Exact Simulation of Complex-Valued Gaussian Stationary Processes Via Circulant Embedding

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Abstract

Circulant embedding is a technique that has been used to generate realizations from certain real-valued Gaussian stationary processes. This technique has two potential advantages over competing methods for simulating time series. First, the statistical properties of the generating procedure are exactly the same as those of the target stationary process. Second, the technique is based upon the discrete Fourier transform and hence is computationally attractive when this transform is computed via a fast Fourier transform (FFT) algorithm. In this paper we show how, when used with a standard 'powers of two' FFT algorithm, circulant embedding can be readily adapted to handle complex-valued Gaussian stationary processes.

Key words: Complex demodulation, Discrete Fourier transform, Fast Fourier transform algorithm, Stochastic processes, Time Series

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1 Introduction

Complex-valued Gaussian stationary processes are used extensively as models for time series in the physical and engineering sciences. Specific applications include time series of (i) reverberation measurements recorded in an underwater environment [10,16], (ii) passive radar returns from the ionosphere [17] and (iii) ultrasound measurements of blood flow [8]. One reason that such models are of interest is that they can be used to generate realistic simulated time series for studying signal processing algorithms. This paper considers the problem of how to generate *exact* realizations from a fully specified complex-valued Gaussian stationary process, where the qualifier 'exact' means that the statistical properties of the process generating the simulated series and of the specified process are theoretically identical.

Many different schemes for simulating stationary processes have been discussed in the literature (see [12,15,18] and references therein). Some of these schemes are computationally efficient, but yield realizations whose statistical properties are only approximately the same as the desired specified process. There are two commonly used exact methods for simulating real-valued Gaussian stationary processes. The first method is based upon the Cholesky decomposition, but this scheme requires $O(N^2)$ numerical operations to generate each time series of length N and hence is computationally inefficient when compared to certain approximate schemes [12]. The second method is known as circulant embedding [5,7,12,19]. This method is less widely known than the Cholesky decomposition, but, when applicable, has the advantage of being based upon the discrete Fourier transform (DFT), which can be efficiently computed via a fast Fourier transform (FFT) algorithm and hence needs only $O(N \log_2(N))$ numerical operations for each generated series. Circulant embedding has been studied extensively for real-valued Gaussian stationary processes. The purpose of this paper is to show that, with a rather simple modification, this method can be adapted to handle complex-valued processes in conjunction with a standard 'powers of two' FFT algorithm.

The remainder of this paper is organized as follows. In Section 2 we present the basic ideas behind circulant embedding and summarize how it has been used to generate exact realizations of real-valued Gaussian stationary processes. We discuss how this method can adapted to work with complex-valued processes in Section 3. We present a small Monte Carlo study in Section 4 to demonstrate that the recipe does indeed generate realizations with the correct statistical properties. We conclude with a small discussion in Section 5. For completeness, Appendix A contains a proof that the procedure given in Section 3 does generate simulations with the correct statistical properties.

2 Background on Circulant Embedding for Real-Valued Processes

Suppose that we want to generate a realization of $\mathbf{X} = [X_0, X_1, \ldots, X_{N-1}]^T$, which is a column vector of length N containing a portion of a real-valued Gaussian stationary process $\{X_t, t \in \mathbb{Z}\}$ with zero mean and autocovariance sequence (ACVS) $\{s_{X,\tau}, \tau \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of all integers; here

$$s_{X,\tau} = \operatorname{cov}\{X_{t+\tau}, X_t\} = E\{X_{t+\tau} X_t\},$$

where $\operatorname{cov}\{X_{t+\tau}, X_t\}$ denotes the covariance between the real-valued random variables (RVs) $X_{t+\tau}$ and X_t, and $E\{\cdot\}$ is the expectation operation. By definition the covariance matrix $\Sigma_{\mathbf{X}}$ for $\mathbf{X}$ has $\operatorname{cov}\{X_j, X_k\}$ as its (j,k)th element, where $0 \le j, k \le N-1$. This matrix is Toeplitz because it has the same value along any diagonal and is symmetric because $\operatorname{cov}\{X_j, X_k\} = \operatorname{cov}\{X_k, X_j\}$.

