Review of Concepts from Fourier & Filtering Theory

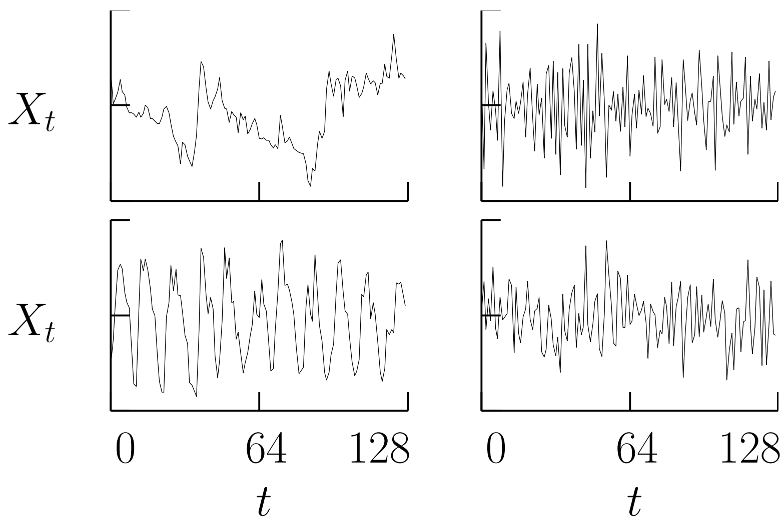
- precise definition of DWT requires a few basic concepts from Fourier analysis and theory of linear filters
- will start with discussion/review of:
  - basic ideas behind Fourier analysis of time series
  - Fourier theory for infinite sequences
  - convolution/filtering of infinite sequences
  - filter cascades
  - Fourier theory for finite sequences
  - circular convolution/filtering of finite sequences
  - periodization of a filter

What is Fourier Analysis?: I

- one of the most widely used methods for data analysis in
  - geophysics
  - oceanography
  - atmospheric science
  - astronomy
  - engineering (all types)
  - etc.
- used to analyze time series (observations collected over time)
- let X_t denote value of time series at time indexed by t
- example: $X_{89} = 65^{\circ}$ = temperature in Loew Hall 105 at 1PM on day 89 of 2018 (30th March)

What is Fourier Analysis?: II

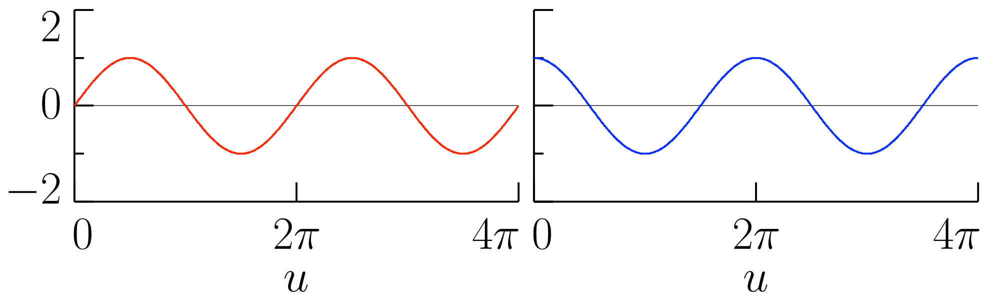
- four examples of time series $X_0, X_1, \ldots, X_{127}$



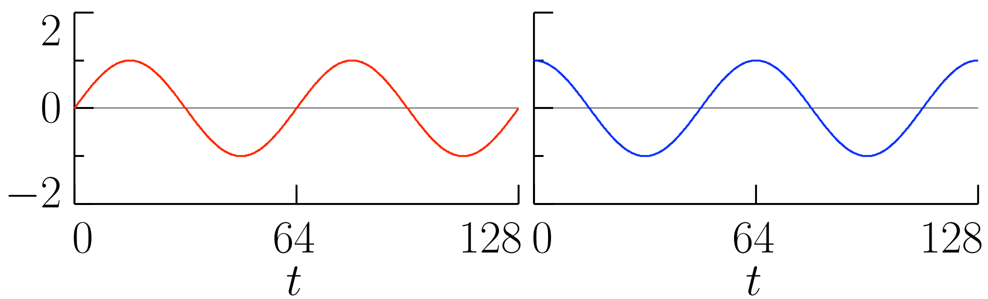
- Q: how would you describe these 4 series?
- Fourier analysis does so by comparing X_t's to sines & cosines (note: will collectively refer to sines & cosines as 'sinusoids')
- Q: what do sines and cosines have to do with time series?

What is Fourier Analysis?: III

- let's plot sin(u) and cos(u) versus u as u goes from 0 to 4π:

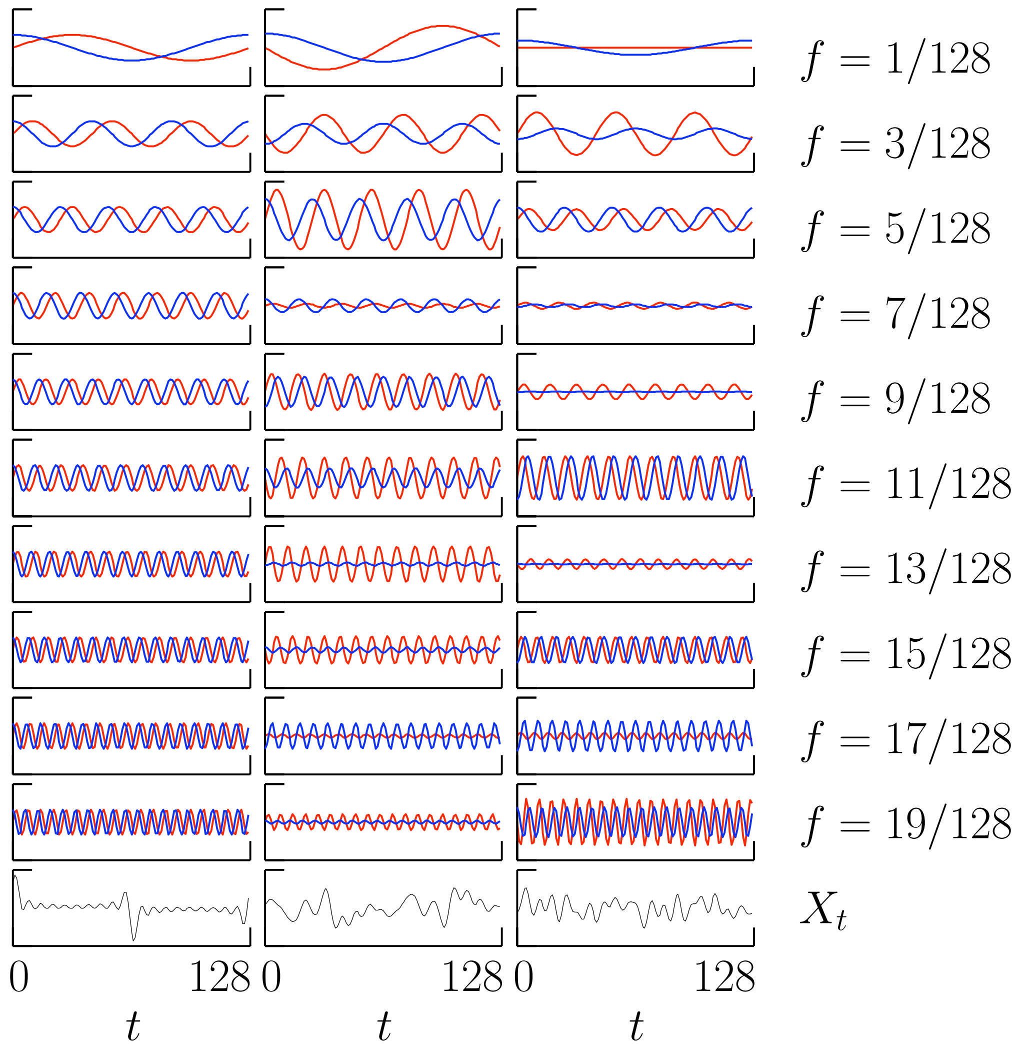


- let $u = 2\pi\frac{2}{128}t$ for $t = 0, 1, \ldots, 127$
- now let's plot sin$(2\pi\frac{2}{128}t)$ and cos$(2\pi\frac{2}{128}t)$ versus t:



