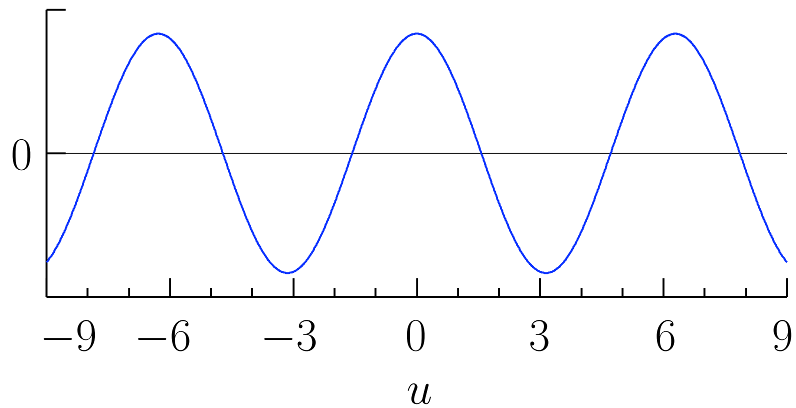
Introduction to Wavelets: Overview

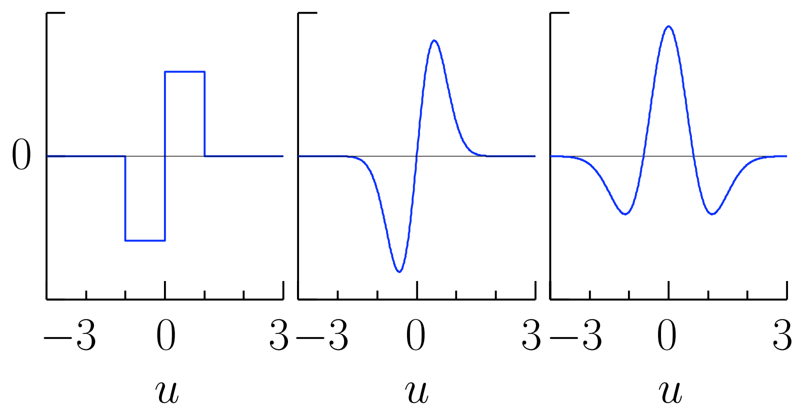
- wavelets are analysis tools for time series and images
- as a subject, wavelets are
  - relatively new (1983 to present)
  - a synthesis of old/new ideas
  - keyword in $50,000+$ articles and books since 1989 (an inundation of material!!!)
- broadly speaking, there have been two waves of wavelets
  - continuous wavelet transform (1983 and on)
  - discrete wavelet transform (1988 and on)
- will introduce subject via CWT & then concentrate on DWT

What is a Wavelet?

- sines & cosines are 'big waves'

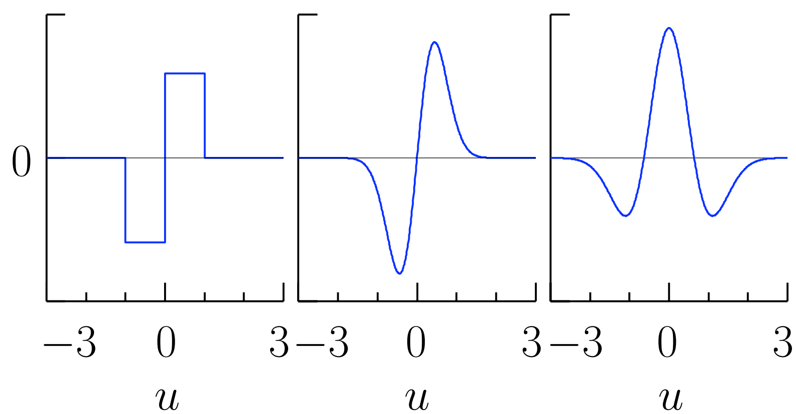


- wavelets are 'small waves' (left-hand is Haar wavelet $\psi^{(\mathrm{H})}(\cdot)$)



Technical Definition of a Wavelet: I

- real-valued function $\psi(\cdot)$ defined over real axis is a wavelet if
  1. integral of $\psi^2(\cdot)$ is unity: $\int_{-\infty}^{\infty} \psi^2(u)\,du = 1$
     (called 'unit energy' property, with apologies to physicists)
  2. integral of $\psi(\cdot)$ is zero: $\int_{-\infty}^{\infty} \psi(u)\,du = 0$
     (technically, need an 'admissibility condition,' but this is almost equivalent to integration to zero)