The key idea behind circulant embedding is to find another real-valued zero mean Gaussian stationary process, say $\{Y_t\}$, such that, for some $M > N$, the RVs in $\mathbf{Y} = [Y_0, Y_1, \ldots, Y_{M-1}]^T$ have a covariance matrix $\Sigma_{\mathbf{Y}}$ that contains $\Sigma_{\mathbf{X}}$ as its upper left-hand submatrix and that is circulant; i.e., the vectors $[Y_0, Y_1, \ldots, Y_{N-1}]^T$ and $\mathbf{X}$ have the same covariance matrix, and the jth row of $\Sigma_{\mathbf{Y}}$ can be obtained by circularly shifting the zeroth row j units to the right. The reason for seeking a circulant embedding is the fact that, if C is an arbitrary $M \times M$ circulant matrix, then we can write $C = \mathcal{F}^H \Lambda \mathcal{F}$, where $\mathcal{F}$ is an $M \times M$ unitary matrix whose (j,k)th entry is $\exp(-i2\pi jk/M)/\sqrt{M}$; Λ is a diagonal matrix; and $\mathcal{F}^H$ is the Hermitian transpose of $\mathcal{F}$ [1]. Thus $\mathcal{F}$ is the eigenvector matrix for any circulant matrix, while the diagonal elements of Λ are the associated eigenvalues. Since $\mathcal{F}\mathcal{F}^H = I$ because $\mathcal{F}$ is unitary, it follows that $\mathcal{F}\Sigma_{\mathbf{Y}}\mathcal{F}^H = \Lambda$, where now Λ has nonnegative entries because all covariance matrices must be nonnegative definite. This implies that $\mathcal{F}\mathbf{Y}$ has Λ as its covariance matrix and hence can be readily simulated [7]. To within a scaling factor, $\mathcal{F}\mathbf{Y}$ is the same as the DFT of $\mathbf{Y}$, and the diagonal elements of Λ are the DFT of the zeroth row of $\Sigma_{\mathbf{Y}}$. Once the DFT of $\mathbf{Y}$ has been simulated, we can take its inverse DFT to form a simulation of $\mathbf{Y}$, after which we can extract the first N values of the simulated $\mathbf{Y}$ to obtain a realization of $\mathbf{X}$. If M is chosen to be a product of small integers (e.g., a power of two), the inverse DFT can be efficiently computed using an FFT algorithm.

In general, in order to generate a time series of length N, the smallest value to which we can set M is $2N-2$, a choice that is referred to as a minimal

embedding [7]. To see this, note that

$$\Sigma_{\mathbf{X}} = \begin{bmatrix} s_{X,0} & s_{X,1} & \cdots & s_{X,(N-2)} & s_{X,(N-1)} \\ s_{X,1} & s_{X,0} & \cdots & s_{X,(N-3)} & s_{X,(N-2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{X,N-2} & s_{X,N-3} & \cdots & s_{X,0} & s_{X,1} \\ s_{X,N-1} & s_{X,N-2} & \cdots & s_{X,1} & s_{X,0} \end{bmatrix}.$$

In order to have $\Sigma_{\mathbf{X}}$ embedded in the upper left-hand corner of $\Sigma_{\mathbf{Y}}$, the first N values in the zeroth row of $\Sigma_{\mathbf{Y}}$ must be equal to the zeroth row of $\Sigma_{\mathbf{X}}$, while the last $N-1$ values must contain – in reverse order – the last $N-1$ values in the zeroth column of $\Sigma_{\mathbf{X}}$, i.e., $s_{X,N-1}, s_{X,N-2}, \ldots, s_{X,1}$; however, the last element in the zeroth row of $\Sigma_{\mathbf{X}}$ is already $s_{X,N-1}$, so we can satisfy our requirement by appending $s_{X,N-2}, \ldots, s_{X,1}$ to the existing elements in the zeroth row of $\Sigma_{\mathbf{X}}$. This leads us to the $(2N-2) \times (2N-2)$ embedding matrix

$$\Sigma_{\mathbf{Y}} = \begin{bmatrix} s_{X,0} & s_{X,1} & \cdots & s_{X,N-2} & s_{X,N-1} & s_{X,N-2} & \cdots & s_{X,1} \\ s_{X,1} & s_{X,0} & \cdots & s_{X,N-3} & s_{X,N-2} & s_{X,N-1} & \cdots & s_{X,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{X,2} & s_{X,3} & \cdots & s_{X,N-2} & s_{X,N-3} & s_{X,N-4} & \cdots & s_{X,1} \\ s_{X,1} & s_{X,2} & \cdots & s_{X,N-1} & s_{X,N-2} & s_{X,N-3} & \cdots & s_{X,0} \end{bmatrix}. \quad (1)$$

