Maximal Overlap Discrete Wavelet Transform

• abbreviation is MODWT (pronounced ‘mod WT’)
• transforms very similar to the MODWT have been studied in the literature under the following names:
  – undecimated DWT (or nondecimated DWT)
  – stationary DWT
  – translation invariant DWT
  – time invariant DWT
  – redundant DWT
• also related to notions of ‘wavelet frames’ and ‘cycle spinning’
• basic idea: use values removed from DWT by downsampling
Quick Comparison of the MODWT to the DWT

• unlike the DWT, MODWT is not orthonormal (in fact MODWT is highly redundant)

• unlike the DWT, MODWT is defined naturally for all samples sizes (i.e., $N$ need not be a multiple of a power of two)

• similar to the DWT, can form multiresolution analyses (MRAs) using MODWT, but with certain additional desirable features; e.g., unlike the DWT, MODWT-based MRA has details and smooths that shift along with $\mathbf{X}$ (if $\mathbf{X}$ has detail $\tilde{\mathcal{D}}_j$, then $\mathcal{T}^m\mathbf{X}$ has detail $\mathcal{T}^m\tilde{\mathcal{D}}_j$)

• similar to the DWT, an analysis of variance (ANOVA) can be based on MODWT wavelet coefficients

• unlike the DWT, MODWT discrete wavelet power spectrum same for $\mathbf{X}$ and its circular shifts $\mathcal{T}^m\mathbf{X}$
Definition of MODWT Wavelet & Scaling Filters: I

- recall that we can obtain DWT wavelet and scaling coefficients directly from $X$ by filtering and downsampling:

$$X \rightarrow \boxed{H_j \left( \frac{k}{N} \right)} \downarrow 2^j \rightarrow W_j \text{ and } X \rightarrow \boxed{G_j \left( \frac{k}{N} \right)} \downarrow 2^j \rightarrow V_j$$

- transfer functions $H_j(\cdot)$ and $G_j(\cdot)$ are associated with impulse response sequences $\{h_{j,l}\}$ and $\{g_{j,l}\}$ via the usual relationships

$$\{h_{j,l}\} \leftrightarrow H_j(\cdot) \text{ and } \{g_{j,l}\} \leftrightarrow G_j(\cdot),$$

and both filters have width $L_j = (2^j - 1)(L - 1) + 1$

- define MODWT filters $\{\tilde{h}_{j,l}\}$ and $\{\tilde{g}_{j,l}\}$ by renormalizing the DWT filters:

$$\tilde{h}_{j,l} = h_{j,l} / 2^j / 2 \text{ and } \tilde{g}_{j,l} = g_{j,l} / 2^j / 2$$
Definition of MODWT Wavelet & Scaling Filters: II

- widths $L_j$ of MODWT and DWT filters are the same
- whereas DWT filters have unit energy, MODWT filters satisfy
  \[
  \sum_{l=0}^{L_j-1} \tilde{h}_{j,l}^2 = \sum_{l=0}^{L_j-1} \tilde{g}_{j,l}^2 = \frac{1}{2^j}
  \]
- let $\tilde{H}_j(\cdot)$ and $\tilde{G}_j(\cdot)$ be the corresponding transfer functions:
  \[
  \tilde{H}_j(f) = \frac{1}{2^{j/2}} H_j(f) \quad \text{and} \quad \tilde{G}_j(f) = \frac{1}{2^{j/2}} G_j(f)
  \]
  so that
  \[
  \{\tilde{h}_{j,l}\} \longleftrightarrow \tilde{H}_j(\cdot) \quad \text{and} \quad \{\tilde{g}_{j,l}\} \longleftrightarrow \tilde{G}_j(\cdot)
  \]
Definition of MODWT Coefficients: I

- level $j$ MODWT wavelet and scaling coefficients are *defined* to be output obtaining by filtering $X$ with $\{\tilde{h}_{j,l}\}$ and $\{\tilde{g}_{j,l}\}$:

\[
X \rightarrow \tilde{H}_j(\frac{k}{N}) \rightarrow \tilde{W}_j \quad \text{and} \quad X \rightarrow \tilde{G}_j(\frac{k}{N}) \rightarrow \tilde{V}_j
\]

- compare the above to its DWT equivalent:

\[
X \rightarrow H_j(\frac{k}{N}) \downarrow 2^j \rightarrow W_j \quad \text{and} \quad X \rightarrow G_j(\frac{k}{N}) \downarrow 2^j \rightarrow V_j
\]

- DWT and MODWT have different normalizations for filters, and there is no downsampling by $2^j$ in the MODWT