- artificial time series exhibiting 2 cycles over time span of 128 (meaning of '$\frac{2}{128}$' – called frequency f of the sinusoid)

Adding Sines & Cosines with Different Frequencies



- sinusoid amplitudes fixed in column 1, but random in 2 & 3

What is Fourier Analysis?: IV

- conclusion: by summing up lots of sinusoids with different amplitudes, can get artificial X_t's that resemble actual X_t's
- goal of Fourier analysis: given a time series X_t, figure out how to construct it using sinusoids; i.e., to write

  $$X_t = \sum_k A_k \sin(2\pi f_k t) + B_k \cos(2\pi f_k t),$$

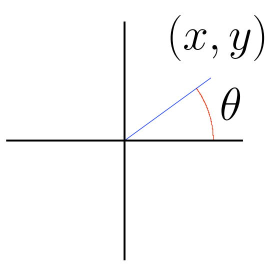
  where f_k's are a collection of different frequencies
- above called 'Fourier representation' for a time series
- allows us to reexpress time series in a standard way
- different time series will need different A_k's and B_k's
- can compare time series by comparing their A_k's and B_k's

Some Notation, Conventions and Basic Facts: I

- easier to do Fourier theory by not dealing with sinusoids directly
- $i \equiv \sqrt{-1}$ and hence $i^2 = -1$, $i^3 = -i$ & $i^4 = 1$
  (note: '$\equiv$' means 'equal by definition')
- if x & y are real-valued variables, $z = x + iy$ is complex-valued
- $z^* \equiv x - iy$, $|z| \equiv \sqrt{x^2 + y^2}$ and $|z|^2 = zz^*$ and
- $e^{ix} \equiv \cos(x) + i\sin(x)$ is definition of a complex exponential
- $|e^{ix}|^2 = 1$ because $\cos^2(x) + \sin^2(x) = 1$
- $e^{i(x+y)} = e^{ix}e^{iy}$ – just expand out both sides
- $(e^{ix})^n = e^{inx}$ for integer n (de Moivre's theorem)
- $\int e^{ix}\,dx = \frac{e^{ix}}{i}$ because
  $\int \cos(x) + i\sin(x)\,dx = \sin(x) - i\cos(x) = \frac{i\sin(x)+\cos(x)}{i}$

Some Notation, Conventions and Basic Facts: II

- since $e^{ix} = \cos(x) + i\sin(x)$ & $e^{-ix} = \cos(x) - i\sin(x)$, have $\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$
- $e^{\pm i\pi} = -1$ (trivial, but useful!)
- can write $z = |z|e^{i\theta}$ (polar representation)
  - $|z|$ is magnitude of z ($|z|$ is nonnegative)
  - $\theta = \arg(z)$ is argument of z (defined if $z \neq 0$); $\theta =$ angle between positive x axis & line to (x, y)
  - by convention $-\pi < \theta \leq \pi$



$z = x + iy$

$|z|$ is length of line from $(0, 0)$ to (x, y)

Fourier Theory for Infinite Sequences: I

- let $\{a_t : t = \ldots, -1, 0, 1, \ldots\} = \{a_t\}$ denote an infinite real-valued sequence satisfying $\sum_t a_t^2 < \infty$ (do *not* need stronger condition $\sum_t |a_t| < \infty$ – addendum to overheads has details)
- discrete Fourier transform (DFT) of $\{a_t\}$:

$$A(f) \equiv \sum_{t=-\infty}^{\infty} a_t e^{-i2\pi ft}$$

- f called frequency: $e^{-i2\pi ft} = \cos(2\pi ft) - i \sin(2\pi ft)$ (controls how fast cosine & sine go up & down as t increases)
- $A(\cdot)$ called Fourier analysis of $\{a_t\}$ (note: $A(\cdot)$ is function, while $A(f)$ is value of $A(\cdot)$ at f)

Fourier Theory for Infinite Sequences: II

- $A(f)$ defined for all f, but $0 \le f \le 1/2$ of main interest
  - because $\{a_t\}$ is real-valued,

$$A(-f) = \sum_{t=-\infty}^{\infty} a_t e^{i2\pi ft} = \left(\sum_{t=-\infty}^{\infty} a_t e^{-i2\pi ft} \right)^* = A^*(f)$$

  - $A(\cdot)$ periodic with unit period; i.e., $A(f+1) = A(f)$ since

$$e^{-i2\pi(f+1)t} = e^{-i2\pi ft} e^{-i2\pi t} = e^{-i2\pi ft} [\cos(2\pi t) - i \sin(2\pi t)] = e^{-i2\pi ft}$$

  - implies $A(f+j) = A(f)$ for any integer j
  - need only know $A(f)$ for $0 \le f \le 1/2$ to know it for all f
- 'low frequencies' are those in lower range of $[0, 1/2]$
- 'high frequencies' are those in upper range of $[0, 1/2]$

Fourier Theory for Infinite Sequences: III

- can reconstruct $\{a_t\}$ from its Fourier transform (Exercise [22c]):

$$\int_{-1/2}^{1/2} A(f)e^{i2\pi ft}\,df = a_t, \quad t = \ldots, -1, 0, 1, \ldots$$

  left-hand side called inverse DFT of $A(\cdot)$
- $\{a_t\}$ and $A(\cdot)$ are two representations for one 'thingy'
- notation: $\{a_t\} \longleftrightarrow A(\cdot)$ means
  - DFT of $\{a_t\}$ is $A(\cdot)$: $A(f) = \sum_t a_t e^{-i2\pi ft}$
  - inverse DFT of $A(\cdot)$ is $\{a_t\}$: $a_t = \int_{-1/2}^{1/2} A(f)e^{i2\pi ft}\,df$
- large $|A(f)|$ says $e^{i2\pi ft}$ important in synthesizing $\{a_t\}$; i.e., $\{a_t\}$ resembles some combination of $\cos(2\pi ft)$ and $\sin(2\pi ft)$

Fourier Theory for Infinite Sequences: IV

- if $\{a_t\} \longleftrightarrow A(\cdot)$ & $\{b_t\} \longleftrightarrow B(\cdot)$, then
$$\sum_{t=-\infty}^{\infty} a_t b_t = \int_{-1/2}^{1/2} A(f) B^*(f)\, df$$
'two sequence' Parseval's theorem (Exercise [23a])

- setting $b_t = a_t$ yields 'one sequence' Parseval:
$$\sum_{t=-\infty}^{\infty} a_t^2 = \int_{-1/2}^{1/2} |A(f)|^2\, df$$
(key to 'energy' decomposition across frequencies)

Fourier Theory for Infinite Sequences: V

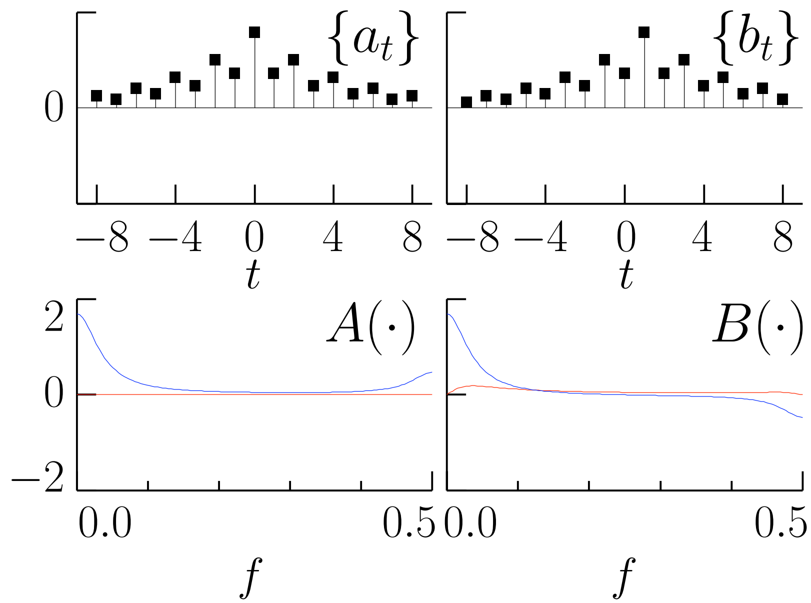
- suppose $\{a_t : t = 0, \ldots, N-1\}$ is finite sequence
- extend to infinite sequence by setting $a_t = 0$ for $t < 0$ & $t \geq N$
- DFT is then

$$A(f) \equiv \sum_{t=-\infty}^{\infty} a_t e^{-i2\pi ft} = \sum_{t=0}^{N-1} a_t e^{-i2\pi ft}$$

