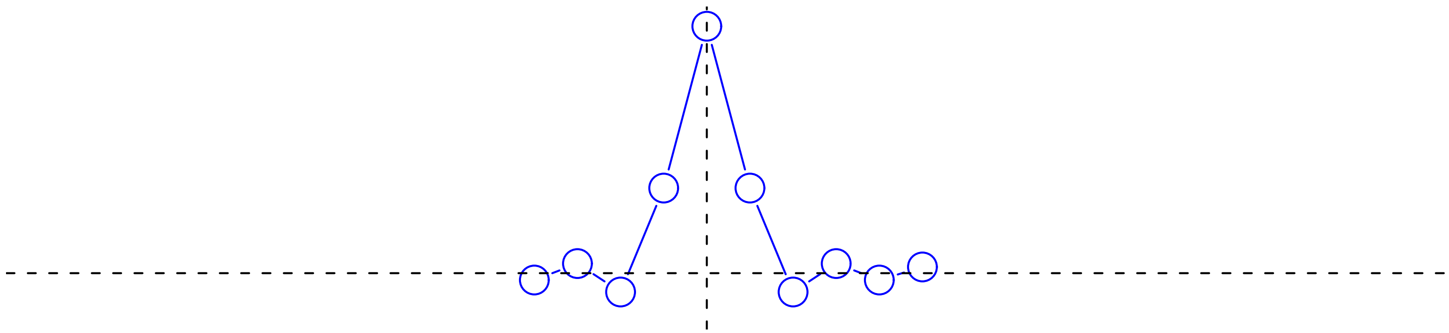
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large $N = 2M$, where

$$S_{l,\pm} \equiv \sin([2l \pm 1]\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm 1}{4})}{\sin(\pi\frac{2l\pm 1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 10$:



Zero-Phase Wavelet (Zephlet) Transform: VI

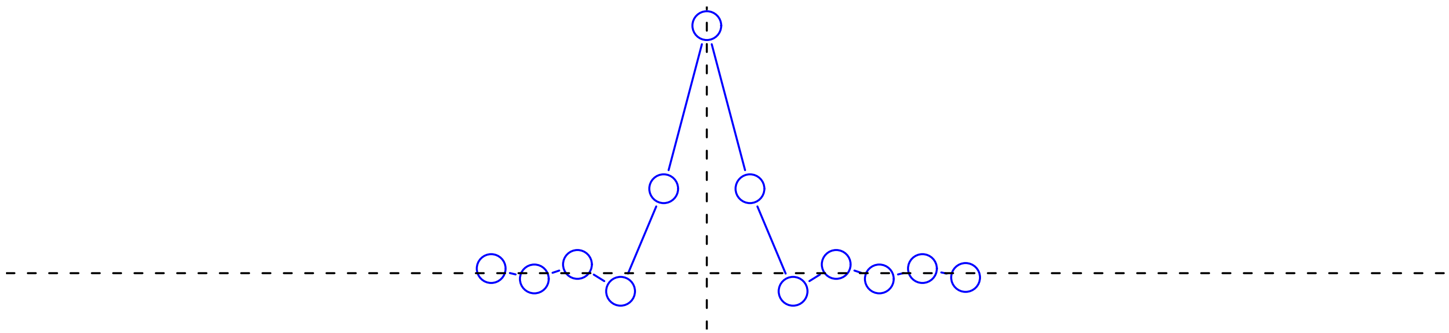
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large $N = 2M$, where

$$S_{l,\pm} \equiv \sin([2l \pm 1]\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm1}{4})}{\sin(\pi\frac{2l\pm1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 12$:



Zero-Phase Wavelet (Zephlet) Transform: VI

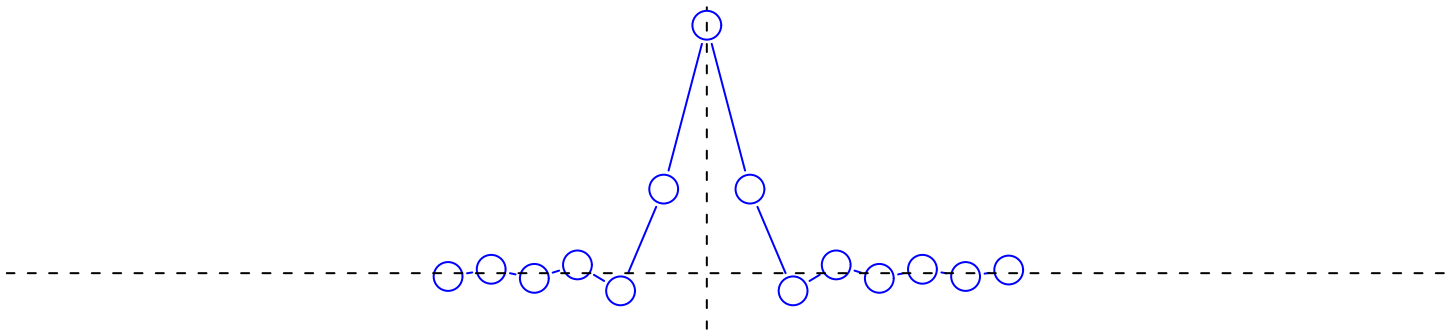
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large $N = 2M$, where

