Savage's Theory and the Anscombe-Aumann Theorem

Philosophy of Statistics
April 28th, 2014

Review

Last Two Weeks:

- Criteria for an "interpretation" of probability
- Four interpretations of probability (briefly summarized)
- Dutch book theorem and representation theorem for comparative probability

Review: Criteria for an Interpretation

Salmon [1967] outlines three questions that an "interpretation" of probability ought to answer:

- Why should probability have particular mathematical properties?
- How do we determine or measure probabilities?
- Why and when is probability useful (especially in the sciences)?

Review: Subjective Interpretation

Question: How do we determine or measure probabilities?

Answer: Thus far, two techniques have been suggested:

- Betting on small sums
- Comparisons of likelihood of two events

Review: Subjective Interpretation

Question: Why should probability have particular mathematical properties?

Answer: Thus far, we have seen two arguments corresponding to the way probability is measured:

- Avoid being a sure loser $\Leftrightarrow$ Degrees of belief $=$ Probabilities
- Satisfy axioms of comparative likelihood $\Leftrightarrow$ Degrees of belief $=$ Probabilities
  - Axioms justified by money pump arguments or introspection

Review: Subjective Interpretation

Question: Why and when is probability useful (especially in the sciences)?

Answer: Probability is one part of expected utility maximization, which is a general theory of decision-making.

Today's Class

Today: Two more arguments for the subjective interpretation

Argumentative Structure

The arguments have the same structure as those encountered in the last class . . .

Argumentative Structure

Structure of Argument:

- If your decisions and/or judgments obey certain principles of rational choice, and
- If your degrees of belief are measured/elicited in particular ways,
- Then your degrees of belief ought to obey the probability axioms.

Argumentative Structure

Differences: The Dutch-Book theorem and axioms for comparative probability **try** to

- Divorce elicitation/measurement of likelihood from that of value.
- Avoid use of objective probabilities.

Argumentative Structure

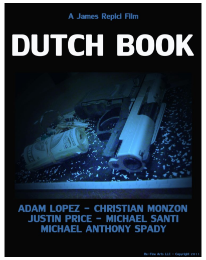
In contrast:

- Both Savage's argument and that of Anscombe and Aumann explicitly require eliciting judgments of value simultaneously with judgments of likelihood.
- Anscombe and Aumann's theorem requires objective probabilities.

Dutch Books and the Value of Money

- In what ways did the Dutch Book argument implicitly use facts about subjects' preferences/values?
- By identifying value with money ...

Dutch Books and Value of Money



- Suppose you are Dutch-Bookable, i.e.
- I can buy and sell bets at your fair prices that guarantee that you lose money for sure.
- However, suppose that given our budgets, I am only guaranteed to take 1 cent from you.

Dutch Books and Risk Seeking

- Finally, suppose you enjoy gambling: you would be willing to pay 1 cent just to bet with me.
- That you will surely lose money does not seem so irrational: you're just paying me 1 cent for the pleasure of gambling.

Moral: Risk-seeking persons may be both rational and yet Dutch-Bookable.

Dutch Books and Risk Aversion

Similar remarks apply to risk averse individuals.

Betting on Utility

Idea: The numerical payoffs/losses in the Dutch Book argument need not represent amounts of money, but could just represent things of value

So being a sure-loser means losing something you value.

Betting on Utility

Question: But why think that value can be numerically quantified?

Outline

Representation Theorem For Utility

Question: But why think that value can be numerically quantified?

Answer: Similar to quantifying judgments of likelihood.

- If your judgments about value can be **measured**/elicited in a particular way,
- and they obey certain axioms of rationality,
- Then there is a numerical utility function quantifying how much you value different items and states of affairs.

Elicitation of Value

> *It is clear that every measurement ... must ultimately be based on some immediate sensation, which possibly cannot and certainly need not be analyzed any further ... Such as the sensations of light, heat, muscular effort, etc., in the corresponding branches of physics ... In the case of utility the immediate sensation of preference - of one object or aggregate of objects as against another - provides this basis*

Von Neumann and Morgenstern [1953]

Elicitation of Value

Question: What are the objects of preference that Von Neumann and Morgenstern ask an individual to compare?

Answer: "Roulette" lotteries.

Roulette Lotteries

Roulette Lotteries

- Let U be any set of objects or state of affairs.
- Roulette Lottery:
  - A probability p of obtaining object w and a $(1-p)$ probability of obtaining v is represented by $pw \oplus (1-p)v$.
  - Heuristic: Think of the probabilities p as the number of slots on a roulette/carnival wheel.
  - One can likewise take roulette lotteries of roulette lotteries. The result is also a roulette lottery.