- notation: $\{a_t : t = 0, \ldots, N-1\} \longleftrightarrow A(\cdot)$; i.e., zero extension is implicit
- will use shorthand $\{a_t\} \longleftrightarrow A(\cdot)$ if $t = 0, \ldots, N-1$ is clear from context

Examples of Fourier Transforms of Infinite Sequences

- consider $a_t = \frac{3}{16}\left(\frac{4}{5}\right)^{|t|} + \frac{1}{20}\left(-\frac{4}{5}\right)^{|t|}$ and $b_t = a_{t-1}$



blue is real part
red is imaginary part

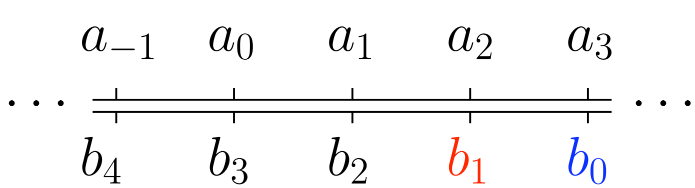
- because $\{a_t\}$ is symmetric about $t = 0$ (i.e., $a_{-t} = a_t$), its DFT $A(\cdot)$ is real-valued
- $\{b_t\}$ is asymmetric, so its DFT $B(\cdot)$ is complex-valued

Convolution of Infinite Sequences: I

- given $\{a_t\} \longleftrightarrow A(\cdot)$ and $\{b_t\} \longleftrightarrow B(\cdot)$, define

$$a * b_t \equiv \sum_{u=-\infty}^{\infty} a_u b_{t-u}, \quad t = \ldots, -1, 0, 1, \ldots$$

- note: '$a * b_t$' is just a fancy variable name (could have used 'c_t')
- sequence $\{a * b_t\}$ is convolution of $\{a_t\}$ and $\{b_t\}$
  - reverse direction of $\{b_t\}$, multiply by $\{a_t\}$ & sum to get $a * b_0$
  - shift $\{b_t\}$ and repeat to get other values $a * b_t$



$a * b_3 = \sum_{u=-\infty}^{\infty} a_u b_{3-u}$

Convolution of Infinite Sequences: II

- DFT of $\{a * b_t\}$ has a simple form, namely,

$$\sum_{t=-\infty}^{\infty} a * b_t e^{-i2\pi ft} = A(f)B(f);$$

i.e., just multiply two DFTs together (Exercise [24])

- can state this result as $\{a * b_t\} \longleftrightarrow A(\cdot)B(\cdot)$
- related concept is (complex) cross-correlation:

$$a^* \star b_t = \sum_{u=-\infty}^{\infty} a_u^* b_{u+t} = \sum_{u=-\infty}^{\infty} a_u b_{u+t} \longleftrightarrow A^*(f)B(f)$$

- letting $b_t = a_t$ yields autocorrelation:

$$\sum_{u=-\infty}^{\infty} a_u a_{u+t} \longleftrightarrow A^*(f)A(f) = |A(f)|^2$$

Basic Concepts of Filtering: I

- convolution & linear time invariant filtering are same concepts:
  - $\{b_t\}$ is input to filter
  - $\{a_t\}$ represents the filter
  - $\{a * b_t\}$ is output from filter
- flow diagram for filtering:

$$\{b_t\} \longrightarrow \boxed{\{a_t\}} \longrightarrow \{a * b_t\} \text{ or } \{b_t\} \longrightarrow \boxed{a_t} \longrightarrow \{a * b_t\}$$

- since $\{a_t\}$ equivalent to $A(\cdot)$, can also express flow diagram as

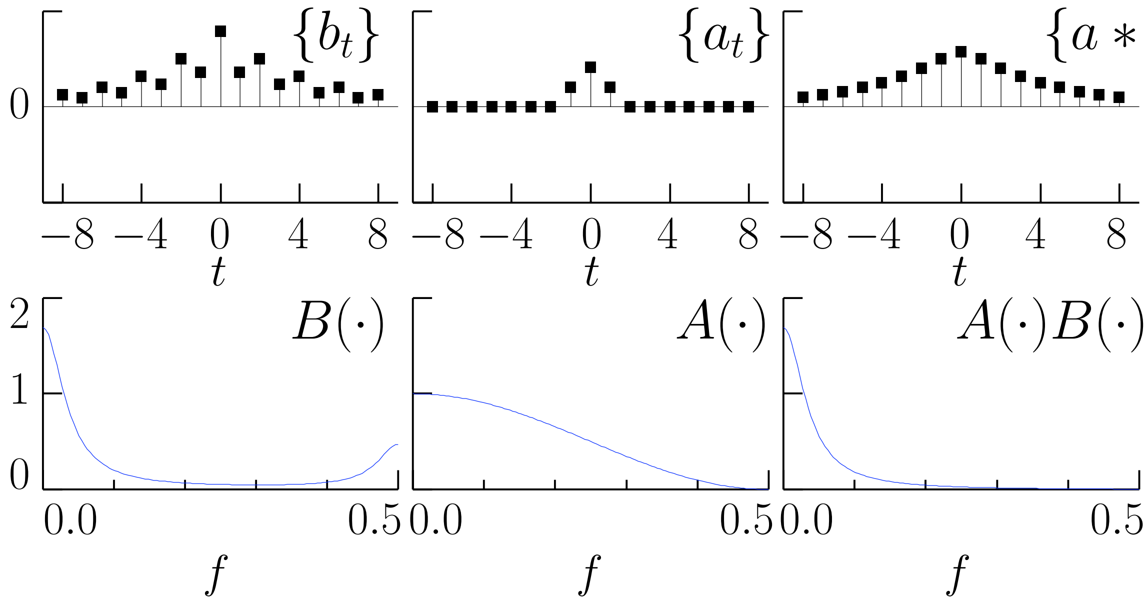
$$\{b_t\} \longrightarrow \boxed{A(\cdot)} \longrightarrow \{a * b_t\}$$

Basic Concepts of Filtering: II

- $\{a_t\}$ called impulse response sequence for filter
- $A(\cdot)$ called transfer function for filter
- in general $A(\cdot)$ is complex-valued, so write $A(f) = |A(f)|e^{i\theta(f)}$
  - $|A(f)|$ defines gain function
  - $\mathcal{A}(f) \equiv |A(f)|^2$ defines squared gain function
  - $\theta(f)$ called phase function (well-defined at f if $|A(f)| > 0$)

Example of a Low-Pass Filter

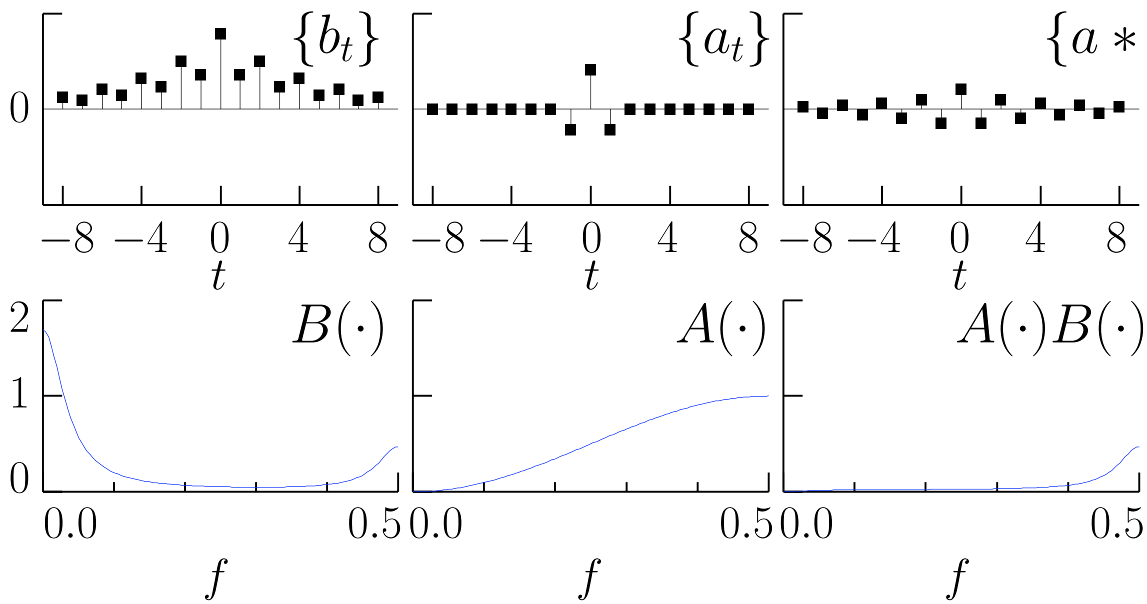
- consider $b_t = \frac{3}{16}\left(\frac{4}{5}\right)^{|t|} + \frac{1}{20}\left(-\frac{4}{5}\right)^{|t|}$ & $a_t = \begin{cases} \frac{1}{2}, & t = 0 \\ \frac{1}{4}, & t = -1 \text{ or } 1 \\ 0, & \text{otherwise} \end{cases}$