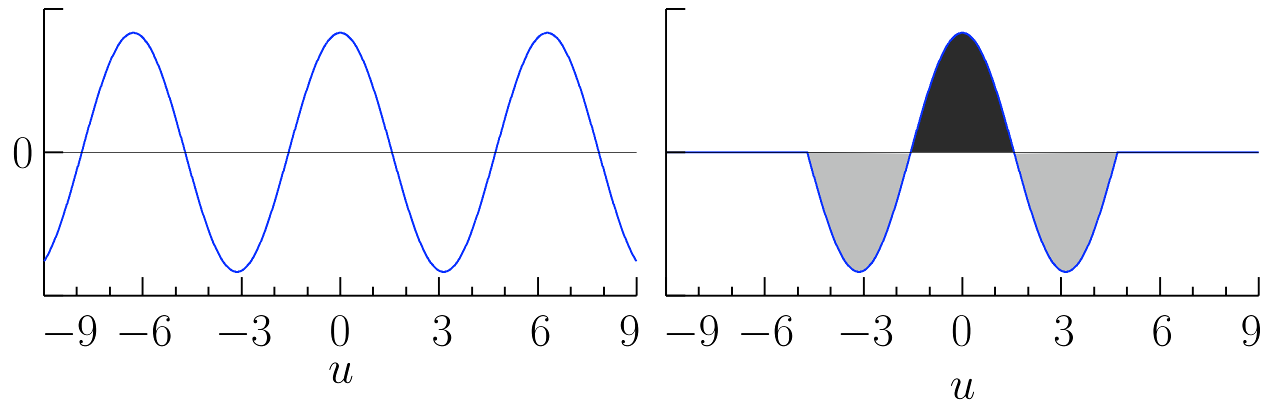


Technical Definition of a Wavelet: II

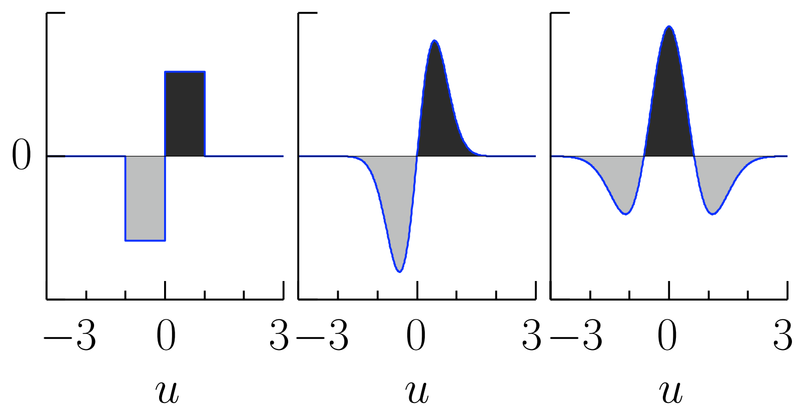
- $\int_{-\infty}^{\infty} \psi^2(u)\,du = 1$ & $\int_{-\infty}^{\infty} \psi(u)\,du = 0$ give a wavelet because:
  - by property 1, for every small $\epsilon > 0$, have
    $$\int_{-\infty}^{-T} \psi^2(u)\,du + \int_{T}^{\infty} \psi^2(u)\,du < \epsilon$$
    for some finite T
  - 'business' part of $\psi(\cdot)$ is over interval $[-T, T]$
  - width $2T$ of $[-T, T]$ might be huge, but will be insignificant compared to $(-\infty, \infty)$
  - by property 2, $\psi(\cdot)$ is balanced above/below horizontal axis
- matches intuitive notion of a 'small' wave

Two Non-Wavelets and Three Wavelets

- two failures: $f(u) = \cos(u)$ & same limited to $[-3\pi/2, 3\pi/2]$:



- Haar wavelet $\psi^{(\mathrm{H})}(\cdot)$ and two of its friends:



What is Wavelet Analysis?

- wavelets tell us about variations in local averages
- to quantify this description, let $x(\cdot)$ be a 'signal'
  - real-valued function of t defined over real axis
  - will refer to t as time (but it need not be such)
- consider 'average value' of $x(\cdot)$ over $[a, b]$:

$$\frac{1}{b-a}\int_a^b x(t)\,dt$$

Approximating Average Value of a Signal

- can approximate integral using Riemann sum
  - break $[a, b]$ into N subintervals of equal width $(b - a)/N$
  - sample $x(\cdot)$ at midpoint of each subinterval:

$$x_j = x\left(a + [j + \tfrac{1}{2}]\tfrac{b-a}{N}\right), \quad j = 0, 1, \ldots, N-1$$

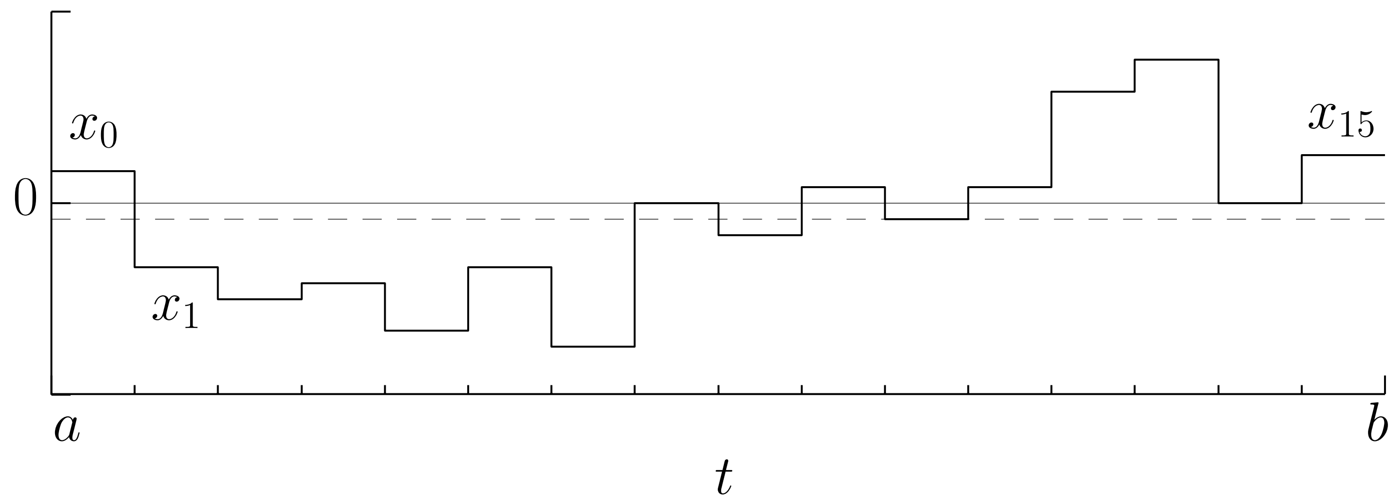
  - Riemann sum = sum of x_j's $\times$ width $(b - a)/N$
  - yields approximation to average value of $x(\cdot)$ over $[a, b]$:

$$\frac{1}{b-a}\int_a^b x(t)\,dt \approx \frac{1}{b-a}\left(\frac{b-a}{N}\sum_{j=0}^{N-1} x_j\right) = \frac{1}{N}\sum_{j=0}^{N-1} x_j$$