- level $J_0$ MODWT consists of $J_0 + 1$ vectors, namely,

\[
\tilde{W}_1, \tilde{W}_2, \ldots, \tilde{W}_{J_0} \quad \text{and} \quad \tilde{V}_{J_0},
\]

each of which has length $N$
Definition of MODWT Coefficients: II

- MODWT of level $J_0$ has $(J_0 + 1)N$ coefficients, whereas DWT has $N$ coefficients for any given $J_0$
- whereas DWT of level $J_0$ requires $N$ to be integer multiple of $2^{J_0}$, MODWT of level $J_0$ is well-defined for any sample size $N$
- when $N$ is divisible by $2^{J_0}$, we can write

$$W_{j,t} = \sum_{l=0}^{L_j-1} h_{j,l} X_{2^j(t+1)-1-l \mod N}$$
$$\tilde{W}_{j,t} = \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l \mod N},$$

and we have the relationship

$$W_{j,t} = 2^{j/2} \tilde{W}_{j,2^j(t+1)-1}$$
$$V_{J_0,t} = 2^{J_0/2} \tilde{V}_{J_0,2^{J_0}(t+1)-1}$$

(here $\tilde{W}_{j,t}$ & $\tilde{V}_{J_0,t}$ denote the $t$th elements of $\tilde{W}_j$ & $\tilde{V}_{J_0}$)
Properties of the MODWT

• as was true with the DWT, we can use the MODWT to obtain
  – a scale-based additive decomposition (MRA) and
  – a scale-based energy decomposition (ANOVA)
• in addition, the MODWT can be computed efficiently via a pyramid algorithm
MODWT Multiresolution Analysis: I

• starting from the definition

\[ \widehat{W}_{j,t} = \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l \mod N}, \quad \text{have} \quad \widetilde{W}_{j,t} = \sum_{l=0}^{N-1} \tilde{h}_{j,l} X_{t-l \mod N}, \]

where \( \{\tilde{h}_{j,l}\} \) is \( \{\tilde{h}_{j,l}\} \) periodized to length \( N \)

• can express the above in matrix notation as \( \widetilde{W}_j = \widetilde{W}_j \mathbf{X} \), where \( \widetilde{W}_j \) is the \( N \times N \) matrix given by

\[
\begin{bmatrix}
\tilde{h}_{j,0} & \tilde{h}_{j,N-1} & \tilde{h}_{j,N-2} & \tilde{h}_{j,N-3} & \cdots & \tilde{h}_{j,3} & \tilde{h}_{j,2} & \tilde{h}_{j,1} \\
\tilde{h}_{j,1} & \tilde{h}_{j,0} & \tilde{h}_{j,N-1} & \tilde{h}_{j,N-2} & \cdots & \tilde{h}_{j,4} & \tilde{h}_{j,3} & \tilde{h}_{j,2} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\tilde{h}_{j,N-2} & \tilde{h}_{j,N-3} & \tilde{h}_{j,N-4} & \tilde{h}_{j,N-5} & \cdots & \tilde{h}_{j,1} & \tilde{h}_{j,0} & \tilde{h}_{j,N-1} \\
\tilde{h}_{j,N-1} & \tilde{h}_{j,N-2} & \tilde{h}_{j,N-3} & \tilde{h}_{j,N-4} & \cdots & \tilde{h}_{j,2} & \tilde{h}_{j,1} & \tilde{h}_{j,0}
\end{bmatrix}
\]
MODWT Multiresolution Analysis: II

- recalling the DWT relationship $\mathcal{D}_j = \mathcal{W}_j^T \mathcal{W}_j$, define $j$th level MODWT detail as $\mathcal{D}_j = \mathcal{W}_j^T \mathcal{W}_j$

- similar development leads to definition for $j$th level MODWT smooth as $\mathcal{S}_j = \mathcal{V}_j^T \mathcal{V}_j$

- will now show that level $J_0$ MODWT-based MRA is given by

$$X = \sum_{j=1}^{J_0} \mathcal{D}_j + \mathcal{S}_{J_0},$$

which is analogous to the DWT-based MRA
MODWT Multiresolution Analysis: III

- since $\tilde{D}_j = \tilde{W}_j^T \tilde{W}_j$, let’s look at $\tilde{W}_j^T$:

$$
\begin{bmatrix}
\tilde{h}_j^0,0 & \tilde{h}_j^0,1 & \tilde{h}_j^0,2 & \tilde{h}_j^0,3 & \cdots & \tilde{h}_j^0,N-3 & \tilde{h}_j^0,N-2 & \tilde{h}_j^0,N-1 \\
\tilde{h}_j^0,N-1 & \tilde{h}_j^0,0 & \tilde{h}_j^0,1 & \tilde{h}_j^0,2 & \cdots & \tilde{h}_j^0,N-4 & \tilde{h}_j^0,N-3 & \tilde{h}_j^0,N-2 \\
\tilde{h}_j^0,N-2 & \tilde{h}_j^0,N-1 & \tilde{h}_j^0,0 & \tilde{h}_j^0,1 & \cdots & \tilde{h}_j^0,N-5 & \tilde{h}_j^0,N-4 & \tilde{h}_j^0,N-3 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\tilde{h}_j^0,2 & \tilde{h}_j^0,3 & \tilde{h}_j^0,4 & \tilde{h}_j^0,5 & \cdots & \tilde{h}_j^0,N-1 & \tilde{h}_j^0,0 & \tilde{h}_j^0,1 \\
\tilde{h}_j^0,1 & \tilde{h}_j^0,2 & \tilde{h}_j^0,3 & \tilde{h}_j^0,4 & \cdots & \tilde{h}_j^0,N-2 & \tilde{h}_j^0,N-1 & \tilde{h}_j^0,0
\end{bmatrix}
$$

- since $\tilde{V}_j^T$ has a similar pattern, elements of $\tilde{D}_j$ & $\tilde{S}_j$ are thus

$$
\tilde{d}_{j,t} = \sum_{l=0}^{N-1} \tilde{h}_{j,l} \tilde{W}_{j,t+l \text{ mod } N} \quad \& \quad \tilde{s}_{j,t} = \sum_{l=0}^{N-1} \tilde{g}_{j,l} \tilde{V}_{j,t+l \text{ mod } N}
$$
MODWT Multiresolution Analysis: IV

- $\tilde{D}_j$ and $\tilde{S}_j$ both formed by cyclic cross-correlations, and hence
  - $\tilde{D}_j$ formed by filtering $\{\tilde{W}_{j,t}\}$ with $\{\tilde{H}^*_j(\frac{k}{N})\}$
  - $\tilde{S}_j$ formed by filtering $\{\tilde{V}_{j,t}\}$ with $\{\tilde{G}^*_j(\frac{k}{N})\}$
- in turn, $\{\tilde{W}_{j,t}\} \& \{\tilde{V}_{j,t}\}$ formed by filtering $\{X_t\} \leftrightarrow \{\chi_k\}$ with $\{\tilde{h}^\circ_{j,l}\} \leftrightarrow \{\tilde{H}_j(\frac{k}{N})\} \& \{\tilde{g}^\circ_{j,l}\} \leftrightarrow \{\tilde{G}_j(\frac{k}{N})\}$
- hence
  
  \[
  \{\tilde{D}_{j,t}\} \leftrightarrow \{\tilde{H}^*_j(\frac{k}{N})\tilde{H}_j(\frac{k}{N})\chi_k\} = \{|\tilde{H}_j(\frac{k}{N})|^2\chi_k\}
  \]
  
  \[
  \{\tilde{S}_{j,t}\} \leftrightarrow \{\tilde{G}^*_j(\frac{k}{N})\tilde{G}_j(\frac{k}{N})\chi_k\} = \{|\tilde{G}_j(\frac{k}{N})|^2\chi_k\}
  \]
MODWT Multiresolution Analysis: V

- since the DFT of a sum is the sum of the individual DFTs,
  \[ \{\tilde{D}_{j,t} + \tilde{S}_{j,t}\} \leftrightarrow \left\{ \left( |\tilde{H}_j(\frac{k}{N})|^2 + |\tilde{G}_j(\frac{k}{N})|^2 \right) \chi_k \right\} \]

- when \( j \geq 2 \), can reduce term in parentheses:

\[
|\tilde{H}_j(\frac{k}{N})|^2 + |\tilde{G}_j(\frac{k}{N})|^2 = |\tilde{H}(2^{j-1}\frac{k}{N})|^2 \prod_{l=0}^{j-2} |\tilde{G}(2^l \frac{k}{N})|^2 + \prod_{l=0}^{j-1} |\tilde{G}(2^l \frac{k}{N})|^2
\]

\[
= \left( |\tilde{H}(2^{j-1}\frac{k}{N})|^2 + |\tilde{G}(2^{j-1}\frac{k}{N})|^2 \right) \prod_{l=0}^{j-2} |\tilde{G}(2^l \frac{k}{N})|^2
\]

\[
= \frac{1}{2} \left( |H(2^{j-1}\frac{k}{N})|^2 + |G(2^{j-1}\frac{k}{N})|^2 \right) |\tilde{G}_{j-1}(\frac{k}{N})|^2
\]

\[
= |\tilde{G}_{j-1}(\frac{k}{N})|^2
\]

since \( |H(f)|^2 + |G(f)|^2 = \mathcal{H}(f) + \mathcal{G}(f) = 2 \)
MODWT Multiresolution Analysis: VI