- note: $A(\cdot)$ & $B(\cdot)$ are both real-valued (equal to gain functions)

Example of a High-Pass Filter

- consider same $\{b_t\}$, but now let $a_t = \begin{cases} \frac{1}{2}, & t = 0 \\ -\frac{1}{4}, & t = -1 \text{ or } 1 \\ 0, & \text{otherwise} \end{cases}$



- might regard $\{a_t\}$ as highly discretized Mexican hat wavelet

Cascade of Filters: I

- idea: output from one filter becomes input to another
- flow diagram for cascade with 2 filters (can have more!):

$$\{b_t\} \longrightarrow \boxed{A_1(\cdot)} \xrightarrow{1.} \boxed{A_2(\cdot)} \xrightarrow{2.} \{a * b_t\}$$

  if $\{b_t\} \longleftrightarrow B(\cdot)$ and $\{a * b_t\} \longleftrightarrow C(\cdot)$, then

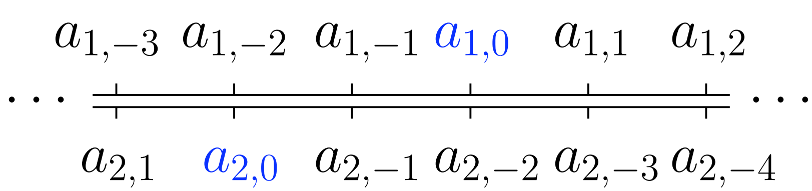
  1. output from $A_1(\cdot)$ has DFT $A_1(f)B(f)$
  2. output from $A_2(\cdot)$ has DFT $A_2(f)A_1(f)B(f)$
     so $C(f) = A_2(f)A_1(f)B(f)$

- let $A(f) \equiv A_2(f)A_1(f)$
- can reexpress overall effect of filter cascade as

$$\{b_t\} \longrightarrow \boxed{A(\cdot)} \longrightarrow \{a * b_t\}$$

Cascade of Filters: II

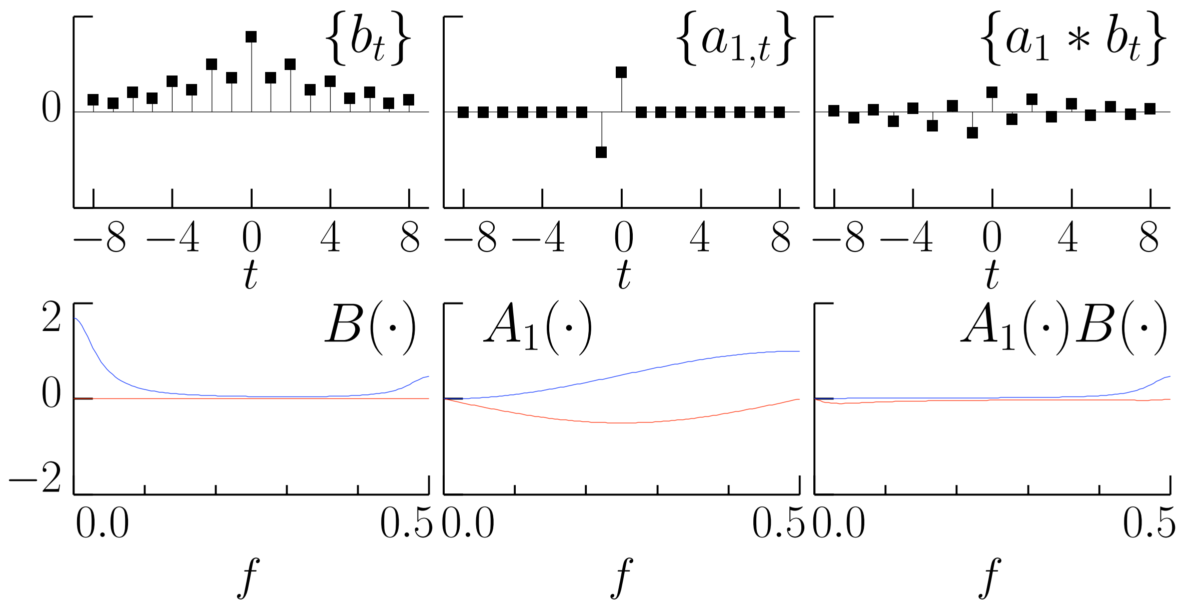
- $A(\cdot)$ is transfer function for equivalent filter for cascade
- let $\{a_t\} \longleftrightarrow A(\cdot)$, $\{a_{1,t}\} \longleftrightarrow A_1(\cdot)$ and $\{a_{2,t}\} \longleftrightarrow A_2(\cdot)$
- to form $\{a_t\}$, just need to convolve $\{a_{1,t}\}$ and $\{a_{2,t}\}$
- example: $a_{1,t} = \begin{cases} -\frac{1}{2}, & t = -1 \\ \frac{1}{2}, & t = 0 \\ 0, & \text{otherwise} \end{cases}$ & $a_{2,t} = \begin{cases} \frac{1}{2}, & t = 0 \\ -\frac{1}{2}, & t = 1 \\ 0, & \text{otherwise} \end{cases}$



$$a_{-2} = \sum_{u=-\infty}^{\infty} a_{1,u} a_{2,-2-u}$$

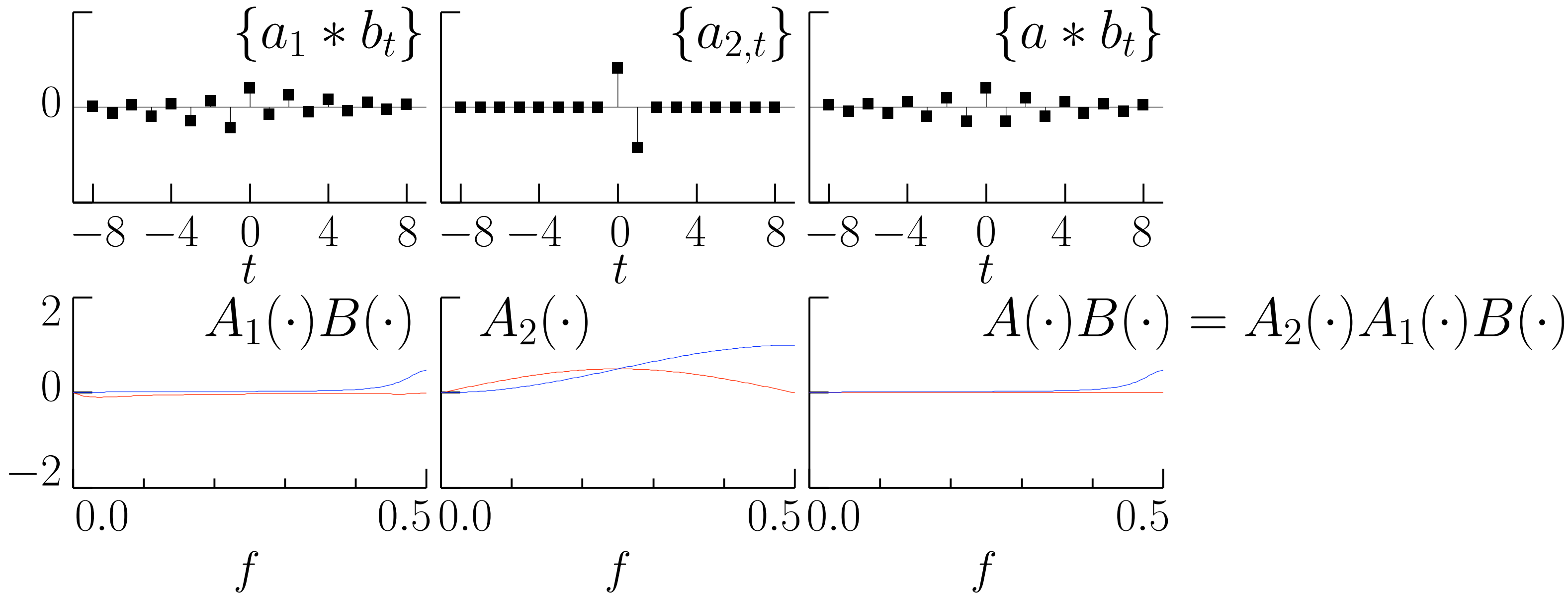
Cascade of Filters: III

- gives high-pass filter seen earlier: $a_t = \begin{cases} \frac{1}{2}, & t = 0 \\ -\frac{1}{4}, & t = -1 \text{ or } 1 \\ 0, & \text{otherwise} \end{cases}$
- filter $\{b_t\}$ with $\{a_{1,t}\}$ to get, say, $\{a_1 * b_t\}$:



Cascade of Filters: IV

- filter $\{a_1 * b_t\}$ with $\{a_{2,t}\}$ to get same $\{a * b_t\}$ as before:



- cascade of M filters of widths $L_1, \ldots, L_M$ has $\{a_t\}$ of width

$$L = \sum_{m=1}^{M} L_m - M + 1 \qquad \text{(Exercise [28a])}$$

(check on above example with $M = 2$: $L = 2 + 2 - 2 + 1 = 3$)

Fourier Theory for Finite Sequences: I

- let $\{a_t : t = 0, 1, \ldots, N-1\} = \{a_t\}$ denote a finite sequence (same shorthand as for infinite sequence – don't get confused!)
- discrete Fourier transform (DFT) of $\{a_t\}$:

$$A_k \equiv \sum_{t=0}^{N-1} a_t e^{-i2\pi f_k t}, \text{ with } f_k \equiv \frac{k}{N} \ \& \ k = 0, 1, \ldots, N-1$$

- note: can define A_k for all k, but $\{A_k : k = 0, 1, \ldots, N-1\}$ is DFT (sequence indexed by all integers k is periodic with a period of N; i.e., $A_{k+N} = A_k$)
- A_k is associated with frequency f_k, and $0 \leq f_k < 1$
- A_k for $0 \leq f_k \leq 1/2$ of main interest because $A_{N-k} = A_k^*$ (if N even, $k = 0, \ldots, N/2$ index the frequencies of interest)

Fourier Theory for Finite Sequences: II

- can recover $\{a_t\}$ from its DFT $\{A_k\}$ (Exercise [29a]):

$$\frac{1}{N}\sum_{k=0}^{N-1} A_k e^{i2\pi f_k t} = a_t, \quad t = 0, 1, \ldots, N-1;$$

  left-hand side called inverse DFT of $\{A_k\}$

- $\{a_t\}$ and $\{A_k\}$ are two representations for one 'thingy'
- relationship between $\{a_t\}$ and $\{A_k\}$ denoted by

$$\{a_t\} \longleftrightarrow \{A_k\} \text{ or, less formally, by } a_t \longleftrightarrow A_k$$

- can define a_t for $t < 0$ & $t \geq N$ via inverse DFT:
  $\{a_t : t = \ldots, -1, 0, 1, \ldots\}$ periodic with period N

Fourier Theory for Finite Sequences: III

- if $\{a_t\} \longleftrightarrow \{A_k\}$ & $\{b_t\} \longleftrightarrow \{B_k\}$, then

$$\sum_{t=0}^{N-1} a_t b_t = \frac{1}{N} \sum_{k=0}^{N-1} A_k B_k^*$$

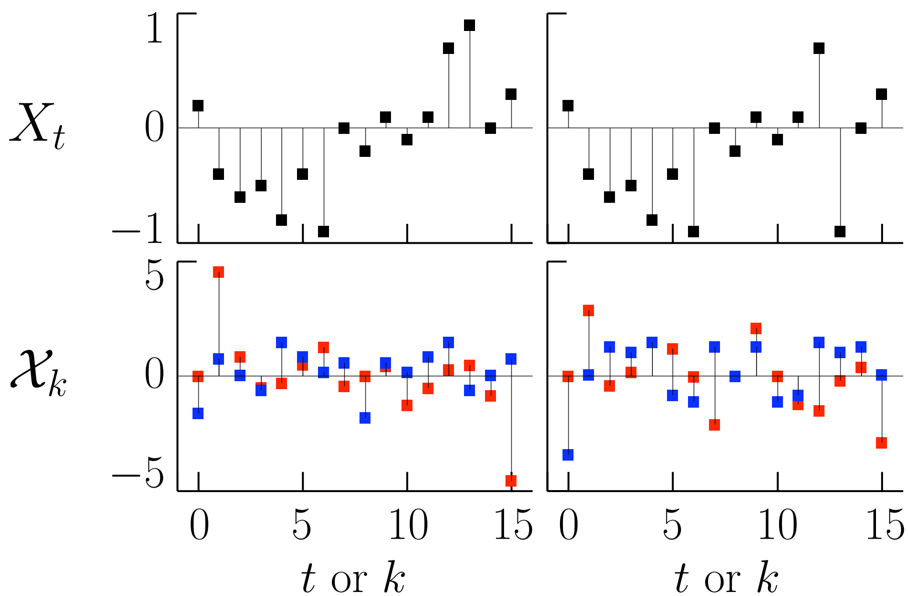
  'two sequence' Parseval's theorem (Exercise [29b])

- setting $b_t = a_t$ yields 'one sequence' Parseval:

$$\sum_{t=0}^{N-1} a_t^2 = \frac{1}{N} \sum_{k=0}^{N-1} |A_k|^2$$

Examples of Fourier Transforms of Finite Sequences

- two time series $\{X_t\}$ of length $N = 16$ and their DFTs $\{\mathcal{X}_k\}$



blue is real part
red is imaginary part

- series differ only at $t = 13$, but their DFTs differ at all k
- note that $\mathcal{X}_{16-k} = \mathcal{X}_k^*$ for $k = 1, 2, 3, 4, 5, 6$ and 7

Convolution/Filtering of Finite Sequences: I

- given $\{a_t\}$ & $\{b_t\}$ of length N with DFTs $\{A_k\}$ & $\{B_k\}$, define

$$a * b_t \equiv \sum_{u=0}^{N-1} a_u b_{t-u}, \quad t = 0, 1, \ldots, N-1$$

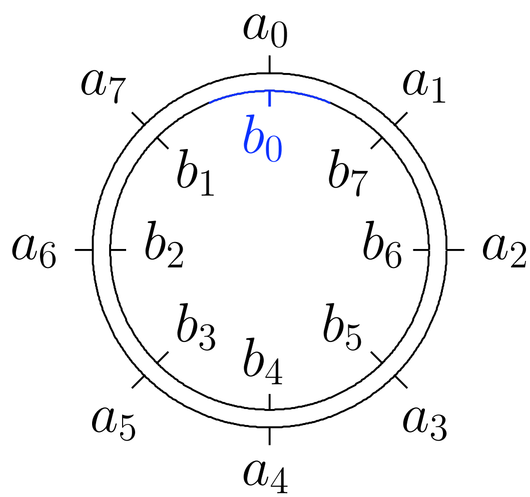
- assumes b_t defined for $t < 0$ by periodic extension; thus $b_{-1} = b_{N-1}$, $b_{-2} = b_{N-2}$, $b_{-3} = b_{N-3}$ etc
- equivalent definition, but with periodicity explicitly stated

$$a * b_t \equiv \sum_{u=0}^{N-1} a_u b_{t-u \bmod N}, \quad t = 0, 1, \ldots, N-1$$

- $k \bmod N \equiv k$ if $0 \leq k \leq N-1$; if not, $k \bmod N \equiv k + nN$, where n is unique integer such that $0 \leq k + nN \leq N-1$; thus $b_{0 \bmod N} = b_0$, $b_{-1 \bmod N} = b_{N-1}$, $b_{-2 \bmod N} = b_{N-2}$ etc.