Roulette Lotteries

> *Probability has often been visualized as a subjective concept more or less in the nature of an estimation. Since we propose to use it in constructing an individual, numerical estimation of utility, the above view of probability would not serve our purpose. The simplest procedure is, therefore, to insist upon the alternative, perfectly well founded interpretation of probability as frequency in long runs. This gives directly the necessary numerical foothold.*

Von Neumann and Morgenstern [1953], pp. 19.

Axioms of Preference

Just as there were plausible axioms for comparative probability, so there are axioms for rational preference.

You tell me: What are V&M's axioms?

Representation Theorem

Theorem

If your preferences over roulette lotteries satisfy V&M's axioms, then there is a function U that takes lotteries as input and returns a numerical value such that

- *If lottery R_1 is preferred to R_2, then $U(R_1) > U(R_2)$.*
- *For a compound lottery $pR_1 \oplus (1-p)R_2$, it must be the case that*

$$U(pR_1 \oplus (1-p)R_2) = pU(R_1) + (1-p)U(R_2).$$

Archimedean Axioms

An important (and new) type of axiom:

Archimedean or continuity axioms:

- If $R_1 \prec R_2 \prec R_3$, then there is some $\alpha \in (0,1)$ such that

$$\alpha R_1 \oplus (1-\alpha)R_3 \prec R_2$$

- If $s_1 \succ s_2 \succ s_3$, then there is some $\alpha \in (0,1)$ such that

$$\alpha R_1 \oplus (1-\alpha)R_3 \succ R_2$$

No Lexicographic Preferences

Although it may not be obvious, these axioms entail that no state can be "infinitely" more valuable than another ...

Elicitation of Value

For example, suppose

- $R_1 =$ Watch your favorite movie,
- $R_2 =$ Watch your favorite movie with popcorn, and
- $R_3 =$ Eternal bliss in Heaven.
- Then clearly, $R_1 \prec R_2 \prec R_3$.
- Is there some probability $\alpha < 1$ such that
  - You prefer R_2 watching your favorite movie with popcorn to
  - A lottery which has a non-zero chance of Eternal bliss but will, with higher probability, only give you the payoff of watching your favorite movie?
  - I.e. Is there some $\alpha < 1$ such that $\alpha R_1 \oplus (1-\alpha) R_3 \prec R_2$

Elicitation of Value

Similarly, suppose

- $R_1 =$ Watch Batman,
- $R_2 =$ Watch Superman, and
- $R_3 =$ Death
- Then clearly, $R_1 \succ R_2 \succ R_3$.
- Is there some probability $\alpha > 0$ such that
  - Rather than watching Superman, you would prefer
  - A lottery with a non-zero chance of death but a higher chance of watching Batman?
  - I.e. $\alpha R_1 \oplus (1-\alpha) R_3 \succ R_2$

From Utility to Probability

Suppose you accept Von Neumann and Morgenstern's theorem.

What can one do with this utility function? What does this have to do with probability?

- Run the Dutch Book Argument with objects of known utility.
- Use Anscombe and Aumann's theorem for subjective probability.

Anscombe Aumann

You tell me: What is a "horse" lottery? In particular, what is the difference between horse and roulette lotteries?

Anscombe Aumann

Horse Lottery L: For each horse h, I will spin a roulette wheel R_h to determine a prize for you if h wins the race. Denote this by:

$$L = [R_1, R_2, \dots R_n]$$

Anscombe Aumann

Horse Lottery L: For each event h in a **mutually exclusive** and **exhaustive** set H, I will spin a roulette wheel R_h to determine a prize for you if $h \in H$ occurs. Denote this by:

$$L = [R_1, R_2, \dots R_n]$$

Anscombe Aumann

Anscombe and Aumann assume:

- Your preferences over roulette lotteries satisfy V&M's axioms.
- You have preferences over horse lotteries satisfying two additional postulates, which state (informally):
  - If I increase the prize that you win if Horse i wins and keep remaining prizes fixed, you prefer the new lottery to the old.
  - When I determine your prize, you don't care if I spin the roulette wheels first, or have the horses run first.

Anscome and Aumann's Theorem

Theorem

If your preferences over horse lotteries obey such axioms, then there is a probability function P that specifies how likely you think each horse is to win.

Outline

Measuring Probability

According to Savage, how can degrees of belief be measured?

Elicitation Procedure: Watch how individuals **act**.

Measuring Probability

Importantly, verbal comparisons of qualitative likelihood need not correspond to behavior.

Savage: Behavior vs. Intuitions

> *Even if the concept (of "more probable than") were so completely intuitive . . . what could such interrogation have to do with the behavior of the person in the face of uncertainty, except of course for his verbal behavior under investigation?*

Savage [1972] pp. 27

The Linda Problem

> *Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.*

Which is more probable?