- implies \( \{ \tilde{D}_{j,t} + \tilde{S}_{j,t} \} \leftrightarrow \{|\tilde{G}_{j-1}(\frac{k}{N})|^2 \chi_k \} \)

- compare the above to \( \{ \tilde{S}_{j,t} \} \leftrightarrow \{|\tilde{G}_j(\frac{k}{N})|^2 \chi_k \} \) and evoke the uniqueness of the DFT to get \( \tilde{S}_{j-1} = \tilde{D}_j + \tilde{S}_j \) for \( j \geq 2 \)

- hence \( \tilde{S}_1 = \tilde{D}_2 + \tilde{S}_2 = \tilde{D}_2 + \tilde{D}_3 + \tilde{S}_3 = \cdots \), leading to

\[
\tilde{S}_1 = \sum_{j=2}^{J_0} \tilde{D}_j + \tilde{S}_{J_0}
\]

and hence \( X = \sum_{j=1}^{J_0} \tilde{D}_j + \tilde{S}_{J_0} \)

if we use Exer. [172]: \( X = \tilde{S}_1 + \tilde{D}_1 \) for all \( N \& L \)
MODWT Multiresolution Analysis: VII

- if we form DWT-based MRAs for $X$ and its circular shifts $T^m X$, $m = 1, \ldots, N - 1$, we can obtain $\tilde{D}_j$ by appropriately averaging all $N$ DWT-based details (‘cycle spinning’)

![Image of MODWT Multiresolution Analysis graphs]

WMTSA: 204 (Exer. [5.1])
• if we form DWT-based MRAs for $X$ and its circular shifts $\mathcal{T}^mX, m = 1, \ldots, N - 1$, we can obtain $\tilde{D}_j$ by appropriately averaging all $N$ DWT-based details (‘cycle spinning’)

\[
\begin{align*}
S_4 & \\
D_4 & \\
D_3 & \\
D_2 & \\
D_1 & \\
\mathcal{T}X & \\
\mathcal{T}^{-1}S_4 & \\
\mathcal{T}^{-1}D_4 & \\
\mathcal{T}^{-1}D_3 & \\
\mathcal{T}^{-1}D_2 & \\
\mathcal{T}^{-1}D_1 & \\
X & \\
\end{align*}
\]
• if we form DWT-based MRAs for $\mathbf{X}$ and its circular shifts $\mathcal{T}^m \mathbf{X}, \ m = 1, \ldots, N - 1$, we can obtain $\tilde{\mathcal{D}}_j$ by appropriately averaging all $N$ DWT-based details (‘cycle spinning’)
MODWT Multiresolution Analysis: VII

- if we form DWT-based MRAs for $X$ and its circular shifts $T^mX, m = 1, \ldots, N - 1,$ we can obtain $\widetilde{D}_j$ by appropriately averaging all $N$ DWT-based details (‘cycle spinning’)

\begin{align*}
S_4 & \quad T^{-3}S_4 \\
D_4 & \quad T^{-3}D_4 \\
D_3 & \quad T^{-3}D_3 \\
D_2 & \quad T^{-3}D_2 \\
D_1 & \quad T^{-3}D_1 \\
T^3X & \quad X
\end{align*}
MODWT Multiresolution Analysis: VII

- if we form DWT-based MRAs for \( X \) and its circular shifts \( T^m X, m = 1, \ldots, N - 1 \), we can obtain \( \tilde{D}_j \) by appropriately averaging all \( N \) DWT-based details ('cycle spinning')
MODWT Multiresolution Analysis: VII

- if we form DWT-based MRAs for \( \mathbf{X} \) and its circular shifts \( T^m \mathbf{X}, m = 1, \ldots, N - 1 \), we can obtain \( \mathbf{\tilde{D}_j} \) by appropriately averaging all \( N \) DWT-based details (‘cycle spinning’)

\[
\begin{align*}
S_4 & \quad | & \quad T^{-5}S_4 \\
D_4 & \quad | & \quad T^{-5}D_4 \\
D_3 & \quad | & \quad T^{-5}D_3 \\
D_2 & \quad | & \quad T^{-5}D_2 \\
D_1 & \quad | & \quad T^{-5}D_1 \\
T^5 \mathbf{X} & \quad | & \quad \mathbf{X}
\end{align*}
\]
MODWT Multiresolution Analysis: VII