$$S_{l,\pm} \equiv \sin([2l \pm 1]\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm1}{4})}{\sin(\pi\frac{2l\pm1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 14$:



Zero-Phase Wavelet (Zephlet) Transform: VI

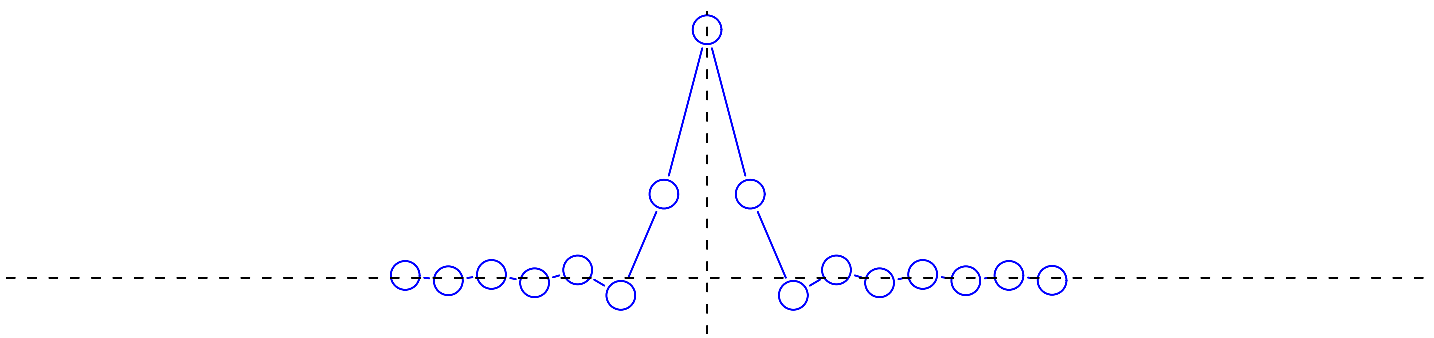
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large $N = 2M$, where

$$S_{l,\pm} \equiv \sin([2l \pm 1]\pi \tfrac{M-1}{4M}) \frac{\sin(\pi \frac{2l\pm 1}{4})}{\sin(\pi \frac{2l\pm 1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 16$:



Zero-Phase Wavelet (Zephlet) Transform: VI

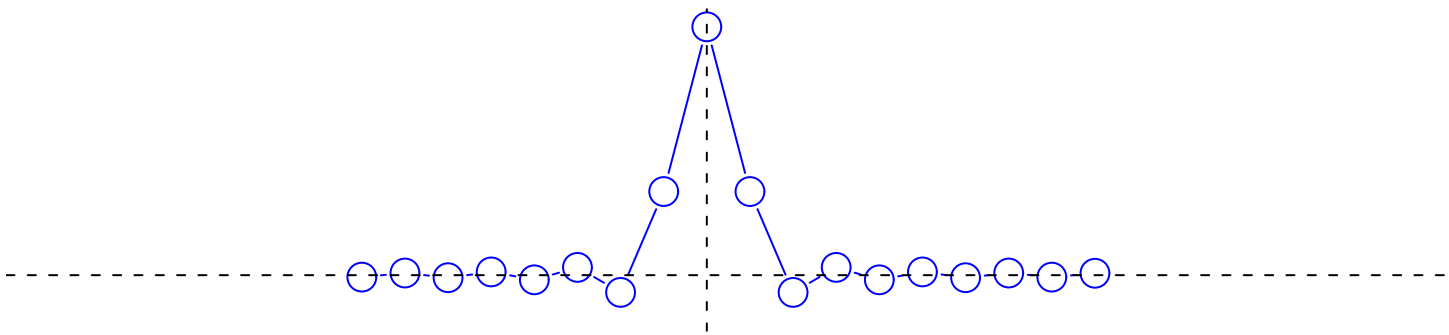
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large $N = 2M$, where

$$S_{l,\pm} \equiv \sin([2l \pm 1]\pi \tfrac{M-1}{4M}) \frac{\sin(\pi \frac{2l\pm 1}{4})}{\sin(\pi \frac{2l\pm 1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 18$:



Zero-Phase Wavelet (Zephlet) Transform: VI

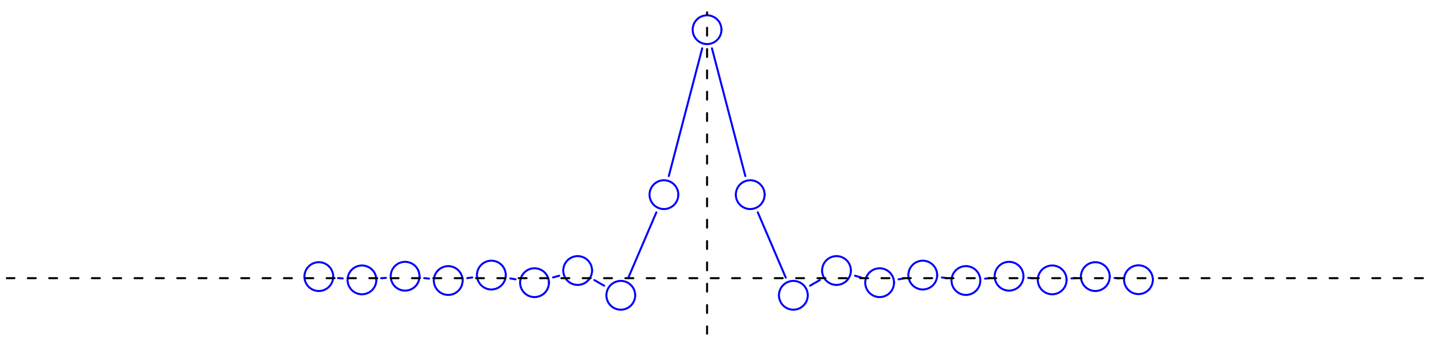
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large $N = 2M$, where

$$S_{l,\pm} \equiv \sin([2l \pm 1]\pi \tfrac{M-1}{4M}) \frac{\sin(\pi \frac{2l\pm 1}{4})}{\sin(\pi \frac{2l\pm 1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 20$:



Zero-Phase Wavelet (Zephlet) Transform: VI

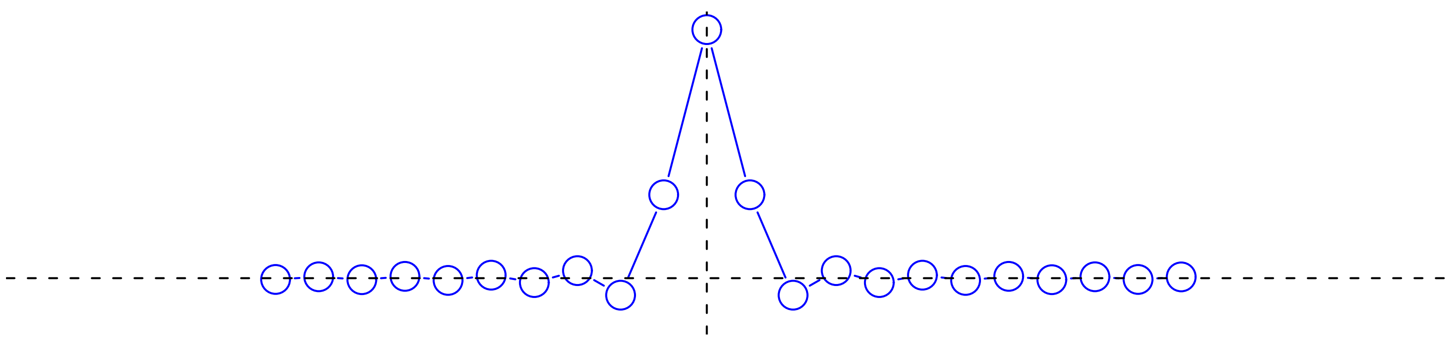
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l \sqrt{2}}{\pi(1 - 4l^2)}$$

for large $N = 2M$, where

$$S_{l,\pm} \equiv \sin([2l \pm 1]\pi \tfrac{M-1}{4M}) \frac{\sin(\pi \frac{2l\pm 1}{4})}{\sin(\pi \frac{2l\pm 1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 22$:



Zero-Phase Wavelet (Zephlet) Transform: VI

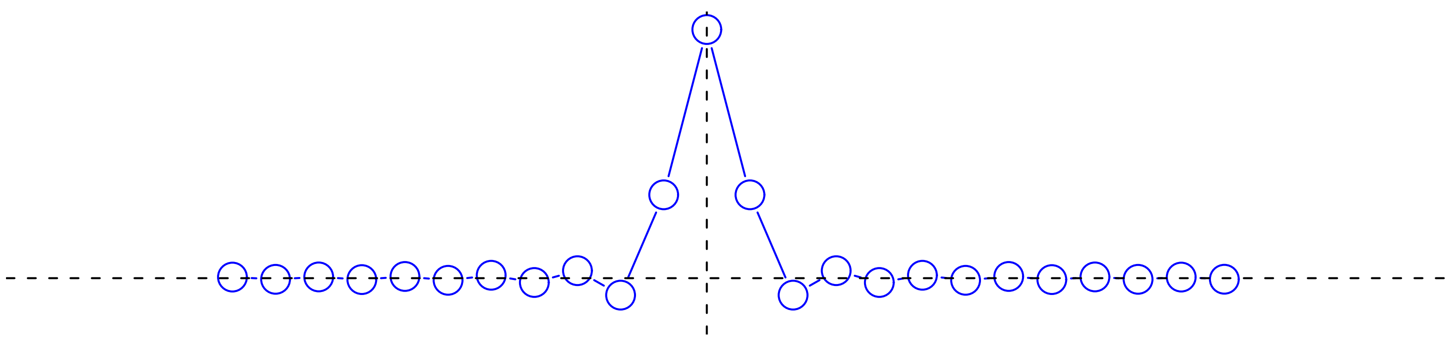
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large $N = 2M$, where

$$S_{l,\pm} \equiv \sin([2l \pm 1]\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm 1}{4})}{\sin(\pi\frac{2l\pm 1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 24$:



Zero-Phase Wavelet (Zephlet) Transform: VI

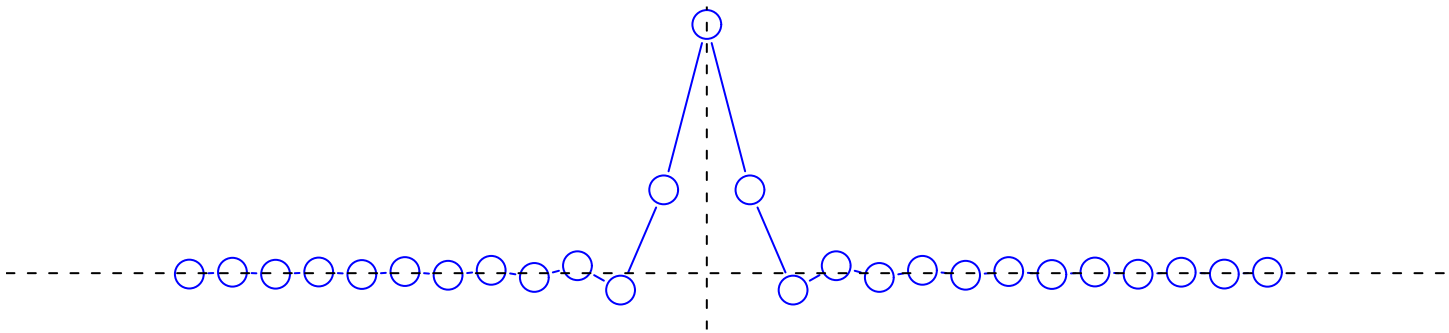
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large $N = 2M$, where

$$S_{l,\pm} \equiv \sin([2l \pm 1]\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm 1}{4})}{\sin(\pi\frac{2l\pm 1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 26$:



Zero-Phase Wavelet (Zephlet) Transform: VI

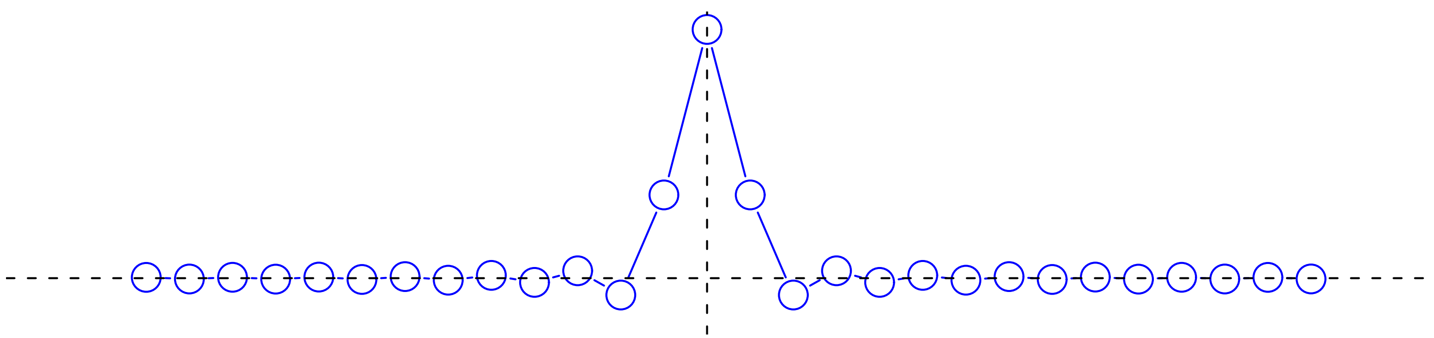
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large $N = 2M$, where