1. Linda is a bank teller.
2. Linda is a bank teller and is active in the feminist movement.

Conjunction Fallacy

Kahneman and Tversky found around 90% chose the second statement as more probable, even though, on first glance, it seems to violate the following rule of probability:

$$P(B\&F) \leq P(B)$$

Conjunction Fallacy

- The experiment has been replicated several times.
- The phenomenon is called the conjunction fallacy.

Experimental Evidence

Charness et al. [2010] repeated Tversky and Kahneman's experiments with several variations. I'll mention three.

- The original experiment.
- The original question. Participants were told there is a correct answer and that they would receive $4 if they answered correctly.
- The original question. Participants discussed the question in groups of three, and then answered independently.

Experimental Evidence

Charness et al. [2010] found the following:

- The original experiment: 85% commit conjunction fallacy.
- With $4 Incentive: 33% commit conjunction fallacy.
- Groups of three with $4 Incentive: 10% commit the conjunction fallacy
  - My Note: If subjects decided independently and voted according to majority rule, one should expect $\sim$ 26% error rate given the individual results.

Experimental Evidence

Still, one in three subjects committed the fallacy even with monetary incentives. What could explain this?

In addition to subject indifference to the experiment (which drives a lot of survey results), there are lots of explanations. Let me note one thing . . .

Experimental Evidence

- Charness et al. [2010]'s experiment comes closer to elicitation methods suggested by Dutch Book arguments and Savage style-representation theorems, **but**
- Subjects are still asked to answer a question rather than choose an action or name a price.
- So a better question in this regard would be the following.

The Linda Problem

> *Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.*

Choose one of the following.

1. I will ask Linda if is a bank teller. If she answers "Yes", then I will give you $1.
2. I will ask Linda if is a bank teller and if she is active in the feminist movement. If she answers "Yes" to both questions, then I will give you $1.

Experimental Evidence

I will bet that very few subjects will choose option 2. Someone should get some grant money and try it.

The options also eliminate ambiguity in the original question concerning the meaning of the sentences (and of "probable") ,for example . . .

Implicatures

> *. . . some apparent biases might occur because the specific words used, or linguistic convention subjects assume the experimenter is following, convey more information than the experimenter intends. In other words, subjects may read between the lines. The potential linguistic problem is this: in the statement "Linda is a feminist bank teller," subjects might think that this statement "Linda is a bank teller" tacitly excludes feminists; they might think it actually means "Linda is a bank teller (and not feminist)." If subjects interpret the wording this way none of the statements are conjunctions of others and no probability rankings are wrong.*

Camerer [1995], pp. 598.

Savage's Framework

Quiz: Explain the following terms:

- State (of the world)
- Consequence
- Act
- Event

Savage's Framework

Formally:

- States and consequences are simply sets.
- An event is a set of states.
- An act is a function from states to consequences.
  - E.g., The act f of "answering true" on a true/false quiz question is the function such that
    - $f(s_{true})$ = right answer ☺
    - $f(s_{false})$ = wrong answer ☹

Savage's Framework

Consequences: The definitions entail that, when deciding among actions, individuals know the consequence of each action in each state of the world.

Savage's Framework

That sounds pretty unintuitive and unhelpful as a model of decision. Consider US foreign policy a few years back.

- Suppose there are two acts: (1) Invade Iraq and (2) Don't.
- Suppose there are two states of the world: Successful and not.
- What are the consequences of the two actions in these two states?

Savage's Framework

Question: What is Savage's solution?
Answer: Specify the states of the world in greater detail.

The argument might be raised that the formal description of decision that has thus been erected seems inadequate because a person may not know the consequences of the acts open to him in each state of the world. He might be so ignorant, for example, as not to be sure whether on rotten egg will spoil a six-egg omelet. But in that case nothing could be simpler than to admit that there are fourst states in the world corresponding to the states of the egg [i.e. rotten and good] and the two conceivable answers to the culinary question whether one bad egg will spoil a six-egg omelet. It seems to me obvious that this solution works in the greatest generality ...

Savage [1972], pp. 15.

Logical Omniscience

Savage's framework, like the Dutch book theorem, makes a "logical omniscience" assumption.

See appendices to these slides.

From Preferences to Probabilities?

Why is this discussion of actions, etc. relevant to subjective probability?

- Individuals have all sorts of preferences among actions. E.g., I prefer eating brussell sprouts to mushrooms.
- Write $f \preceq g$ to mean that the decision-maker finds doing g at least as preferable as doing f.
- Savage will use preferences among actions to define a (qualitative) probability relation $\leq$ among events:
  - Given events E and F, the claim that the agent considers A more likely to B will be represented by $E \leq F$.

From Preferences to Probabilities?

Importantly, Savage's theory is normative.

The claim he aims to prove is:

- If an agent has rational preferences among actions, then
- She can be modeled as if she assigns probabilities to events.

Rational Preferences

So what axioms ought a rational agent's preferences satisfy?

Savage's Framework

P1: The set of all actions ought to be simply ordered.