- if we form DWT-based MRAs for \( \mathbf{X} \) and its circular shifts \( \mathcal{T}^m \mathbf{X} \), \( m = 1, \ldots, N - 1 \), we can obtain \( \tilde{D}_j \) by appropriately averaging all \( N \) DWT-based details (‘cycle spinning’)

\[
\begin{align*}
\mathcal{S}_4 & \quad \mathbb{1} & \quad \mathcal{T}^{-6} \mathcal{S}_4 \\
\mathcal{D}_4 & \quad \mathbb{1} & \quad \mathcal{T}^{-6} \mathcal{D}_4 \\
\mathcal{D}_3 & \quad \mathbb{1} & \quad \mathcal{T}^{-6} \mathcal{D}_3 \\
\mathcal{D}_2 & \quad \mathbb{1} & \quad \mathcal{T}^{-6} \mathcal{D}_2 \\
\mathcal{D}_1 & \quad \mathbb{1} & \quad \mathcal{T}^{-6} \mathcal{D}_1 \\
\mathcal{T}^6 \mathbf{X} & \quad \mathbb{1} & \quad \mathbf{X}
\end{align*}
\]
MODWT Multiresolution Analysis: VII

- if we form DWT-based MRAs for $X$ and its circular shifts $T^mX$, $m = 1, \ldots, N - 1$, we can obtain $\tilde{D}_j$ by appropriately averaging all $N$ DWT-based details (‘cycle spinning’)

![Diagram showing MODWT Multiresolution Analysis]

WMTSA: 204 (Exer. [5.1])
MODWT Multiresolution Analysis: VII

- if we form DWT-based MRAs for $\mathbf{X}$ and its circular shifts $\mathcal{T}^m \mathbf{X}$, $m = 1, \ldots, N - 1$, we can obtain $\mathbf{\hat{D}_j}$ by appropriately averaging all $N$ DWT-based details (‘cycle spinning’)

\[ \begin{align*}
\mathcal{S}_4 & \quad \mathcal{T}^{-8} \mathcal{S}_4 \\
\mathcal{D}_4 & \quad \mathcal{T}^{-8} \mathcal{D}_4 \\
\mathcal{D}_3 & \quad \mathcal{T}^{-8} \mathcal{D}_3 \\
\mathcal{D}_2 & \quad \mathcal{T}^{-8} \mathcal{D}_2 \\
\mathcal{D}_1 & \quad \mathcal{T}^{-8} \mathcal{D}_1 \\
\mathcal{T}^8 \mathbf{X} & \quad \mathbf{X} \\
\end{align*} \]
MODWT Multiresolution Analysis: VII

- if we form DWT-based MRAs for $X$ and its circular shifts $\mathcal{T}^m X$, $m = 1, \ldots, N - 1$, we can obtain $\tilde{D}_j$ by appropriately averaging all $N$ DWT-based details ('cycle spinning')
MODWT Multiresolution Analysis: VII

- if we form DWT-based MRAs for $\mathbf{X}$ and its circular shifts $T^m\mathbf{X}, m = 1, \ldots, N - 1$, we can obtain $\tilde{D}_j$ by appropriately averaging all $N$ DWT-based details (‘cycle spinning’)

![Diagram showing MODWT Multiresolution Analysis]
MODWT Multiresolution Analysis: VII

- if we form DWT-based MRAs for $\mathbf{X}$ and its circular shifts $\mathcal{T}^m \mathbf{X}$, $m = 1, \ldots, N - 1$, we can obtain $\tilde{D}_j$ by appropriately averaging all $N$ DWT-based details (‘cycle spinning’)
• if we form DWT-based MRAs for $X$ and its circular shifts $T^m X$, $m = 1, \ldots, N - 1$, we can obtain $\tilde{D}_j$ by appropriately averaging all $N$ DWT-based details (‘cycle spinning’)

\[ S_4, D_4, D_3, D_2, D_1, T^{12}X, T^{-12}S_4, T^{-12}D_4, T^{-12}D_3, T^{-12}D_2, T^{-12}D_1, X \]
• if we form DWT-based MRAs for $\mathbf{X}$ and its circular shifts $T^m \mathbf{X}, m = 1, \ldots, N - 1$, we can obtain $\tilde{D}_j$ by appropriately averaging all $N$ DWT-based details (‘cycle spinning’).
if we form DWT-based MRAs for $X$ and its circular shifts $\mathcal{T}^m X$, $m = 1, \ldots, N - 1$, we can obtain $\hat{D}_j$ by appropriately averaging all $N$ DWT-based details ('cycle spinning')
MODWT Multiresolution Analysis: VII

- if we form DWT-based MRAs for \( \mathbf{X} \) and its circular shifts \( \mathcal{T}^m \mathbf{X}, m = 1, \ldots, N - 1 \), we can obtain \( \tilde{D}_j \) by appropriately averaging all \( N \) DWT-based details (‘cycle spinning’)

\[
\begin{align*}
&\mathcal{T}^{15} \mathbf{X} \\
&\begin{array}{c}
\mathcal{S}_4 \\
\mathcal{D}_4 \\
\mathcal{D}_3 \\
\mathcal{D}_2 \\
\mathcal{D}_1
\end{array} \\
&1 \\
&0 \\
&-1
\end{align*}
\]

\[
\begin{align*}
&\begin{array}{c}
\mathcal{T}^{-15} \mathcal{S}_4 \\
\mathcal{T}^{-15} \mathcal{D}_4 \\
\mathcal{T}^{-15} \mathcal{D}_3 \\
\mathcal{T}^{-15} \mathcal{D}_2 \\
\mathcal{T}^{-15} \mathcal{D}_1
\end{array} \\
&1 \\
&0 \\
&-1
\end{align*}
\]
MODWT Multiresolution Analysis: VIII