- average value of $x(\cdot) \approx$ sample mean of sampled values

Example of Average Value of a Signal

- let $x(\cdot)$ be step function taking on values $x_0, x_1, \ldots, x_{15}$ over 16 equal subintervals of $[a, b]$:



- here we have

$$\frac{1}{b-a}\int_a^b x(t)\,dt = \frac{1}{16}\sum_{j=0}^{15} x_j = \text{ height of dashed line}$$

Average Values at Different Scales and Times

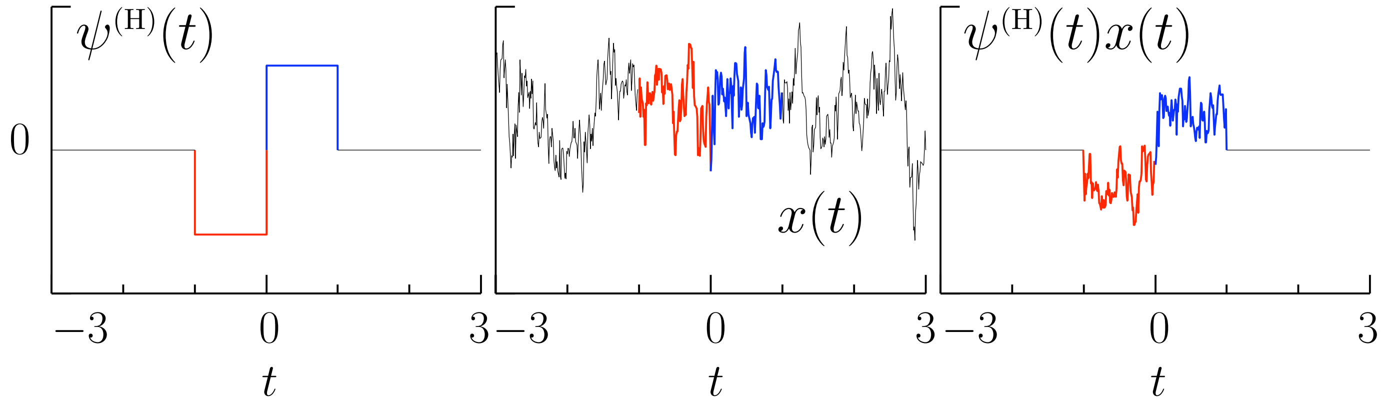
- define the following function of λ and t

$$A(\lambda, t) \equiv \frac{1}{\lambda} \int_{t-\frac{\lambda}{2}}^{t+\frac{\lambda}{2}} x(u)\, du$$

  – λ is width of interval – refered to as 'scale'
  – t is midpoint of interval
- $A(\lambda, t)$ is average value of $x(\cdot)$ over scale λ centered at t
- average values of signals have wide-spread interest
  – one second average temperatures over forest
  – ten minute rainfall rate during severe storm
  – yearly average temperatures over central England

Defining a Wavelet Coefficient W

- multiply Haar wavelet & time series $x(\cdot)$ together:



- integrate resulting function to get 'wavelet coefficient' $W(1,0)$:

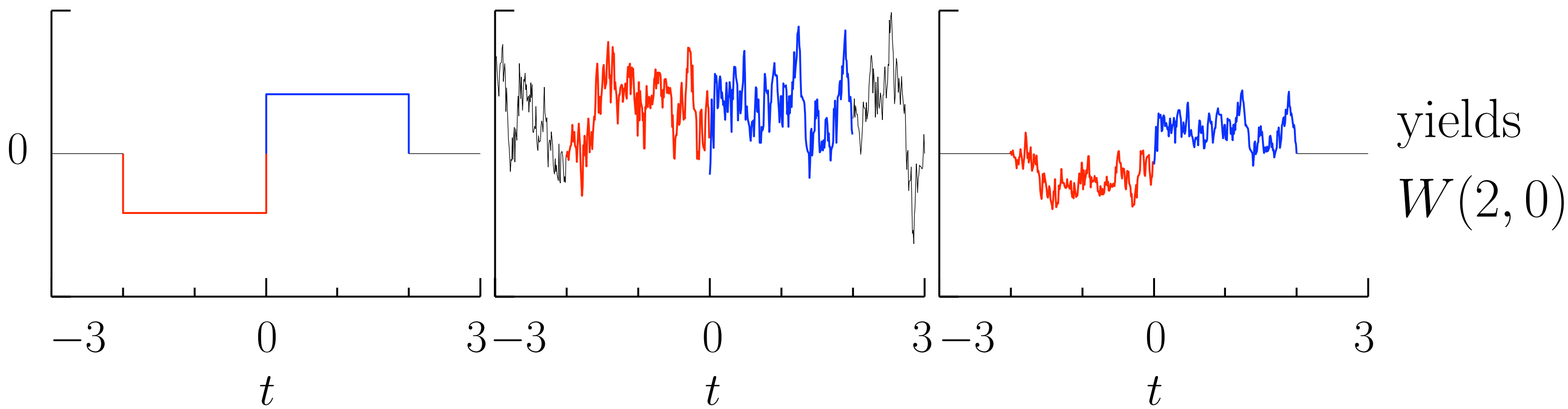
$$\int_{-\infty}^{\infty} \psi^{(\mathrm{H})}(t)x(t)\,dt = W(1,0)$$