The one difficulty we can experience with a minimal embedding is that the right-hand side of Equation (1) might not be a valid covariance matrix because its construction does not guarantee that it is positive semidefinite. There exist certain stationary processes for which it is known that (1) is not positive semidefinite for certain sample sizes N, but there are also other processes for which (1) is guaranteed to be positive semidefinite for all sample sizes [4,7,11]; however, if we are allowed to increase N, we can find a corresponding $\Sigma_{\mathbf{Y}}$ that is a valid covariance matrix for all stationary processes [6,7]. It is easy to determine if $\Sigma_{\mathbf{Y}}$ for a particular stationary process and choice of N is positive semidefinite. We just need to take the DFT of the first row of $\Sigma_{\mathbf{Y}}$ and check to see if all its values are nonnegative (see steps (1) and (2) of the recipe presented in the next section). In cases where $\Sigma_{\mathbf{Y}}$ is positive semidefinite, circulant embedding thus provides us an efficient means of generating realizations from a real-valued Gaussian stationary process (see [7,19] for details).

3 Adaptation to Complex-Valued Processes

Suppose now that $\mathbf{X}$ contains a portion of a complex-valued Gaussian stationary process with zero mean and ACVS $\{s_{X,\tau}, \tau \in \mathbb{Z}\}$, where

$$s_{X,\tau} = \operatorname{cov}\{X_{t+\tau}, X_t\} = E\{X_{t+\tau} X_t^*\},$$

(in the above X_t^* is the complex conjugate of X_t). The covariance matrix $\Sigma_{\mathbf{X}}$ for $\mathbf{X}$ is now a Hermitian Toeplitz matrix; i.e., $\Sigma_{\mathbf{X}}^H = \Sigma_{\mathbf{X}}$. As in the case of a real-valued process, if we can find another complex-valued zero mean stationary process $\{Y_t\}$ such that $\mathbf{Y} = [Y_0, Y_1, \ldots, Y_{M-1}]^T$ has a circulant covariance matrix with $\Sigma_{\mathbf{X}}$ embedded in it, then the DFT of $\mathbf{Y}$ has a diagonal covariance matrix and provides a computationally efficient method for generating realizations of $\mathbf{X}$. In contrast to the real-valued case, the smallest value to which we can set M is in general $2N-1$ rather than $2N-2$. To see this, note that

$$\Sigma_{\mathbf{X}} = \begin{bmatrix} s_{X,0} & s_{X,1}^* & \cdots & s_{X,(N-2)}^* & s_{X,(N-1)}^* \\ s_{X,1} & s_{X,0} & \cdots & s_{X,(N-3)}^* & s_{X,(N-2)}^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{X,N-2} & s_{X,N-3} & \cdots & s_{X,0} & s_{X,1}^* \\ s_{X,N-1} & s_{X,N-2} & \cdots & s_{X,1} & s_{X,0} \end{bmatrix}.$$

In order to have $\Sigma_{\mathbf{X}}$ embedded in its upper left-hand corner, the last $N-1$ values in the zeroth row of $\Sigma_{\mathbf{Y}}$ must contain – in reverse order – the last $N-1$ values in the zeroth column of $\Sigma_{\mathbf{X}}$. This requirement dictates that the smallest that $\Sigma_{\mathbf{Y}}$ can be is a $(2N-1) \times (2N-1)$ matrix taking the form

$$\begin{bmatrix} s_{X,0} & s_{X,1}^* & \cdots & s_{X,N-2}^* & s_{X,N-1}^* & s_{X,N-1} & s_{X,N-2} & \cdots & s_{X,1} \\ s_{X,1} & s_{X,0} & \cdots & s_{X,N-3}^* & s_{X,N-2}^* & s_{X,N-1}^* & s_{X,N-1} & \cdots & s_{X,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{X,2}^* & s_{X,3}^* & \cdots & s_{X,N-1} & s_{X,N-2} & s_{X,N-3} & s_{X,N-4} & \cdots & s_{X,1}^* \\ s_{X,1}^* & s_{X,2}^* & \cdots & s_{X,N-1}^* & s_{X,N-1} & s_{X,N-2} & s_{X,N-3} & \cdots & s_{X,0} \end{bmatrix}.$$

The fact that $2N-1$ is odd is inconvenient for use with a standard 'powers of two' FFT algorithm. Note, however, that, if $s_{X,N-1}$ happens to be real-valued so that $s_{X,N-1}^* = s_{X,N-1}$, then $\Sigma_{\mathbf{Y}}$ can reduced down to a $(2N-2) \times (2N-2)$

matrix of the form

$$\begin{bmatrix} s_{X,0} & s_{X,1}^* & \cdots & s_{X,N-2}^* & s_{X,N-1} & s_{X,N-2} & \cdots & s_{X,1} \\ s_{X,1} & s_{X,0} & \cdots & s_{X,N-3}^* & s_{X,N-2}^* & s_{X,N-1} & \cdots & s_{X,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{X,2}^* & s_{X,3}^* & \cdots & s_{X,N-2} & s_{X,N-3} & s_{X,N-4} & \cdots & s_{X,1}^* \\ s_{X,1}^* & s_{X,2}^* & \cdots & s_{X,N-1}^* & s_{X,N-2} & s_{X,N-3} & \cdots & s_{X,0} \end{bmatrix}.$$