Convolution/Filtering of Finite Sequences: II

- sequence $\{a * b_t\}$ called circular (cyclic) convolution



$a * b_0 = \sum_{u=0}^{7} a_u b_{-u \bmod 8}$

- DFT of $\{a * b_t\}$ again has a simple form (Exercise [30]):

$$\sum_{t=0}^{N-1} a * b_t e^{-i2\pi f_k t} = A_k B_k;$$

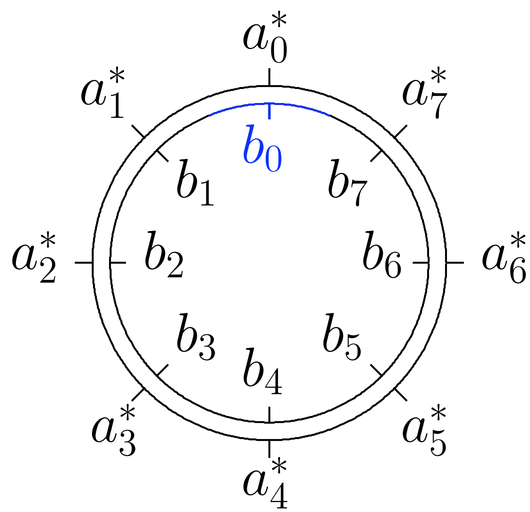
i.e., $\{a * b_t\} \longleftrightarrow \{A_k B_k\}$

Convolution/Filtering of Finite Sequences: III

- related concept is complex cross-correlation:

$$a^* \star b_t \equiv \sum_{u=0}^{N-1} a_u^* b_{u+t \bmod N} \qquad t = 0, 1, \ldots, N-1,$$

for which $\{a^* \star b_t\} \longleftrightarrow \{A_k^* B_k\}$



$a^* \star b_0 = \sum_{u=0}^{7} a_u^* b_u$

Convolution/Filtering of Finite Sequences: IV

- with $\{a_t\} \longleftrightarrow \{A_k\}$, can obtain $\{a^* \star b_t\}$ by filtering $\{b_t\}$ with filter $\{A_k^*\}$ (Exercise [31])
- flow diagram for circular filtering:

$$\{b_t\} \longrightarrow \boxed{\{a_t\}} \longrightarrow \{a * b_t\} \text{ or } \{b_t\} \longrightarrow \boxed{\{A_k\}} \longrightarrow \{a * b_t\}$$

(latter cannot be mistaken for infinite sequence case)

- sometimes convenient to simplify the above to

$$\{b_t\} \longrightarrow \boxed{a_t} \longrightarrow \{a * b_t\} \text{ or } \{b_t\} \longrightarrow \boxed{A_k} \longrightarrow \{a * b_t\}$$

or to just

$$b_t \longrightarrow \boxed{a_t} \longrightarrow a * b_t \text{ or } b_t \longrightarrow \boxed{A_k} \longrightarrow a * b_t$$

Periodized Filters: I

- circular filters of length N often constructed implicitly
- let $\{b_t : t = 0, \ldots, N-1\}$ be a finite sequence, and consider using $\{a_t : t = 0, 1, \ldots, M-1\}$ to form

$$a * b_t = \sum_{u=0}^{M-1} a_u b_{t-u \bmod N}, \quad t = 0, \ldots, N-1$$

- resembles circular filtering: input $\{b_t\}$, output $\{a * b_t\}$, but $\{a_t\}$ is a sequence of width M (need not be equal to N)
- if $M < N$, can write

$$a * b_t = \sum_{u=0}^{M-1} a_u b_{t-u \bmod N} = \sum_{u=0}^{N-1} a_u b_{t-u \bmod N}$$

by *defining* $a_t = 0$ for $t = M, \ldots, N-1$

Periodized Filters: II

- if $M > N$, define $a_t = 0$ for all $t \geq M$ so that

$$a * b_t = \sum_{u=0}^{M-1} a_u b_{t-u \bmod N} = \sum_{u=0}^{\infty} a_u b_{t-u \bmod N}$$

- split infinite sum into sum of sums over N terms:

$$a * b_t = \sum_{u=0}^{N-1} a_u b_{t-u \bmod N} + \sum_{u=N}^{2N-1} a_u b_{t-u \bmod N} + \cdots$$

- rewrite each sum so that u goes from 0 to $N-1$:

$$a * b_t = \sum_{u=0}^{N-1} a_u b_{t-u \bmod N} + \sum_{u=0}^{N-1} a_{u+N} b_{t-u-N \bmod N} + \cdots$$

Periodized Filters: III

- use fact that, for any integer n,

$$t - u - nN \bmod N = t - u \bmod N$$

to get

$$a * b_t = \sum_{u=0}^{N-1} a_u b_{t-u \bmod N} + \sum_{u=0}^{N-1} a_{u+N} b_{t-u \bmod N} + \cdots$$

- collect multipliers of $b_{t-u \bmod N}$ together & call their sum a_u°:

$$a * b_t = \sum_{u=0}^{N-1} \left(\sum_{n=0}^{\infty} a_{u+nN} \right) b_{t-u \bmod N} \equiv \sum_{u=0}^{N-1} a_u^{\circ} b_{t-u \bmod N}$$

Periodized Filters: IV

- $\{a_t^\circ\}$ is $\{a_t\}$ periodized to length N and is formed by
  - chopping zero-padded $\{a_t\}$ into finite sequences of length N:

$$\underbrace{a_0, a_1, \ldots, a_{N-1}}_{\text{block } n=0}, \underbrace{a_N, a_{N+1}, \ldots, a_{2N-1}}_{\text{block } n=1}, \ldots$$

  - adding finite sequences element by element:

$$\begin{array}{lcccc}
\text{block } n = 0: & a_0 & a_1 & \cdots & a_{N-1} \\
 & + & + & \cdots & + \\
\text{block } n = 1: & a_N & a_{N+1} & \cdots & a_{2N-1} \\
 & + & + & \cdots & + \\
 & \vdots & \vdots & \vdots & \vdots \\
 & \Downarrow & \Downarrow & \cdots & \Downarrow \\
\text{periodized filter:} & a_0^\circ & a_1^\circ & \cdots & a_{N-1}^\circ
\end{array}$$

Periodized Filters: V

- as example, let's periodize $\{a_0, a_1, a_2, a_3, a_4, a_5\}$ to length 4
- extend with zeros and chop into blocks of length 4:

$$\underbrace{a_0, a_1, a_2, a_3}_{\text{block } n=0}, \underbrace{a_4, a_5, 0, 0}_{\text{block } n=1}, \underbrace{0, 0, 0, 0}_{\text{block } n=2}, \ldots$$

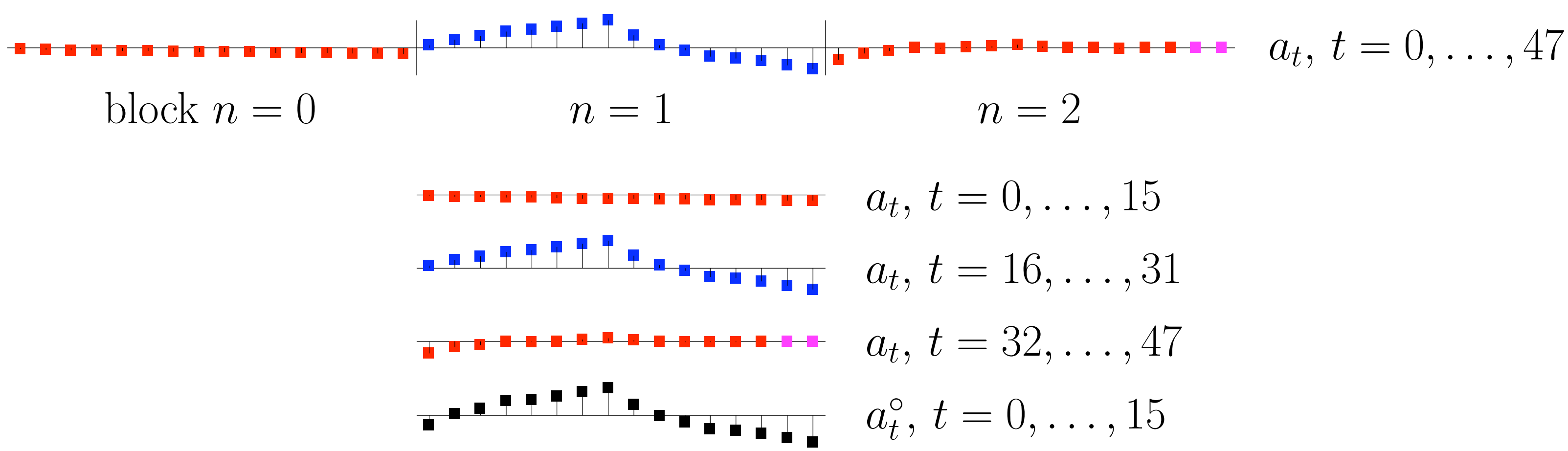
- add blocks element by element:

$$\begin{array}{lcccc}
\text{block } n = 0: & a_0 & a_1 & a_2 & a_3 \\
 & + & + & + & + \\
\text{block } n = 1: & a_4 & a_5 & 0 & 0 \\
 & \Downarrow & \Downarrow & \Downarrow & \Downarrow \\
\text{periodized filter:} & a_0 + a_4 & a_1 + a_5 & a_2 & a_3
\end{array}$$