- to see what $W(1,0)$ is telling us about $x(\cdot)$, note that

$$W(1,0) \propto \frac{1}{1}\int_0^1 x(t)\,dt - \frac{1}{1}\int_{-1}^0 x(t)\,dt = A(1,\tfrac{1}{2}) - A(1,-\tfrac{1}{2})$$

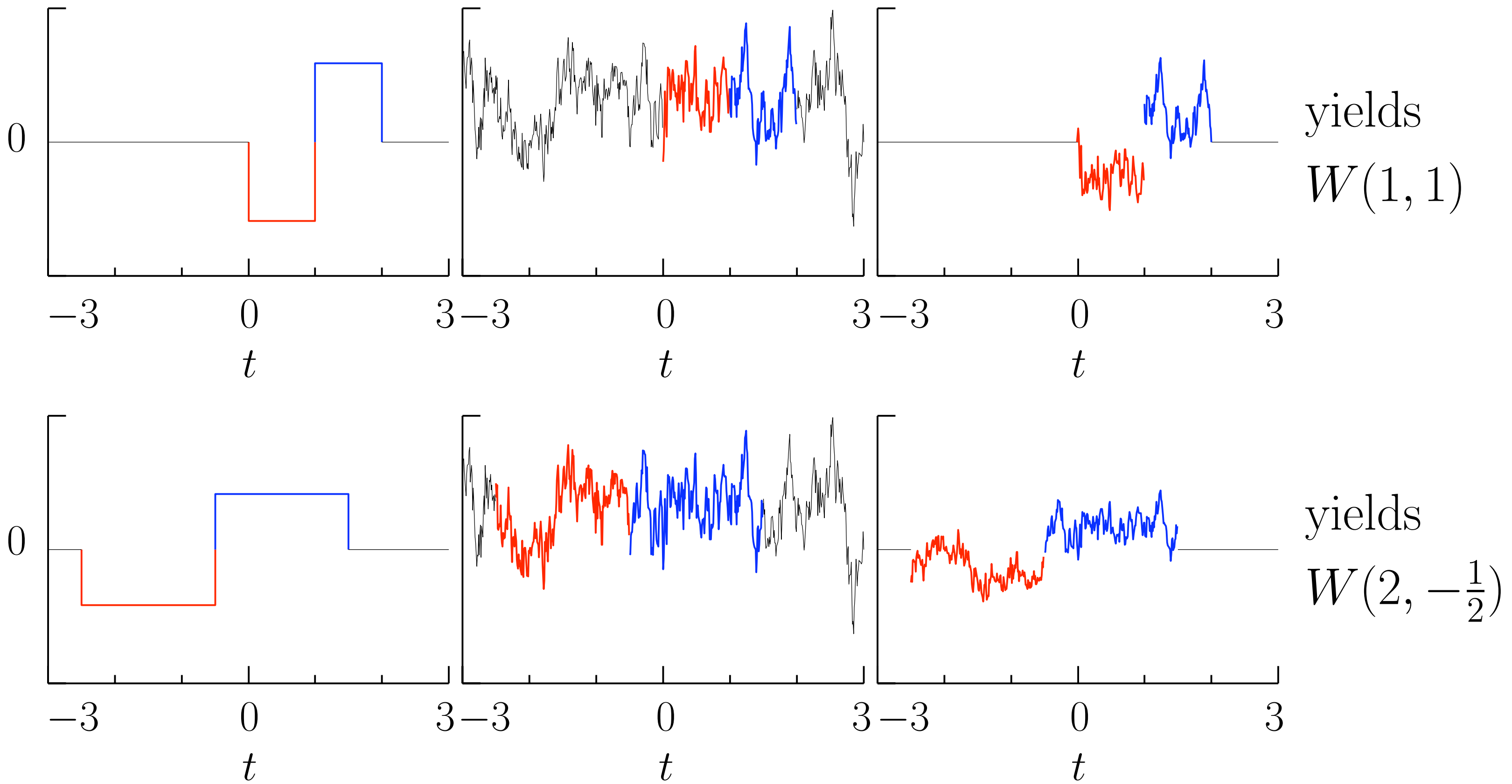
Defining Wavelet Coefficients for Other Scales

- $W(1,0)$ proportional to difference between averages of $x(\cdot)$ over $[-1,0]$ & $[0,1]$, i.e., two unit scale averages before/after $t=0$
  - '1' in $W(1,0)$ denotes scale 1 (width of each interval)
  - '0' in $W(1,0)$ denotes time 0 (center of combined intervals)
- stretch or shrink wavelet to define $W(\tau,0)$ for other scales τ:



Defining Wavelet Coefficients for Other Locations

- relocate to define $W(\tau, t)$ for other times t:



Haar Continuous Wavelet Transform (CWT)

- for all $\tau > 0$ and all $-\infty < t < \infty$, can write

$$W(\tau, t) = \frac{1}{\sqrt{\tau}} \int_{-\infty}^{\infty} x(u) \psi^{(\mathrm{H})} \left(\frac{u - t}{\tau} \right) du$$

  - $\frac{u-t}{\tau}$ does the stretching/shrinking and relocating
  - $\frac{1}{\sqrt{\tau}}$ needed so $\psi^{(\mathrm{H})}_{\tau,t}(u) \equiv \frac{1}{\sqrt{\tau}} \psi^{(\mathrm{H})} \left(\frac{u-t}{\tau} \right)$ has unit energy
  - since it also integrates to zero, $\psi^{(\mathrm{H})}_{\tau,t}(\cdot)$ is a wavelet
- $W(\tau, t)$ over all $\tau > 0$ and all t is Haar CWT for $x(\cdot)$
- analyzes/breaks up/decomposes $x(\cdot)$ into components
  - associated with a scale and a time
  - physically related to a difference of averages