- left-hand plots show $\tilde{D}_j$, while right-hand plots show average of $\mathcal{T}^{-m}D_j$ in MRA for $\mathcal{T}^mX$, $m = 0, 1, \ldots, 15$
MODWT Analysis of Variance: I

• for any $J_0 \geq 1 \& N \geq 1$, will now show that

$$\|X\|^2 = \sum_{j=1}^{J_0} \|\tilde{W}_j\|^2 + \|\tilde{V}_{J_0}\|^2,$$

leading to an analysis of the sample variance of $X$:

$$\hat{\sigma}_X^2 = \frac{1}{N} \sum_{j=1}^{J_0} \|\tilde{W}_j\|^2 + \frac{1}{N} \|\tilde{V}_{J_0}\|^2 - \overline{X}^2,$$

which is analogous to the DWT-based analysis of variance.
MODWT Analysis of Variance: II

• as before, let \( \{X_k\} \) be the DFT of \( \{X_t\} \) so that
  \[
  \{\tilde{W}_{j,t}\} \longleftrightarrow \{\tilde{H}_j(\frac{k}{N})X_k\} \quad \& \quad \{\tilde{V}_{j,t}\} \longleftrightarrow \{\tilde{G}_j(\frac{k}{N})X_k\}
  \]

• Parseval’s theorem says:
  \[
  \|\tilde{W}_j\|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |\tilde{H}_j(\frac{k}{N})|^2 |X_k|^2 \quad \& \quad \|\tilde{V}_j\|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |\tilde{G}_j(\frac{k}{N})|^2 |X_k|^2
  \]

• since \( |\tilde{H}_j(\frac{k}{N})|^2 + |\tilde{G}_j(\frac{k}{N})|^2 = |\tilde{G}_{j-1}(\frac{k}{N})|^2, j \geq 2 \), adding yields
  \[
  \|\tilde{W}_j\|^2 + \|\tilde{V}_j\|^2 = \frac{1}{N} \sum_{k=0}^{N-1} \left( |\tilde{H}_j(\frac{k}{N})|^2 + |\tilde{G}_j(\frac{k}{N})|^2 \right) |X_k|^2
  \]
  \[
  = \frac{1}{N} \sum_{k=0}^{N-1} |\tilde{G}_{j-1}(\frac{k}{N})|^2 |X_k|^2 = \|\tilde{V}_{j-1}\|^2
  \]
MODWT Analysis of Variance: III

- using $\|\tilde{V}_{j-1}\|^2 = \|\tilde{W}_j\|^2 + \|\tilde{V}_j\|^2$ for $j = 2, 3, \ldots, J_0$ yields
  \[
  \|\tilde{V}_1\|^2 = \|\tilde{W}_2\|^2 + \|\tilde{V}_2\|^2 \\
  = \|\tilde{W}_2\|^2 + \|\tilde{W}_3\|^2 + \|\tilde{V}_3\|^2 \\
  \vdots \\
  = \sum_{j=2}^{J_0} \|\tilde{W}_j\|^2 + \|\tilde{V}_J\|^2
  \]

- desired result
  \[
  \|X\|^2 = \sum_{j=1}^{J_0} \|\tilde{W}_j\|^2 + \|\tilde{V}_J\|^2,
  \]

now follows if we can show that $\|X\|^2 = \|\tilde{W}_1\|^2 + \|\tilde{V}_1\|^2$, and this is the subject of Exer. [171a]
MODWT Pyramid Algorithm: I