- yields $a_0^\circ = a_0 + a_4$, $a_1^\circ = a_1 + a_5$, $a_2^\circ = a_2$ and $a_3^\circ = a_3$

Periodized Filters: VI

- as a second example, let's periodize a filter of width $M = 46$ to length $N = 16$, which, after padding with two zeros, goes as follows:



Periodized Filters: VII

- have set $a_t = 0$ for all $t \geq M$; now set $a_t = 0$ for all $t < 0$ also
- DFT of infinite sequence $\{a_t\}$ given by

$$A(f) = \sum_{t=-\infty}^{\infty} a_t e^{-i2\pi ft} = \sum_{t=0}^{M-1} a_t e^{-i2\pi ft}$$

- Exercise [33]: DFT of $\{a_t^\circ : t = 0, \ldots, N-1\}$ is given by $\{A(\frac{k}{N}) : k = 0, \ldots, N-1\}$
- periodization equivalent to sampling in frequency domain
- result holds for $M < N$, $M = N$ or $M > N$ (and for starting values of t other than 0)

Periodized Filters: VIII

- in terms of a flow diagram, can thus express

$$a * b_t = \sum_{u=0}^{M-1} a_u b_{t-u \bmod N}, \quad t = 0, \ldots, N-1,$$

as

$$\{b_t\} \longrightarrow \boxed{\{A(\tfrac{k}{N})\}} \longrightarrow \{a*b_t\} \text{ or } \{b_t\} \longrightarrow \boxed{A(\tfrac{k}{N})} \longrightarrow \{a*b_t\}$$

- variation on the above:
  - place N elements of $\{b_t\}$ into vector $\mathbf{B}$
  - place N elements of $\{a * b_t\}$ into vector $\mathbf{C}$
  - can then reexpress flow diagram as

$$\mathbf{B} \longrightarrow \boxed{A(\tfrac{k}{N})} \longrightarrow \mathbf{C}$$

  - above is most common form of flow diagram

Summary of Fourier/Filtering Theory: I

- $\{a_t : t = \ldots, -1, 0, 1, \ldots\} = \{a_t\}$ has DFT

$$A(f) \equiv \sum_{t=-\infty}^{\infty} a_t e^{-i2\pi f t}$$

- inverse DFT says that

$$a_t = \int_{-1/2}^{1/2} A(f) e^{i2\pi f t}\, df$$

- relationship between $\{a_t\}$ and $A(\cdot)$ denoted by

$$\{a_t\} \longleftrightarrow A(\cdot) \text{ or, less formally, by } a_t \longleftrightarrow A(f)$$

Summary of Fourier/Filtering Theory: II

- given $\{a_t\} \longleftrightarrow A(\cdot)$ and $\{b_t\} \longleftrightarrow B(\cdot)$, their convolution

$$a * b_t \equiv \sum_{u=-\infty}^{\infty} a_u b_{t-u}, \quad t = \ldots, -1, 0, 1, \ldots,$$

  has a DFT given by

$$\sum_{t=-\infty}^{\infty} a * b_t e^{-i2\pi f t} = A(f)B(f)$$

- $\{a * b_t\}$ is output from filter with impulse response sequence $\{a_t\}$ and transfer function $A(\cdot)$ related by $\{a_t\} \longleftrightarrow A(\cdot)$
- can express filtering operation in a flow diagram as either

$$\{b_t\} \longrightarrow \boxed{\{a_t\}} \longrightarrow \{a * b_t\} \text{ or } \{b_t\} \longrightarrow \boxed{A(\cdot)} \longrightarrow \{a * b_t\}$$

Summary of Fourier/Filtering Theory: III

- $\{a_t : t = 0, 1, \ldots, N-1\} = \{a_t\}$ has DFT

$$A_k \equiv \sum_{t=0}^{N-1} a_t e^{-i2\pi f_k t}, \text{ with } f_k \equiv \frac{k}{N} \;\&\; k = 0, 1, \ldots, N-1$$

- inverse DFT says that

$$a_t = \frac{1}{N} \sum_{k=0}^{N-1} A_k e^{i2\pi f_k t}, \quad t = 0, 1, \ldots, N-1$$

- relationship between $\{a_t\}$ and $\{A_k\}$ denoted by

$$\{a_t\} \longleftrightarrow \{A_k\} \text{ or, less formally, by } a_t \longleftrightarrow A_k$$

Summary of Fourier/Filtering Theory: IV

- given $\{a_t\}$ & $\{b_t\}$ of length N with DFTs $\{A_k\}$ & $\{B_k\}$, their circular convolution

$$a * b_t \equiv \sum_{u=0}^{N-1} a_u b_{t-u \bmod N}, \quad t = 0, 1, \ldots, N-1,$$

has a DFT given by

$$\sum_{t=0}^{N-1} a * b_t e^{-i2\pi f_k t} = A_k B_k$$

- $\{a * b_t\}$ is output from circular filtering operation expressed as

$$\{b_t\} \longrightarrow \boxed{a_t} \longrightarrow \{a * b_t\} \text{ or } \{b_t\} \longrightarrow \boxed{A_k} \longrightarrow \{a * b_t\}$$

Summary of Fourier/Filtering Theory: V

- given $\{a_t\}$ of width M & $\{b_t\}$, can express

$$a * b_t = \sum_{u=0}^{M-1} a_u b_{t-u \bmod N}, \quad t = 0, \ldots, N-1,$$

as (assuming $a_t \equiv 0$ for $t < 0$ and $t \geq M$)

$$a * b_t = \sum_{u=0}^{N-1} a_u^{\circ} b_{t-u \bmod N}, \text{ where } a_u^{\circ} \equiv \sum_{n=-\infty}^{\infty} a_{u+nN}$$

- DFT of $\{a_t^{\circ}\}$ given by $A(\frac{k}{N})$, $k = 0, \ldots, N-1$, where

$$A(f) \equiv \sum_{t=-\infty}^{\infty} a_t e^{-i2\pi ft} = \sum_{t=0}^{M-1} a_t e^{-i2\pi ft}$$

Summary of Fourier/Filtering Theory: VI

- can represent this type of filtering operation as either

$$\{b_t\} \longrightarrow \boxed{A(\tfrac{k}{N})} \longrightarrow \{a * b_t\} \text{ or } \mathbf{B} \longrightarrow \boxed{A(\tfrac{k}{N})} \longrightarrow \mathbf{C}$$

where $\mathbf{B}$ & $\mathbf{C}$ are vectors of length N containing $\{b_t\}$ & $\{a * b_t\}$

Summary of Fourier/Filtering Theory: VII

- can achieve effect of cascade with L filters

$$\{b_t\} \longrightarrow \boxed{A_1(\cdot)} \longrightarrow \boxed{A_2(\cdot)} \longrightarrow \cdots \longrightarrow \boxed{A_L(\cdot)} \longrightarrow \{a * b_t\}$$

by using a single equivalent filter

$$\{b_t\} \longrightarrow \boxed{A(\cdot)} \longrightarrow \{a * b_t\}, \ \text{where} \ A(f) = \prod_{l=1}^{L} A_l(f)$$