Other Continuous Wavelet Transforms: I

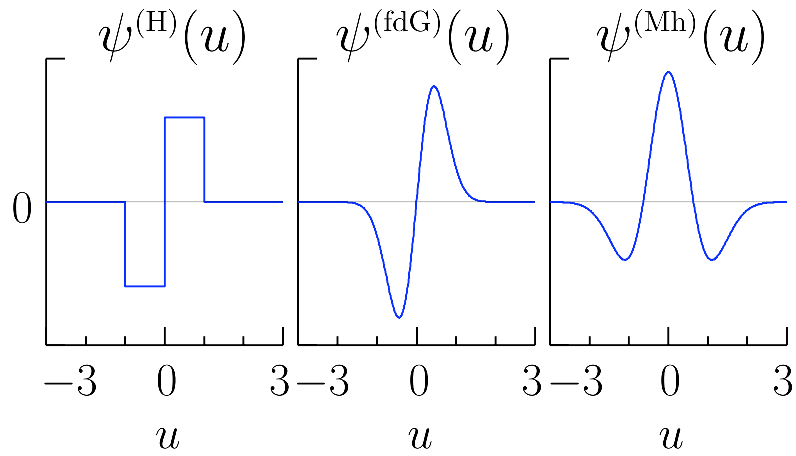
- can do the same for wavelets other than the Haar
- start with basic wavelet $\psi(\cdot)$
- use $\psi_{\tau,t}(u) = \frac{1}{\sqrt{\tau}}\psi\left(\frac{u-t}{\tau}\right)$ to stretch/shrink & relocate
- define CWT via

$$W(\tau, t) = \int_{-\infty}^{\infty} x(u)\psi_{\tau,t}(u)\,du = \frac{1}{\sqrt{\tau}}\int_{-\infty}^{\infty} x(u)\psi\left(\frac{u-t}{\tau}\right)du$$

- analyzes/breaks up/decomposes $x(\cdot)$ into components
  - associated with a scale and a time
  - physically related to a difference of *weighted* averages

Other Continuous Wavelet Transforms: II

- consider two friends of Haar wavelet



- $\psi^{(\mathrm{fdG})}(\cdot)$ proportional to 1st derivative of Gaussian PDF
- 'Mexican hat' wavelet $\psi^{(\mathrm{Mh})}(\cdot)$ proportional to 2nd derivative
- $\psi^{(\mathrm{fdG})}(\cdot)$ looks at difference of adjacent weighted averages
- $\psi^{(\mathrm{Mh})}(\cdot)$ looks at difference between weighted average and sum of weighted averages occurring before & after

First Scary-Looking Equation

- CWT equivalent to $x(\cdot)$ because we can write
$$x(t) = \int_0^\infty \left[\frac{1}{C\tau^2} \int_{-\infty}^{\infty} W(\tau, u) \frac{1}{\sqrt{\tau}} \psi\left(\frac{t-u}{\tau}\right) du \right] d\tau,$$
where C is a constant depending on specific wavelet $\psi(\cdot)$
- can synthesize (put back together) $x(\cdot)$ given its CWT; i.e., nothing is lost in reexpressing signal $x(\cdot)$ via its CWT
- regard stuff in brackets as defining 'scale τ' signal at time t
- says we can reexpress $x(\cdot)$ as integral (sum) of new signals, each associated with a particular scale
- similar additive decompositions will be one central theme

Second Scary-Looking Equation

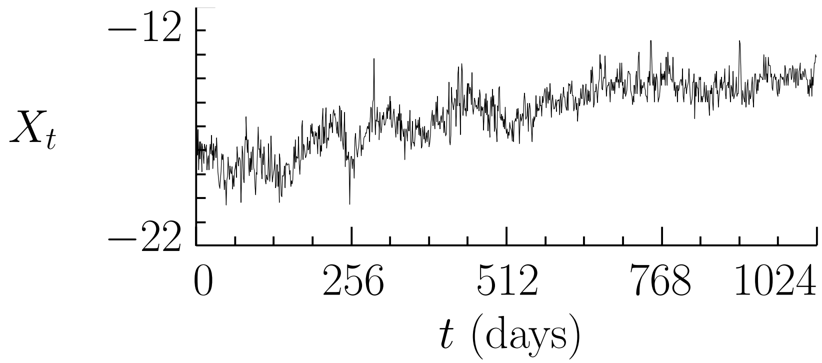
- energy in $x(\cdot)$ is reexpressed in CWT because

$$\text{energy} = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_0^{\infty} \left[\frac{1}{C\tau^2} \int_{-\infty}^{\infty} W^2(\tau, t)\,dt \right] d\tau$$

- can regard $x^2(t)$ versus t as breaking up the energy across time (i.e., an 'energy density' function)
- regard stuff in brackets as breaking up the energy across scales
- says we can reexpress energy as integral (sum) of components, each associated with a particular scale
- function defined by $W^2(\tau, t)/C\tau^2$ is an energy density across both time and scale
- similar energy decompositions will be a second central theme