$$S_{l,\pm} \equiv \sin([2l \pm 1]\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm1}{4})}{\sin(\pi\frac{2l\pm1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 28$:



Zero-Phase Wavelet (Zephlet) Transform: VI

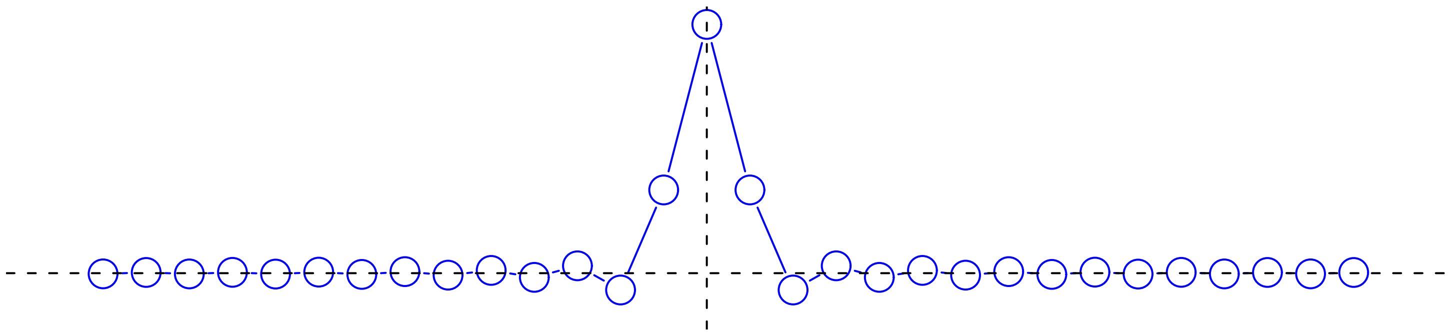
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large $N = 2M$, where

$$S_{l,\pm} \equiv \sin([2l \pm 1]\pi \tfrac{M-1}{4M}) \frac{\sin(\pi \frac{2l\pm 1}{4})}{\sin(\pi \frac{2l\pm 1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 30$:



Zero-Phase Wavelet (Zephlet) Transform: VI

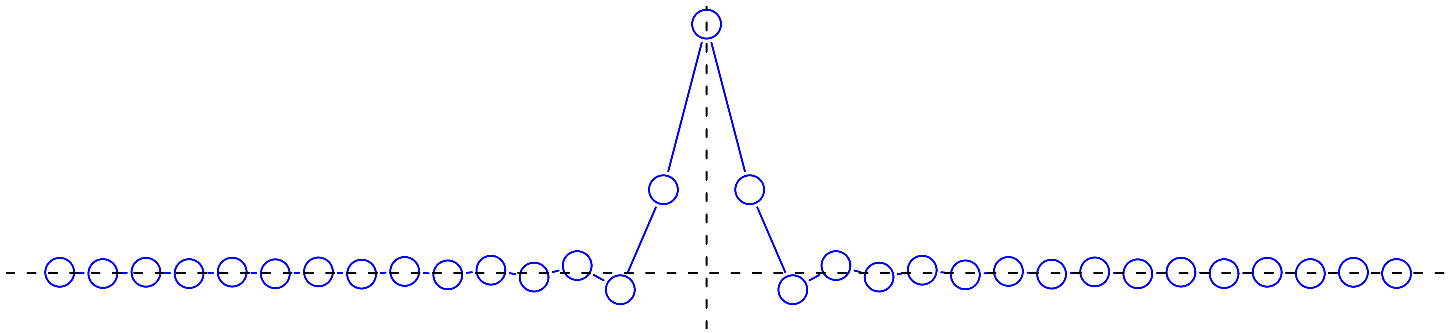
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large $N = 2M$, where

$$S_{l,\pm} \equiv \sin([2l \pm 1]\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm1}{4})}{\sin(\pi\frac{2l\pm1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 32$:



Comparison of Outputs from LA(8) & Zephlet Scaling Filters (Input is Doppler Signal)