In general, $s_{X,N-1}$ is not real-valued, but, when it is not, we can force this desirable condition by modulating $\{X_t\}$. This is based upon the fact that, if $\{X_t\}$ is a zero mean stationary process with ACVS $\{s_{X,\tau}\}$, then $\widetilde{X}_t = X_t e^{-i2\pi\nu t}$ defines a similar process with an ACVS dictated by $\tilde{s}_{X,\tau} = s_{X,\tau} e^{-i2\pi\nu\tau}$. If we express $s_{X,N-1}$ in polar notation as $|s_{X,N-1}|e^{i\phi}$ and let $\nu = \frac{\phi}{2(N-1)\pi}$, then $\{\widetilde{X}_t\}$ has an ACVS such that $\tilde{s}_{X,N-1}$ is real-valued because

$$\tilde{s}_{X,N-1} = s_{X,N-1} e^{-i2\pi\nu(N-1)} = |s_{X,N-1}| e^{i\phi} e^{-i2\pi\frac{\phi}{2(N-1)\pi}(N-1)} = |s_{X,N-1}|.$$

The scheme then is to simulate the modulated process $\{\widetilde{X}_t\}$ and then to demodulate the simulation to obtain the desired simulated $\mathbf{X}$. This leads to the following recipe for use with a 'powers of two' FFT algorithm.

(1) Let M be any power of two such that $M \geq 2N-2$. If $s_{X,\frac{M}{2}}$ is real-valued, let $\nu = 0$; if not, express $s_{X,\frac{M}{2}}$ in polar notation as $|s_{X,\frac{M}{2}}|e^{i\phi}$, and let $\nu = \frac{\phi}{M\pi}$. Define $\tilde{s}_{X,\tau} = s_{X,\tau} e^{-i2\pi\nu\tau}$, and form the sequence

$$\tilde{s}_{X,0}, \tilde{s}_{X,1}^*, \ldots, \tilde{s}_{X,\frac{M}{2}-1}^*, \tilde{s}_{X,\frac{M}{2}}, \tilde{s}_{X,\frac{M}{2}-1}, \ldots, \tilde{s}_{X,1}. \tag{2}$$

Compute the DFT of the sequence displayed above:

$$S_k = \sum_{\tau=0}^{\frac{M}{2}-1} \tilde{s}_{X,\tau}^* e^{-i2\pi k\tau/M} + \sum_{\tau=\frac{M}{2}}^{M-1} \tilde{s}_{X,M-\tau} e^{-i2\pi k\tau/M}, \tag{3}$$

$k = 0, 1, \ldots, M-1$. By construction, S_k must be real-valued.

(2) Check that $S_k \geq 0$ for all k, which is required for the embedding matrix to be positive semidefinite. If this nonnegativity condition is not satisfied, there are two straight-forward recourses, one of which we assume to be taken before going to step (3). The first is to increase M. In theory we can obtain nonnegative S_k's by making M large enough [6,7]; however, there is no guarantee that this can be done without making M impractically large. The second recourse is to set any negative S_k's to zero, in which case the realizations will only have approximately the correct statistical

properties [19]. If we define $S_k^+ = \max\{S_k, 0\}$, the quality of approximation can be quantified by comparing the ACVS of the process generating the realizations, namely,

$$\tilde{s}'_{X,\tau} = \frac{1}{M} \sum_{k=0}^{M-1} S_k^+ e^{i2\pi k\tau/M}, \;\; \tau = 0, \ldots, \frac{M}{2},$$

to the corresponding parts of the target ACVS $\tilde{s}_{X,\tau}$.

(3) Let $Z_0, \ldots, Z_{2M-1}$ be a random sample of $2M$ standard real-valued Gaussian RVs; i.e., the Z_k's are uncorrelated, and their means and variances are, respectively, zero and unity. Compute the complex-valued sequence

$$\widetilde{\mathcal{Y}}_k = (Z_{2k} + iZ_{2k+1}) \sqrt{\frac{S_k}{2M}}, \quad k = 0, \ldots, M-1.$$

(4) Finally use the DFT to compute the complex-valued sequence

$$\widetilde{Y}_t = \sum_{k=0}^{M-1} \widetilde{\mathcal{Y}}_k e^{-i2\pi kt/M}, \quad t = 0, \ldots, M-1.$$

Let $Y_t = \widetilde{Y}_t e^{i2\pi\nu t}$. The desired simulation of $\mathbf{X}$ is given by $[Y_0, \ldots, Y_{N-1}]^T$.