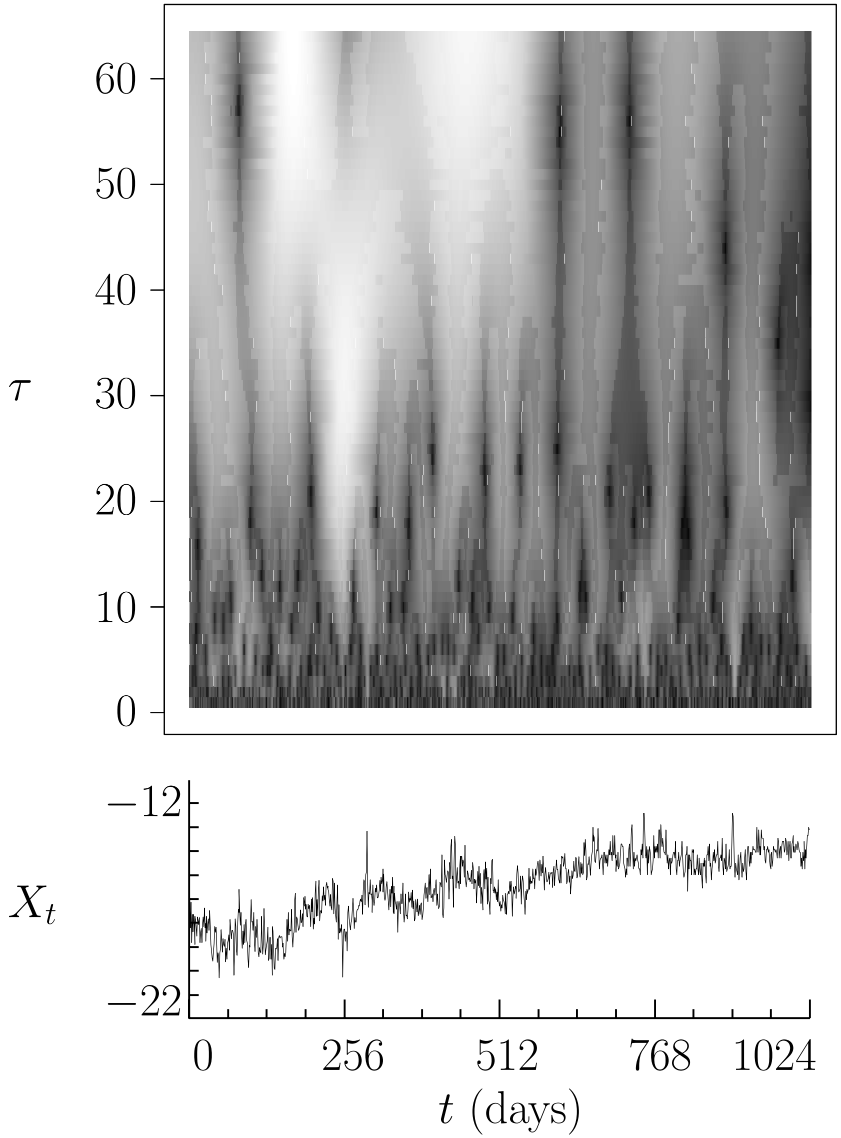
Example: Atomic Clock Data

- example: average daily frequency variations in clock 571

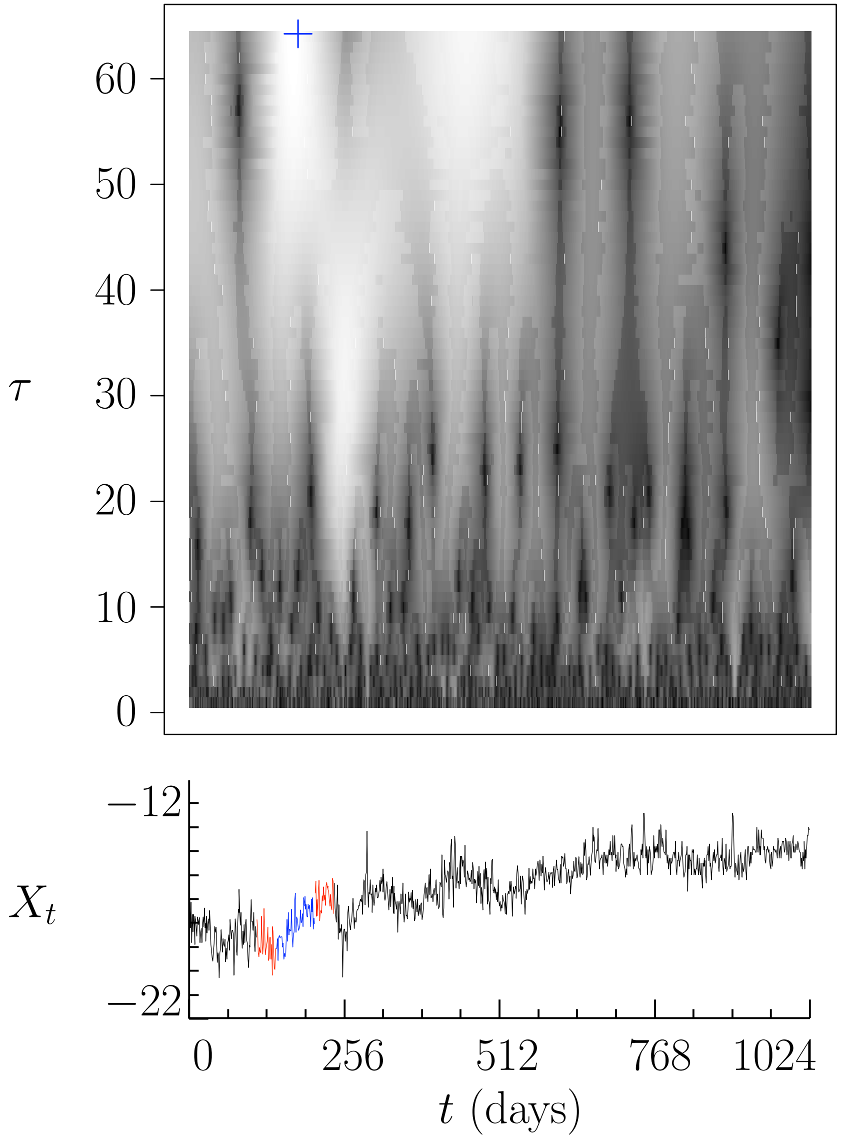


- t is measured in days (one measurement per day)
- plot shows X_t versus integer t
- $X_t = 0$ would mean that clock 571 could keep time perfectly
- $X_t < 0$ implies that clock is losing time systematically
- can easily adjust clock if X_t were constant
- inherent quality of clock related to changes in averages of X_t