- similarly, effect of cascade with L circular filters

$$\mathbf{B} \longrightarrow \boxed{A_1(\tfrac{k}{N})} \longrightarrow \boxed{A_2(\tfrac{k}{N})} \longrightarrow \cdots \longrightarrow \boxed{A_L(\tfrac{k}{N})} \longrightarrow \mathbf{C}$$

can be achieved using a single equivalent circular filter

$$\mathbf{B} \longrightarrow \boxed{A(\tfrac{k}{N})} \longrightarrow \mathbf{C}, \ \text{where} \ A(\tfrac{k}{N}) = \prod_{l=1}^{L} A_l(\tfrac{k}{N})$$

Do We Need $\sum_t |a_t| < \infty$ for DFT to Exist?: I

- for real-valued infinite sequence $\{a_t : t = \ldots, -1, 0, 1, \ldots\}$, have stated that $\sum_t a_t^2 < \infty$ is sufficient for DFT

$$A(f) \equiv \sum_{t=-\infty}^{\infty} a_t e^{-i2\pi f t}$$

  to exist and to be well-defined

- note that $\sum_t a_t^2 < \infty$ does *not* imply $\sum_t |a_t| < \infty$
- might seem we need stronger condition $\sum_t |a_t| < \infty$ since

$$A(0) = \sum_{t=-\infty}^{\infty} a_t = \sum_{t=-\infty}^{\infty} |a_t|$$

  if $a_t \geq 0$ for all t, opening up possibility $A(0) = \infty$ if we only assume $\sum_t a_t^2 < \infty$

Do We Need $\sum_t |a_t| < \infty$ for DFT to Exist?: II

- in fact, $\sum_t a_t^2 < \infty$ is sufficient, as per following argument (see, e.g., Section 1.3 of L.H. Koopmans, *The Spectral Analysis of Time Series*, Academic Press, 1974)
- let $L^2(-\frac{1}{2}, -\frac{1}{2})$ denote collection of all complex-valued functions $A(\cdot)$ such that

$$\int_{-1/2}^{1/2} |A(f)|^2 \, df < \infty$$

(need to interpret above integral as Lebesgue integral)

- can regard $L^2(-\frac{1}{2}, -\frac{1}{2})$ as Hilbert space with inner product

$$\langle A(\cdot), B(\cdot) \rangle = \int_{-1/2}^{1/2} A(f) B^*(f) \, df$$

Do We Need $\sum_t |a_t| < \infty$ for DFT to Exist?: III

- can argue that $E_t(f) \equiv e^{-i2\pi ft}$ for $t = 0, \pm 1, \ldots$ form a complete orthonormal sequence in $L^2(-\frac{1}{2}, -\frac{1}{2})$
- hence $A(\cdot) \in L^2(-\frac{1}{2}, -\frac{1}{2})$ if and only if there exists a sequence of complex numbers $\{a_t, t = 0, \pm 1, \ldots\}$ with $\sum_t |a_t|^2 < \infty$ such that

$$A(f) = \sum_{t=-\infty}^{\infty} a_t E_t(f) = \sum_{t=-\infty}^{\infty} a_t e^{-i2\pi ft}$$

where

$$a_t = \langle A(\cdot), E_t(\cdot)\rangle = \int_{-1/2}^{1/2} A(f) e^{i2\pi ft}\, df$$

Do We Need $\sum_t |a_t| < \infty$ for DFT to Exist?: IV

- let ℓ^2 be set all complex-valued sequences $\{a_t\}$ such that

$$\sum_{t=-\infty}^{\infty} |a_t|^2 < \infty$$

- can regard ℓ^2 as Hilbert space with inner product

$$\langle \{a_t\}, \{b_t\} \rangle = \sum_{t=-\infty}^{\infty} a_t b_t^*$$

- thus

$$A(f) = \sum_{t=-\infty}^{\infty} a_t e^{-i2\pi f t} \text{ and } a_t = \int_{-1/2}^{1/2} A(f) e^{i2\pi f t}\, df$$

give a one-to-one mapping (the DFT) from $L^2(-\frac{1}{2}, -\frac{1}{2})$ onto ℓ^2 that can be shown to preserve inner products

Do We Need $\sum_t |a_t| < \infty$ for DFT to Exist?: V

- second heuristic proof (not based on Hilbert space theory)
- for integer $m \geq 0$, let
$$A_m(f) \equiv \sum_{t=-m}^{m} a_t e^{-i2\pi ft},$$
i.e., DFT of finite sequence $\{a_t : t = -m, \ldots, m\}$
- one-sequence Parseval's theorem says
$$\sum_{t=-m}^{m} a_t^2 = \int_{-1/2}^{1/2} |A_m(f)|^2 \, df$$
(solution to Exercise [23a] gives rigorous proof for finite sums)

Do We Need $\sum_t |a_t| < \infty$ for DFT to Exist?: VI

- hence

$$\sum_{t=-\infty}^{\infty} a_t^2 = \lim_{m\to\infty} \sum_{t=-m}^{m} a_t^2 = \lim_{m\to\infty} \int_{-1/2}^{1/2} |A_m(f)|^2 \, df$$

$$= \int_{-1/2}^{1/2} \lim_{m\to\infty} |A_m(f)|^2 \, df$$

$$= \int_{-1/2}^{1/2} |A(f)|^2 \, df$$

(note: need to justify interchange of limit and integration using argument such as provided by Vitali convergence theorem)

Do We Need $\sum_t |a_t| < \infty$ for DFT to Exist?: VII

- hence $A(\cdot)$ is square-integrable over interval $[-\frac{1}{2}, \frac{1}{2}]$
- if $B(\cdot)$ is also square-integrable over $[-\frac{1}{2}, \frac{1}{2}]$, Cauchy–Schwarz inequality (CSI) says

$$\left| \int_{-1/2}^{1/2} A(f) B^*(f)\, df \right|^2 \leq \int_{-1/2}^{1/2} |A(f)|^2\, df \int_{-1/2}^{1/2} |B(f)|^2\, df$$

- letting $B(f) = e^{-i2\pi ft}$ in above says that

$$\left| \int_{-1/2}^{1/2} A(f) e^{i2\pi ft}\, df \right|^2 \leq \int_{-1/2}^{1/2} |A(f)|^2\, df < \infty$$

Do We Need $\sum_t |a_t| < \infty$ for DFT to Exist?: VIII

- hence
$$\int_{-1/2}^{1/2} A(f) e^{i2\pi f t}\, df \equiv \tilde{a}_t$$
is finite for all t
- final step is to argue that we must have $\tilde{a}_t = a_t$
- for DFT $A_m(\cdot)$ of finite sequence $\{a_t : t = -m, \ldots, m\}$, have
$$\int_{-1/2}^{1/2} A_m(f) e^{i2\pi f t}\, df = a_t$$
for all $m \geq |t|$ (solution to Exercise [22c] gives rigorous proof for finite sums)

Do We Need $\sum_t |a_t| < \infty$ for DFT to Exist?: IX

- thus, for $m \geq |t|$,

$$
\begin{aligned}
|\tilde{a}_t - a_t|^2 &= \left| \int_{-1/2}^{1/2} \left(A(f) - A_m(f) \right) e^{i2\pi f t}\, df \right|^2 \\
&\leq \left| \int_{-1/2}^{1/2} |A(f) - A_m(f)|\, df \right|^2 \\
&\leq \int_{-1/2}^{1/2} |A(f) - A_m(f)|^2\, df \quad \text{(using CSI)} \\
&= \sum_{u=-\infty}^{-m} a_u^2 + \sum_{u=m}^{\infty} a_u^2 \to 0
\end{aligned}
$$

as $m \to \infty$, which completes the proof

Do We Need $\sum_t |a_t| < \infty$ for DFT to Exist?: X

- thus stronger condition $\sum_t |a_t| < \infty$ is sufficient but not necessary for DFT to exist
- example of real-valued sequence for which $\sum_t |a_t| = \infty$ but $\sum_t a_t^2 < \infty$ is

$$a_t = \frac{\Gamma(\frac{1}{2})\Gamma(|t| + \frac{1}{4})}{\sqrt{2}\pi\Gamma(|t| + \frac{3}{4})},$$

  for which

$$A(f) = \frac{1}{\sqrt{2|\sin(\pi f)|}}$$

  - note that $A(0) = \infty$ since $\sin(0) = 0$
  - above $\{a_t\}$ is autocovariance sequence for a fractionally differenced (FD) process with parameter $\delta = \frac{1}{4}$ (we'll be discussing FD processes later on)