A proof that the RVs $Y_0, \ldots, Y_{N-1}$ have exactly the same statistical properties as $X_0, \ldots, X_{N-1}$ is given in Appendix A.

Two notes about this recipe are in order. First, to generate a second realization, we need only repeat steps (3) and (4), so multiple realizations can be easily generated once the weights $\sqrt{\frac{S_k}{2M}}$ have been determined. Second, we could have defined $\widetilde{\mathcal{Y}}_k$ in step (3) to be $(Z_{2k} + iZ_{2k+1}) \sqrt{\frac{MS_k}{2}}$, in which case step (4) would need the inverse DFT rather than the DFT to generate the realizations. The DFT is preferred here because it does not involve a multiplication by $1/M$ and hence can be computed somewhat faster.

4 Examples

As an example, let us consider a complex-valued Gaussian stationary process that is constructed by summing together a real-valued Gaussian fractionally differenced (FD) process and a complex demodulated FD process (the two FD processes are taken to be pairwise uncorrelated). By definition an FD process that is dictated by the parameters $0 < \delta < \frac{1}{2}$ and $\sigma_\epsilon^2 > 0$ is stationary with a strictly positive ACVS given by

$$s_{\delta,\tau} = \frac{\sigma_\epsilon^2 \sin(\pi\delta)\Gamma(1-2\delta)\Gamma(\tau+\delta)}{\pi\Gamma(\tau+1-\delta)}$$

(see [2] for details). The process variance is given by $s_{\delta,0}$, for which the above expression reduces somewhat to

$$s_{\delta,0} = \frac{\sigma_\epsilon^2 \Gamma(1-2\delta)}{\Gamma^2(1-\delta)}.$$

Once $s_{\delta,0}$ has been computed, the remaining part of the ACVS can be computed easily using the recursion

$$s_{\delta,\tau} = s_{\delta,\tau-1}\frac{\tau+\delta-1}{\tau-\delta}, \;\; \tau = 1, 2, \ldots.$$

The constructed complex-valued process $\{X_t\}$ has an ACVS given by

$$s_{X,\tau} = s_{0.45,\tau} + e^{i2\pi f_0 \tau} s_{0.3,\tau}, \tag{4}$$

where $f_0 = 0.12121$, and σ_ϵ^2 was set for the two FD processes in such a way that $s_{0.45,0} = 1$ and $s_{0.3,0} = 4$, thus yielding $s_{X,0} = 5$. The rationale for considering this example is that it has a slowly decaying ACVS, and use of a Cholesky decomposition to generate simulations is computationally expensive [13]. The corresponding spectral density function $S(f)$, $f \in [-\frac{1}{2}, \frac{1}{2}]$, for $\{X_t\}$ diverges to infinity as $f \to 0$ and $f \to f_0$, but these singularities are integrable and similar in form to those occurring in Gegenbauer processes [9].

Figure 1 summarizes the results of computer experiments using $\{X_t\}$. Plot (a) shows the true ACVS for lags $\tau = 0, 1, \ldots, 512$ (the thick and thin curves denote, respectively, the real and imaginary parts). Taking $N = 513$ and $M = 1024$, we formed the spectral weights $\sqrt{\frac{S_k}{2M}}$, $k = 0, 1, \ldots, M-1$, and these are shown in (e). Plots (c) and (d) show, respectively, the real and imaginary parts of one realization generated via the circulant embedding method. We generated $10,000$ such realizations and computed an unbiased estimate of the ACVS from each realization. Plot (b) shows the average of all $10,000$ ACVS estimates, which is virtually indistinguishable from the true ACVS in (a). The root mean square difference between the true ACVS $s_{X,\tau}$ and its averaged estimate $\hat{s}_{X,\tau}$, i.e.,

$$\left(\frac{1}{513} \sum_{\tau=0}^{512} |\hat{s}_{X,\tau} - s_{X,\tau}|^2 \right)^{\frac{1}{2}},$$

is slightly less than 0.01. This experiment thus supports the claim that the circular embedding is an exact method.