Mexican Hat CWT of Clock Data: I



Mexican Hat CWT of Clock Data: II



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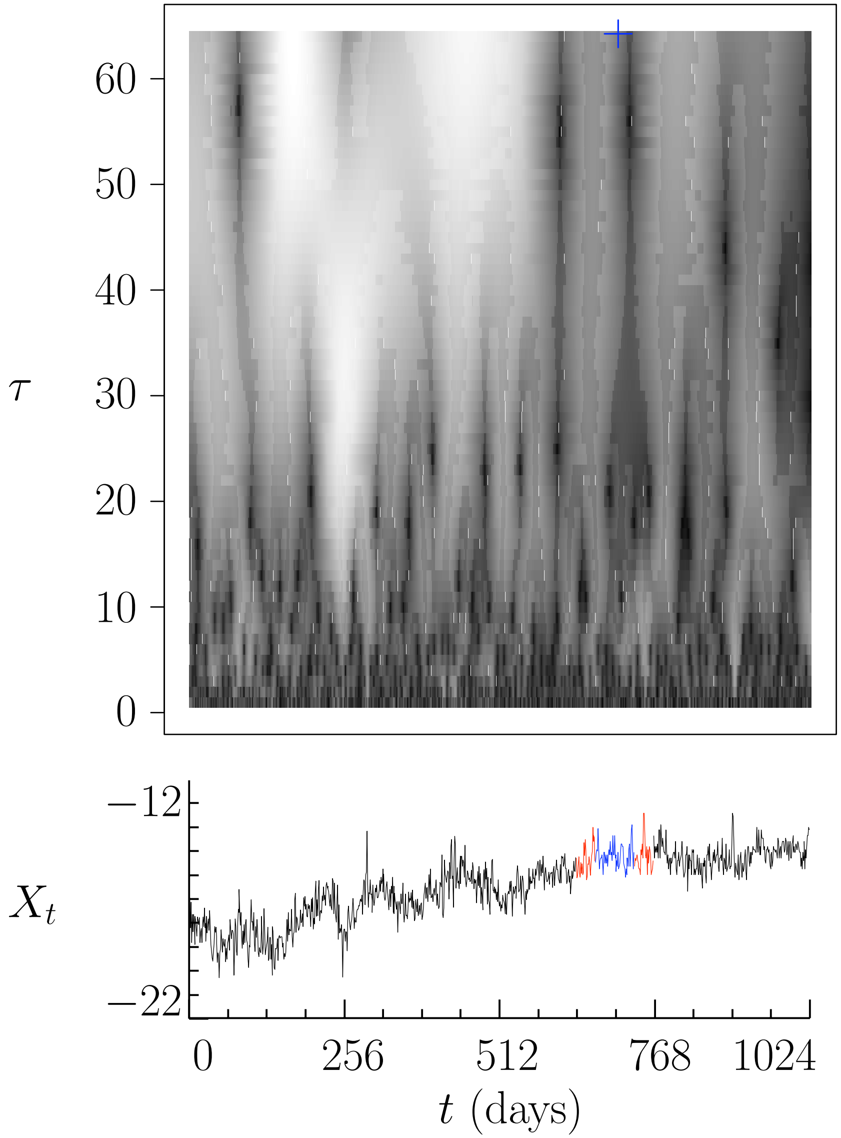
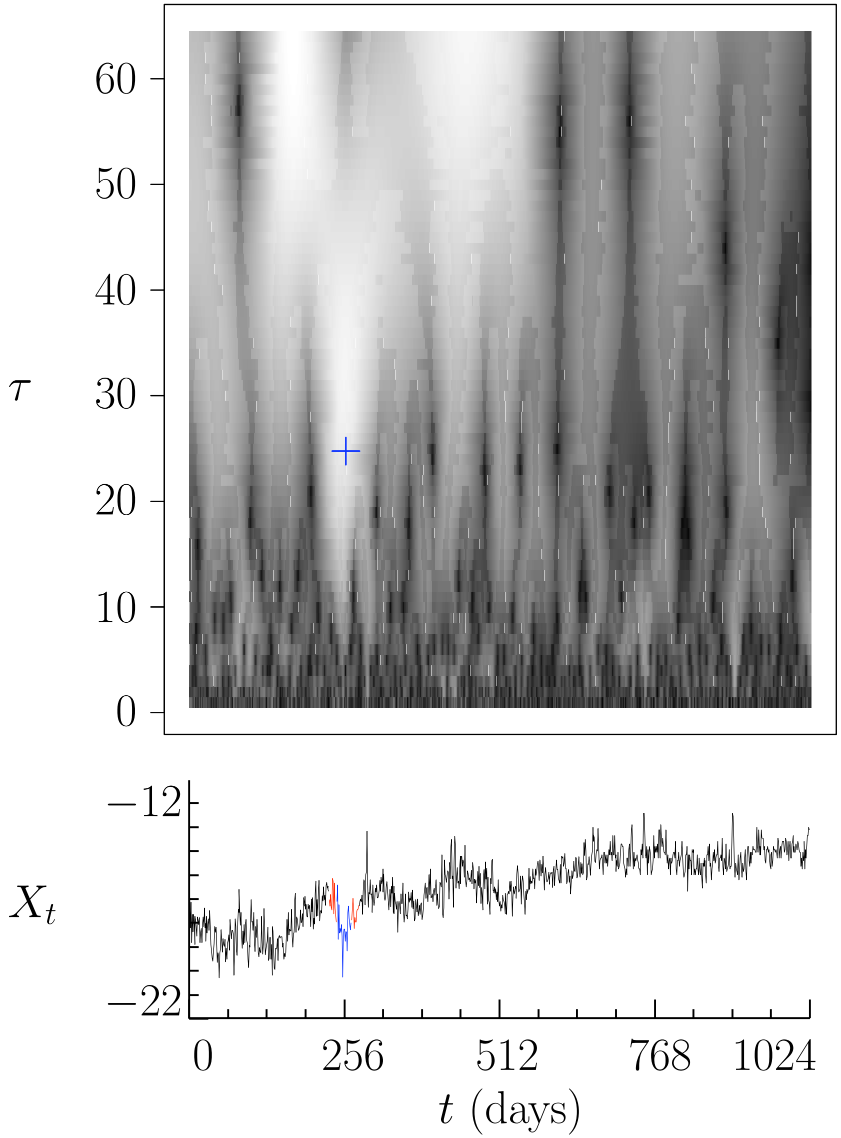
Mexican Hat CWT of Clock Data: III