- goal: compute $\tilde{W}_j \& \tilde{V}_j$ using $\tilde{V}_{j-1}$ rather than $X$
- can obtain all 3 by filtering $X$ directly:
  - to get $\tilde{V}_j$, use $\{\tilde{G}_j\left(\frac{k}{N}\right) = \tilde{G}_{j-1}\left(\frac{k}{N}\right)\tilde{G}(2^{j-1}\frac{k}{N})\}$
  - to get $\tilde{W}_j$, use $\{\tilde{H}_j\left(\frac{k}{N}\right) = \tilde{G}_{j-1}\left(\frac{k}{N}\right)\tilde{H}(2^{j-1}\frac{k}{N})\}$
  - to get $\tilde{V}_{j-1}$, use $\{\tilde{G}_{j-1}\left(\frac{k}{N}\right)\}$
- can get $\tilde{V}_j \& \tilde{W}_j$ using $\tilde{G}(2^{j-1}\frac{k}{N}) \& \tilde{H}(2^{j-1}\frac{k}{N})$ on $\tilde{V}_{j-1}$
- Exer. [91]: if $\{\tilde{h}_l\} \leftrightarrow \tilde{H}(f)$, the inverse DFT of $\tilde{H}(2^{j-1}f)$ is
  $$\{\tilde{h}_0, \underbrace{0, \ldots, 0}_{2^{j-1}-1 \text{ zeros}}, \tilde{h}_1, \underbrace{0, \ldots, 0}_{2^{j-1}-1 \text{ zeros}}, \ldots, \tilde{h}_{L-2}, \underbrace{0, \ldots, 0}_{2^{j-1}-1 \text{ zeros}}, \tilde{h}_{L-1}\}$$
MODWT Pyramid Algorithm: II

• letting $\tilde{V}_{0,t} \equiv X_t$, implies that, for all $j \geq 1$,

$$\tilde{W}_{j,t} = \sum_{l=0}^{L-1} \tilde{h}_l \tilde{V}_{j-1,t-2j^{-1}l \mod N} \quad \& \quad \tilde{V}_{j,t} = \sum_{l=0}^{L-1} \tilde{g}_l \tilde{V}_{j-1,t-2j^{-1}l \mod N}$$

• algorithm requires $N \log_2(N)$ multiplications, which is the same as needed by fast Fourier transform algorithm

• inverse pyramid algorithm is given by

$$\tilde{V}_{j-1,t} = \sum_{l=0}^{L-1} \tilde{h}_l \tilde{W}_{j,t+2j^{-1}l \mod N} + \sum_{l=0}^{L-1} \tilde{g}_l \tilde{V}_{j,t+2j^{-1}l \mod N}$$

(proof of this statement is the subject of Exer. [175])
MODWT Pyramid Algorithm: III

- pyramid algorithm summarized in following flow diagram:

\[
\begin{align*}
\tilde{G}(2^{j-1} \frac{k}{N}) & \rightarrow \tilde{V}_j & \rightarrow \tilde{G}^*(2^{j-1} \frac{k}{N}) \\
\tilde{V}_{j-1} & \leftarrow & \tilde{V}_j & \rightarrow \tilde{V}_{j-1} \\
\end{align*}
\]

\[
\begin{align*}
\tilde{H}(2^{j-1} \frac{k}{N}) & \rightarrow \tilde{W}_j & \rightarrow \tilde{H}^*(2^{j-1} \frac{k}{N}) \\
\end{align*}
\]

- item [1] of Comments and Extensions to Sec. 5.5 has pseudo code for MODWT pyramid algorithm
MODWT Pyramid Algorithm: IV

- similar to DWT, can describe transform from \( \tilde{V}_{j-1} \) to \( \tilde{W}_j \) & \( \tilde{V}_j \) as \( \tilde{W}_j = \tilde{B}_j \tilde{V}_{j-1} \) & \( \tilde{V}_j = \tilde{A}_j \tilde{V}_{j-1} \), where now \( \tilde{B}_j \) & \( \tilde{A}_j \) are \( N \times N \) matrices
- rows of \( \tilde{B}_j \) contain inverse DFT of \( \{ \tilde{H}(2^{j-1} \frac{k}{N}) \} \)
- rows of \( \tilde{A}_j \) contain inverse DFT of \( \{ \tilde{G}(2^{j-1} \frac{k}{N}) \} \)
- example of \( \tilde{B}_j \) with \( j = 2, N = 12 \) & \( L = 4 \):

\[
\tilde{B}_2 \equiv \begin{bmatrix}
\tilde{h}_0 & 0 & 0 & 0 & 0 & 0 & \tilde{h}_3 & 0 & \tilde{h}_2 & 0 & \tilde{h}_1 & 0 \\
0 & \tilde{h}_0 & 0 & 0 & 0 & 0 & 0 & \tilde{h}_3 & 0 & \tilde{h}_2 & 0 & \tilde{h}_1 \\
\tilde{h}_1 & 0 & \tilde{h}_0 & 0 & 0 & 0 & 0 & 0 & \tilde{h}_3 & 0 & \tilde{h}_2 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \tilde{h}_3 & 0 & \tilde{h}_2 & 0 & \tilde{h}_1 & 0 & \tilde{h}_0 & 0 \\
0 & 0 & 0 & 0 & \tilde{h}_3 & 0 & \tilde{h}_2 & 0 & \tilde{h}_1 & 0 & \tilde{h}_0 & 0
\end{bmatrix}
\]
MODWT Pyramid Algorithm: V