As a second example, let us consider a complex-valued Gaussian stationary process with a Gaussian-shaped ACVS, namely,

$$s_{X,\tau} = 5e^{-0.005\tau^2 + i2\pi f_0 \tau}, \tag{5}$$

where f_0 is as before. This ACVS is similar to ones that have been used to model radar clutter noise and Doppler measurements of blood flow [14]. Figure 2 summarizes the computer experiments for this process and is organized like Figure 1 (the theoretical and estimates ACVSs are only shown for $\tau = 0, \ldots, 64$ because these are virtually zero for $\tau > 64$). For this example, four of the S_k's were negative, but these values were numerically very close to zero (the most negative value was -1.8×10^{-14}). We set these to zero prior to computing the weights. The theoretical and estimated ACVSs are again virtually the same (cf. plots (a) and (b)), and the root mean square difference between the true and estimated ACVS estimates is now 0.015.

5 Discussion

We have shown how the circulant embedding method can be adapted to generate realizations of a complex-valued Gaussian stationary process via a standard 'powers of two' FFT algorithm. This method can also be adapted to handle multichannel complex-valued Gaussian stationary fields using a straightforward extension of the techniques discussed in Chan and Wood [3]; however, in general the device of using complex demodulation to allow us to generate realizations with a 'powers of two' FFT algorithm can no longer be used in the multichannel case. As formulated by Chan and Wood [3], the multichannel adaptation requires that M be odd, and they suggest use of a 'powers of three' FFT algorithm.

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A Proofs

Here we provide a proof that the recipe stated in Section 3 generates realizations with the stated statistical properties. Note first that S_k is a real-valued sequence because, in the summations in Equation (3), the $\tau = 0$ and $\tau = \frac{M}{2}$ terms are real-valued while the remaining $M - 2$ terms can be added in pairs that are real-valued; i.e.,

$$\tilde{s}^*_{X,\tau} e^{-i2\pi k\tau/M} + \tilde{s}_{X,\tau} e^{-i2\pi k(M-\tau)/M} = \tilde{s}^*_{X,\tau} e^{-i2\pi k\tau/M} + \tilde{s}_{X,\tau} e^{i2\pi k\tau/M},$$

where the right-hand sum is real-valued because it is equal to its complex conjugate.

We now show that $\widetilde{Y}_t$, $t = 0, \ldots, N-1$, has the same statistical properties as $\widetilde{X}_t$, $t = 0, \ldots, N-1$; i.e., both sequences of RVs are Gaussian with zero mean and with an ACVS given by $\tilde{s}_{X,\tau}$. We first note that

$$E\{\widetilde{Y}_t\} = \sum_{k=0}^{M-1} E\{\widetilde{\mathcal{Y}}_k\} e^{-i2\pi kt/M};$$

however, by definition, $\mathcal{Y}_k$ is a linear combination of two Gaussian RVs with mean zero, and hence $E\{\widetilde{\mathcal{Y}}_k\} = 0$ also, from which we can conclude that $E\{\widetilde{Y}_t\} = 0$. Next note that

$$\begin{aligned}
\operatorname{cov}\{\widetilde{Y}_{t+\tau}, \widetilde{Y}_t\} &= \sum_{j=0}^{M-1} \sum_{k=0}^{M-1} E\{\widetilde{\mathcal{Y}}_j \widetilde{\mathcal{Y}}^*_k\} e^{-i2\pi j(t+\tau)/M} e^{i2\pi kt/M} \\
&= \sum_{k=0}^{M-1} E\{|\widetilde{\mathcal{Y}}_k|^2\} e^{-i2\pi k\tau/M}
\end{aligned}$$

since, by construction, $E\{\widetilde{\mathcal{Y}}_j \widetilde{\mathcal{Y}}^*_k\} = 0$ when $j \neq k$. Since $E\{|\widetilde{\mathcal{Y}}_k|^2\} = \frac{S_k}{M}$, it follows that, for $\tau = 0, \ldots, \frac{M}{2}$,

$$\operatorname{cov}\{\widetilde{Y}_{t+\tau}, \widetilde{Y}_t\} = \frac{1}{M} \sum_{k=0}^{M-1} S_k e^{-i2\pi k\tau/M} = \left(\frac{1}{M} \sum_{k=0}^{M-1} S_k e^{i2\pi k\tau/M} \right)^* = \tilde{s}_{X,\tau}$$

since the term in the parentheses is the inverse DFT of $\{S_k\}$, and $\{S_k\}$ is the DFT of the sequence displayed in Equation (2). Since $N - 1 \leq \frac{M}{2}$, the desired result follows.