- **Reflexivity:** $f \preceq f$
- **Transitivity:** $f \preceq g$ and $g \preceq h$ entails that $f \preceq h$.
- **Totality:** For any pair of actions f and g, either $f \preceq g$ or $g \preceq f$.

Savage's Framework

Having preferences among actions produces preferences among consequences. How?

Constant Acts

Let c be any consequence. Define $\widetilde{c}$ to be the constant act (i.e. function) such that

$$\widetilde{c}(s) = c$$

for all states of the world s.

Preferences among Consequences

Define an ordering $\trianglelefteq$ among consequences as follows:

$$c \trianglelefteq d \Leftrightarrow \widetilde{c} \preceq \widetilde{d}$$

From Preferences to Probabilities

Here's the intuitive idea of Savage's definition of probability.

From Preferences to Probabilities

Here's the intuitive idea of Savage's definition of probability.

- Take two "prizes" (i.e. consequences) that you have preferences among.
- For example, let c be eat celery for breakfast and d be eat a donut.
- Clearly, $c \prec d$.

From Preferences to Probabilities

- Let E and F bet two events. E.g.
  - E is the event the Euro is worth less than the dollar next year.
  - F is the event that Americans get Fatter (on average) next year.
- Clearly, you think E is less likely than F.

From Preferences to Probabilities

- Consider two acts:
  - Act 1: You eat a donut d if E occurs and eat celery c otherwise.
  - Act 2: You eat a donut d if F occurs and eat celery c otherwise.
- Which act do you prefer?

From Preferences to Probabilities

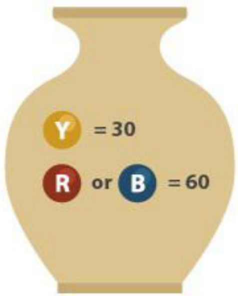
- If you aren't silly, you should prefer the second to the first.
- Because you prefer donuts to celery, and
- You think the Euro becoming less valuable than the dollar is less likely than increased American obesity,
- You increase your chances at the delicious prize if you choose the second act.

Rational Preferences

To be more specific, I'll need to use the blackboard.

Notes are available on the course website.

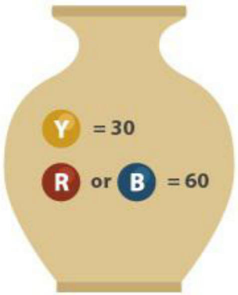
Ellsberg



Suppose an urn contains ninety balls.

- 30 are yellow
- 60 are red or black: you don't know the proportions.

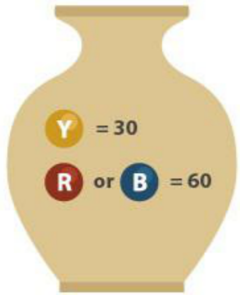
Ellsberg



Which do you prefer?

- Bet on yellow.
- Bet on red.

Ellsberg



Which do you prefer?

- Bet on yellow or black.
- Bet on red or black.

Objections to Additivity

If you are like most subjects, you said the following:

- $Y \succ R$
- $Y \cup B \prec R \cup B$.

And your preferences also contradict Savage's Sure-Thing principle.

Ellsberg and the Sure-Thing Principle

- If Black:
  - Y agrees with R (both lose), and
  - $Y \vee B$ agrees with $R \vee B$ (both win).
- If Not Black:
  - Y agrees with $Y \vee B$ (both win iff Yellow is drawn)
  - R agrees with $R \vee B$ (both win iff Red is drawn)
- You prefer Y to B.

By Savage's Sure-Thing principle, you ought to prefer $Y \vee B$ to $R \vee B$.

Rational Preferences

The same experiment can be used to argue the additivity assumption in the comparative probability theorems is suspect.

Preview

Upcoming Weeks:

- Discussion of updating one's degrees of belief by conditionalization
- The propensity interpretation
- Finally: Statistics

Course Mechanics

Course Mechanics: On the website, download:

- Notes on Savage
- Paper Topics Available
- Revised syllabus if you have not already.

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Logical Omniscience

Savage [1972] claims that his definitions entail that the decision-maker is logically omniscient: she must believe all the logical consequences of her beliefs.

> *In so far as 'rational' means logical, there is no live question . . . In particular, such a person cannot be uncertain about decidable mathematical propositions.*

[Savage, 1972]

Logical Omniscience

Question: Why is logical omniscience a consequence of Savage's framework?

- Actions are functions from states of the world to consequences.
- A \$1 bet on any tautology (e.g. $p \vee \neg p$) is the same action as betting \$1 on a different tautology no matter how complex (e.g. that set theory entails the fundamental theorem of calculus).

Logical Omniscience

Moral: In Savage's framework, actions are the same, regardless of how they are described.

- In Philosophical Jargon: Equivalence among acts is extensional, not intensional.