- Exer. [175]: like DWT, can express $\widetilde{W}_j \& \widetilde{V}_j \longrightarrow \widetilde{V}_{j-1}$ as
  \[ \widetilde{V}_{j-1} = \widetilde{B}_j^T \widetilde{W}_j + \widetilde{A}_j^T \widetilde{V}_j \]

- starting with $\widetilde{V}_0 = X$, $J_0$ recursions yield
  \[
  X = \left( \begin{array}{c}
  \widetilde{D}_1 \\
  \widetilde{D}_2 \\
  \widetilde{D}_3 \\
  \vdots \\
  \widetilde{D}_{J_0}
  \end{array} \right) \begin{array}{c}
  \tilde{B}_1^T \tilde{W}_1 \\
  \tilde{A}_1^T \tilde{B}_2^T \tilde{W}_2 \\
  \tilde{A}_1^T \tilde{B}_3^T \tilde{W}_3 \\
  \tilde{A}_1^T \cdots \tilde{A}_{J_0-1}^T \tilde{B}_{J_0}^T \tilde{W}_{J_0} \\
  \tilde{A}_1^T \cdots \tilde{A}_{J_0-1}^T \tilde{A}_{J_0}^T \tilde{V}_{J_0}
  \end{array}
  \]

- since $\tilde{D}_j \equiv \tilde{W}_j^T \tilde{W}_j$ and $\tilde{S}_{J_0} \equiv \tilde{V}_{J_0}^T \tilde{V}_{J_0}$, we evidently have
  $\tilde{W}_j = \tilde{B}_j \tilde{A}_{j-1} \cdots \tilde{A}_1$ and $\tilde{V}_{J_0} = \tilde{A}_{J_0} \tilde{A}_{J_0-1} \cdots \tilde{A}_1$
Example of $J_0 = 4$ LA(8) MODWT

- oxygen isotope records $X$ from Antarctic ice core
Relationship Between MODWT and DWT

- bottom plot shows $W_4$ from DWT after circular shift $T^{-3}$ to align coefficients properly in time (more about $T$ later)
- top plot shows $\widetilde{W}_4$ from MODWT and subsamples that, upon rescaling, yield $W_4$ via $W_{4,t} = 4\widetilde{W}_{4,16(t+1)-1}$

\[ T^{-3} \widetilde{W}_4 \]

\[ T^{-53} \widetilde{W}_4 \]
Example of $J_0 = 4$ LA(8) MODWT MRA

- oxygen isotope records $\mathbf{X}$ from Antarctic ice core
Example of Variance Decomposition

- decomposition of sample variance from MODWT

\[ \hat{\sigma}^2_X \equiv \frac{1}{N} \sum_{t=0}^{N-1} (X_t - \bar{X})^2 = \sum_{j=1}^{4} \frac{1}{N} \| \tilde{W}_j \|^2 + \frac{1}{N} \| \tilde{V}_4 \|^2 - \bar{X}^2 \]

- LA(8)-based example for oxygen isotope records
  - 0.5 year changes: \[ \frac{1}{N} \| \tilde{W}_1 \|^2 \doteq 0.145 (\doteq 4.5\% \text{ of } \hat{\sigma}^2_X) \]
  - 1.0 years changes: \[ \frac{1}{N} \| \tilde{W}_2 \|^2 \doteq 0.500 (\doteq 15.6\%) \]
  - 2.0 years changes: \[ \frac{1}{N} \| \tilde{W}_3 \|^2 \doteq 0.751 (\doteq 23.4\%) \]
  - 4.0 years changes: \[ \frac{1}{N} \| \tilde{W}_4 \|^2 \doteq 0.839 (\doteq 26.2\%) \]
  - 8.0 years averages: \[ \frac{1}{N} \| \tilde{V}_4 \|^2 - \bar{X}^2 \doteq 0.969 (\doteq 30.2\%) \]
  - sample variance: \[ \hat{\sigma}^2_X \doteq 3.204 \]
Summary of Key Points about the MODWT

• similar to the DWT, the MODWT offers
  – a scale-based multiresolution analysis
  – a scale-based analysis of the sample variance
  – a pyramid algorithm for computing the transform efficiently
• unlike the DWT, the MODWT is
  – defined for all sample sizes (no ‘power of 2’ restrictions)
  – unaffected by circular shifts to \( \mathbf{X} \) in that coefficients, details and smooths shift along with \( \mathbf{X} \) (example coming later)
  – highly redundant in that a level \( J_0 \) transform consists of \((J_0 + 1)N\) values rather than just \(N\)
• as we shall see, the MODWT can eliminate ‘alignment’ artifacts, but its redundancies are problematic for some uses