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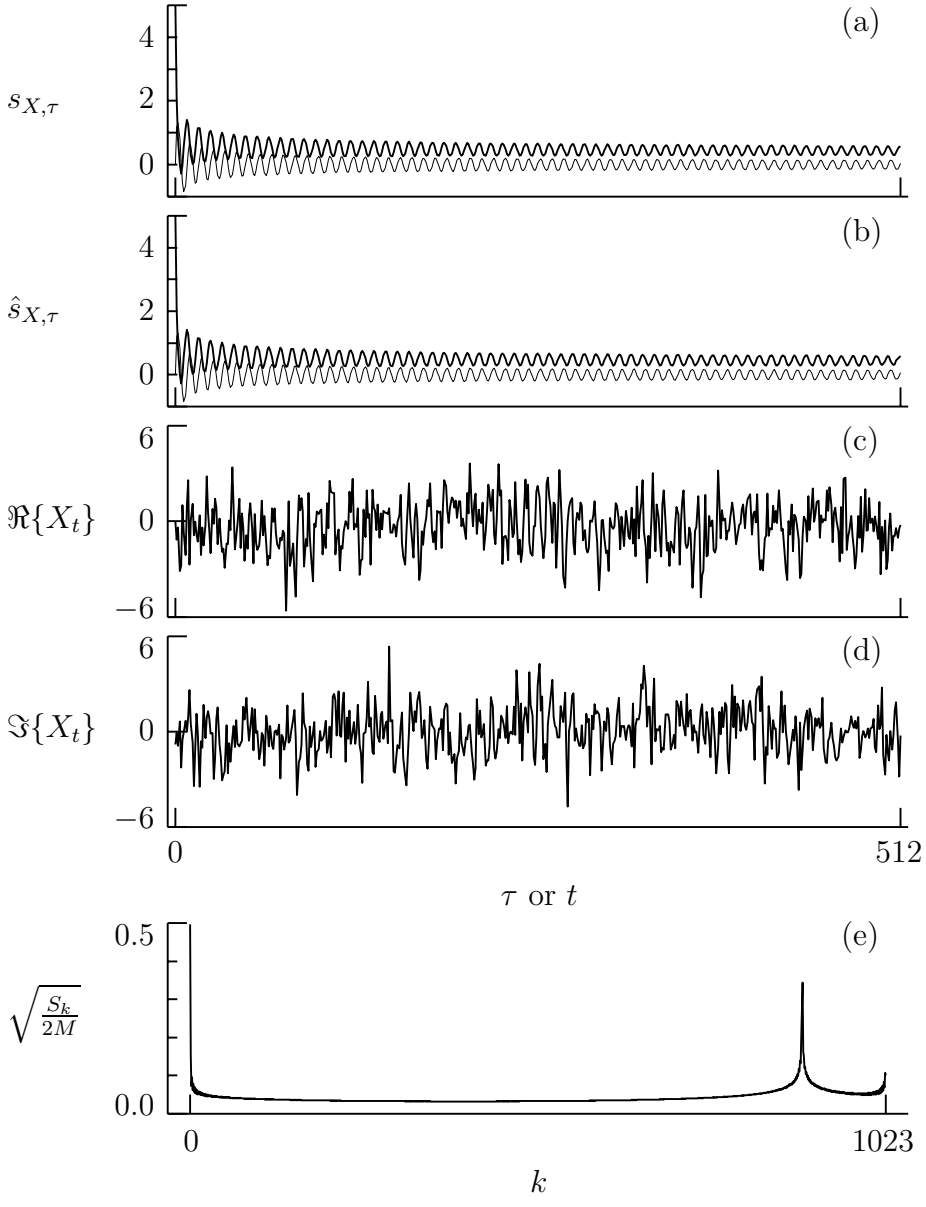
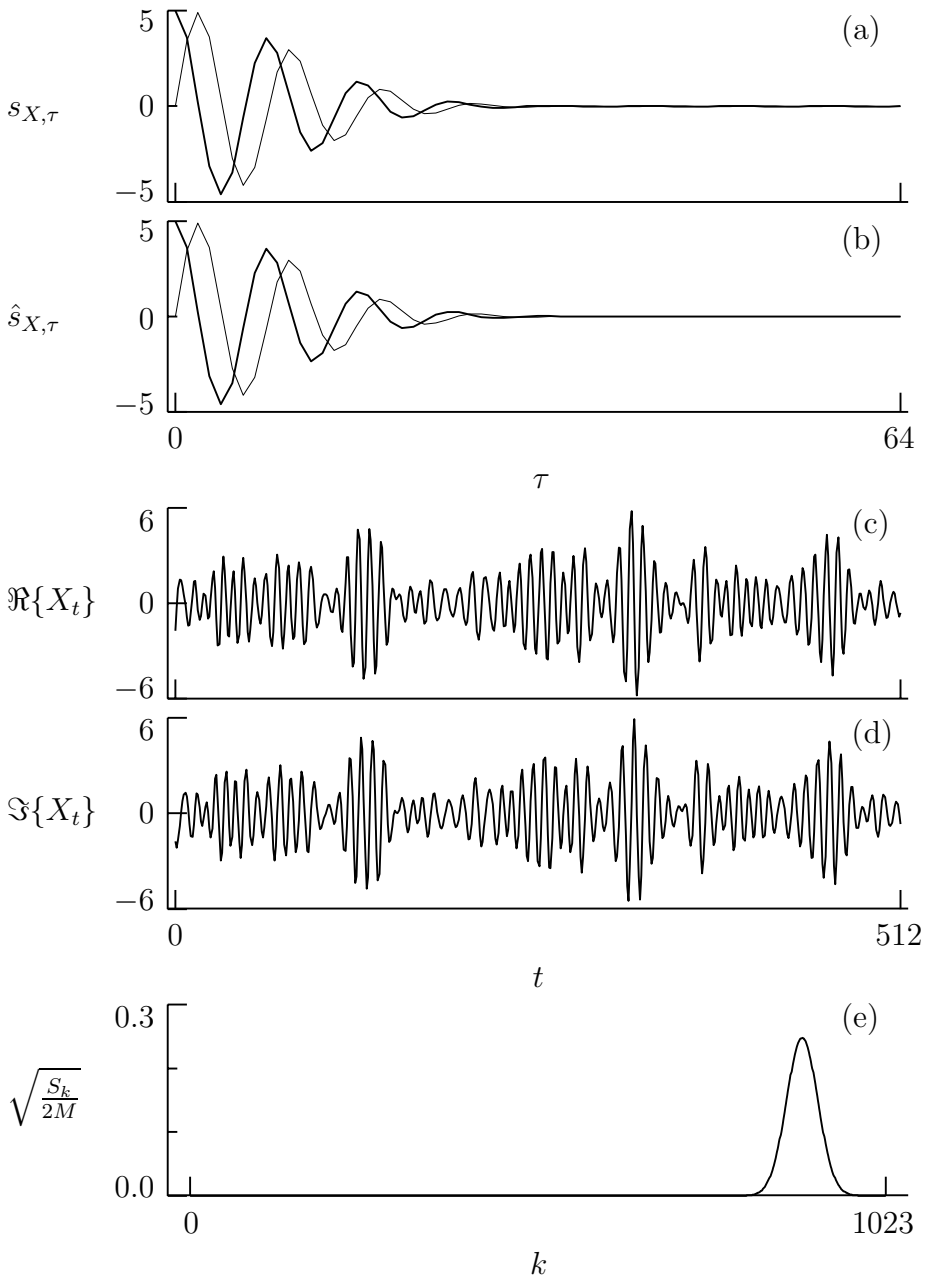
Figure Captions