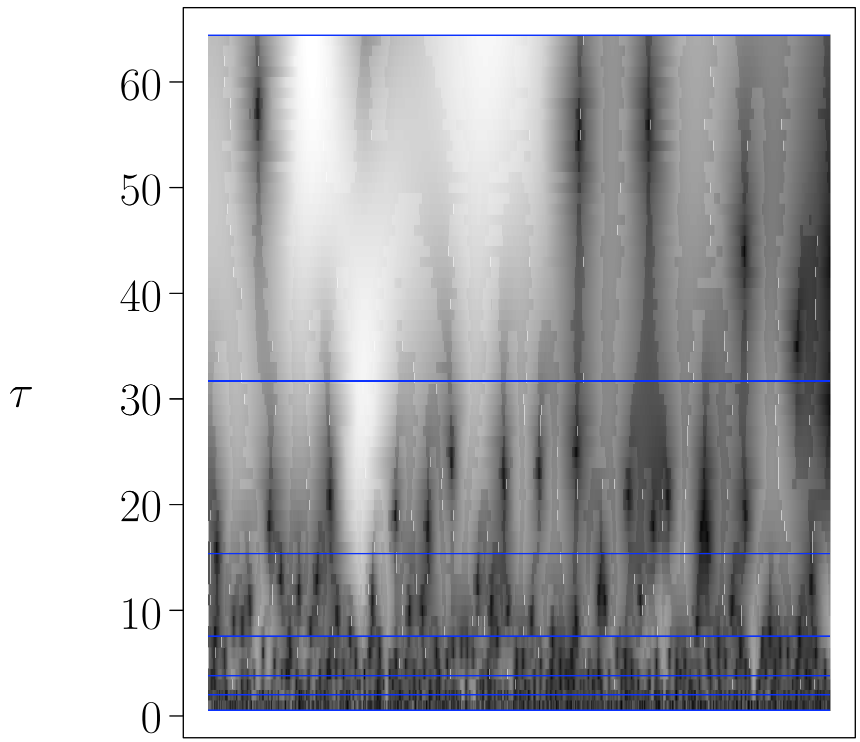


Mexican Hat CWT of Clock Data: IV



Beyond the CWT: the DWT

- can often get by with subsamples of $W(\tau, t)$
- leads to notion of discrete wavelet transform (DWT) (can regard as discretized 'slices' through CWT)



Rationale for the DWT

- DWT has appeal in its own right
  - most time series are sampled as discrete values (can be tricky to implement CWT)
  - can formulate as orthonormal transform (makes meaningful statistical analysis possible)
  - tends to decorrelate certain time series
  - standardization to dyadic scales often adequate
  - generalizes to notion of wavelet packets
  - can be faster than the fast Fourier transform
- will concentrate primarily on DWT for remainder of course

Addendum on First Scary-Looking Equation: I

- can synthesize signal $x(\cdot)$ from its CWT $W(\cdot, \cdot)$:

$$x(t) = \int_0^\infty \left[\frac{1}{C\tau^2} \int_{-\infty}^\infty W(\tau, u) \frac{1}{\sqrt{\tau}} \psi\left(\frac{t-u}{\tau}\right) du \right] d\tau, \quad (*)$$

  where C is a constant depending on specific wavelet $\psi(\cdot)$

- Q: what is the constant C all about?
- as mentioned on overhead I–3, for a function $\psi(\cdot)$ to be a wavelet, it must satisfy a so-called 'admissibility condition'
- to state admissibility condition, let $\Psi(\cdot)$ denote Fourier transform of $\psi(\cdot)$ (assumed to be a square-integrable function):

$$\Psi(f) = \int_{-\infty}^\infty \psi(u) e^{-i2\pi f u} \, du$$

Addendum on First Scary-Looking Equation: II

- admissibility condition says that
$$C \equiv \int_0^\infty \frac{|\Psi(f)|^2}{f}\,df \text{ must be such that } 0 < C < \infty$$
(note: above implies that $\psi(\cdot)$ must integrate to zero)
- C above is same C appearing in $(*)$
- as to *why* C appears, need to work through proof of $(*)$, which is not trivial
  - see Mallat, 1998, §4.3 for a clear proof
  - proof in the wavelet literature due to Grossman and Morlet, 1984, who discuss why admissibility condition is needed
  - Grossman and Morlet's result actually appeared earlier in 1964 paper by Calderón