Figure 1. Summary of computer experiments involving a complex-valued Gaussian stationary process whose real and imaginary components are formed from two pairwise uncorrelated real-valued fractionally differenced processes. Plot (a) shows the true autocovariance sequence (ACVS) for this process, which is defined in Equation (4). The real and imaginary parts are, respectively, the thick and thin curves. Plot (b) shows an estimate of this ACVS formed by averaging together unbiased ACVS estimates computed for $10,000$ realizations (each of length $N = 513$) from this process, where each realization was formed using the circulant embedding method. Plots (c) and (d) show the real and imaginary components of one of these realizations. Plot (e) shows the $M = 1024$ spectral weights that are needed by circulant embedding method to generate each realization.

Figure 2. As in Figure 1, but now for the complex-valued Gaussian stationary process whose ACVS is given in Equation (5). Here the true and estimated ACVSs are only shown out to lag $\tau = 64$ since they are virtually zero at all larger lags.


(a)
4
2
0
$s_{X,\tau}$
(b)
4
2
0
$\hat{s}_{X,\tau}$
(c)
6
0
−6
$\Re\{X_t\}$
(d)
6
0
−6
$\Im\{X_t\}$
0
512
τ or t
(e)
0.5
0.0
$\sqrt{\frac{S_k}{2M}}$
0
1023
k


5
0
−5
(a)
$s_{X,\tau}$
5
0
−5
(b)
$\hat{s}_{X,\tau}$
0
64
τ
6
0
−6
(c)
$\Re\{X_t\}$
6
0
−6
(d)
$\Im\{X_t\}$
0
512
t
0.3
0.0
(e)
$\sqrt{\frac{S_k}{2M}}$
0
1023
k