Dutch Books and Comparative Probability

Philosophy of Statistics
April 15th, 2014

Review

Last Class:

- Criteria for an "interpretation" of probability
- Four interpretations of probability (briefly summarized)

Review: Criteria for an Interpretation

Salmon [1967] outlines three questions that an "interpretation" of probability ought to answer:

- Why should probability have particular mathematical properties?
- How do we determine or measure probabilities?
- Why and when is probability useful (especially in the sciences)?

Today's Class

Today: Subjective interpretation

Argumentative Structure

We'll see two arguments for the claim that degrees of belief ought to satisfy the probability axioms.

Argumentative Structure

Structure of Argument:

- If your decisions and/or judgments obey certain principles of rational choice, and
- If your degrees of belief are measured/elicited in particular ways,
- Then your degrees of belief ought to obey the probability axioms.

Outline

1. Decision Theory 101
2. Dutch Book Theorem
3. Comparative Probabilities
4. Preview
5. References

Decision Matrices

	Sun	Rain
Read	2	3
Biergarten	4	-2
Watch "Glee"	-10	-10

Decision Matrices:

- Rows = Actions
- Columns = States of the world
  - Act-State Independence - We will assume states are unaffected by the decision-maker's actions.
- Cells = Outcomes of actions in various states of the world

Strict Dominance

	Sun	Rain
Read	2	3
Biergarten	4	-2
Watch "Glee"	-10	-10

Strict Dominance: If the outcome of some action a_1 (e.g., Watch Glee) is strictly worse than that of another a_2 (e.g., Read) regardless of the state of the world, do not choose a_1.

Strict Dominance

	Sun	Rain
Read	2	3
Biergarten	4	-2
Frisbee	3	-2

Weak Dominance: If

- The outcome of action a_1 (e.g., Frisbee) is strictly worse than that of another a_2 (e.g., Biergarten) in some state of the world (Sun), **and**
- The outcome of a_2 is at least as good as a_1 in all states of the world, then
- Do not choose a_1.

Worst-Case

	Sun	Rain
Read	2	3
Biergarten	4	-3

Worst-Case: Each action has a worst-case payoff.

E.g., For Read, it's 2. For Biergarten, it's -3.

Minimax

	Sun	Rain
Read	2	3
Biergarten	4	-3

Minimax: Pick the action with the best worst-case payoff. Here, it's Read.

Probability and Decision-Making

- But suppose you look outside, and it's a beautiful spring day in Munich.
- You read the weather forecast, which claims the chance of rain is .5%.
- Minimax ignores the probability of rain.
- We'd like some decision rule that simultaneously considers payoffs/losses and probability.

Decision Matrices

Suppose you fully believe the weather forecast, which claims the chance of rain is .5%.

	Sun	Rain
Read	2	3
Biergarten	4	-3

The expected utility of Biergarten is:

$$\begin{aligned} \textsc{seu}(\mathit{Biergarten}) &= p(\mathit{Sun}) \cdot 4 + p(\mathit{Rain}) \cdot -3 \\ &= 995 \cdot 4 + .005 \cdot -3 \\ &= 3.965 \end{aligned}$$

Decision Matrices

Suppose you fully believe the weather forecast, which claims the chance of rain is .5%.

	Sun	Rain
Read	2	3
Biergarten	4	-3

In contrast, expected utility of Read is:

$$\begin{aligned} \textsc{seu}(\mathit{Read}) &= p(\mathit{Sun}) \cdot 2 + p(\mathit{Rain}) \cdot 3 \\ &= 995 \cdot 2 + .005 \cdot 3 \\ &= 2.005 \end{aligned}$$

Three Decision Rules

- Maximize (subjective) expected utility (SEU)
- Dominance
- Minimax

Rationality and Expected Utility

- The Standard in Economics: An agent is rational if she acts **as if** she were maximizing expected utility.
- That is, the agent may not act **with the intent** of maximizing expected utility. She may happen to do maximize utility accidentally or unconsciously (due to practice and training, or genetic predisposition).
- There are a number of arguments for the claim that expected utility maximization is the **unique** rational decision rule; we won't discuss them in this class.

Three Decision Rules

Observation: Dominance and minimax are well-defined decision rules even if

- One does not assign states of the world probabilities; in fact, neither rule requires even the qualitative comparison of the likelihood of outcomes.
- One does not assign outcomes numerical payoffs; the decision rule makes sense even if payoffs are only qualitatively ordered.

Three Decision Rules

Moral: We can use dominance or minimax-based reasoning to justify the claim that subjective degrees of belief ought to obey the probability axioms.

Measuring Probability

How can degrees of belief be measured?

Elicitation Procedure 1: Betting behavior (with small sums)

Bets and Probabilities

Suppose we want to bet whether HIllary Clinton will be the next US president.

I ask you for your fair price for a $1 gamble.

Fair Prices

That is, consider a ticket that entitles its owner to $1 if Hillary Clinton is the next US president.

Your fair price $Pr(H)$ is the price you are willing to pay for such a ticket and sell it for.

Dutch Book - Upper Bound

Claim 1: Your fair price $Pr(H)$ should be less than $1.

Otherwise, buying the bet is strictly dominated by abstaining:

	Hilary Wins	Hilary Loses
Abstain	0	0
Buy Bet	$1 - Pr(H)$	$-Pr(H)$

Dutch Book - Lower Bound

Claim 2: Your fair price $Pr(H)$ should not be negative.

Otherwise, selling the bet is strictly dominated by abstaining:

	Hilary Wins	Hilary Loses
Abstain	0	0
Sell Bet	$Pr(H) - 1$	$Pr(H)$

Dutch Book - Additivity

Suppose we want to bet whether HIllary Clinton or Rand Paul will be the next US president.

Dutch Book - Additivity

I ask you for your fair prices for three types of tickets:

- A ticket that pays \$1 if Hillary Clinton is the next president,
- A ticket that pays \$1 if Rand Paul is the next president,
- A ticket that pays \$1 if either Hillary Clinton or Rand Paul is the next president.

Dutch Book - Additivity

Denote these fair prices:

- $Pr(H)$
- $Pr(R)$
- $Pr(H \cup R)$.

Dutch Book - Additivity

Suppose your fair prices are such that $Pr(H) + Pr(R) < Pr(H \cup R)$.

Dutch Book

In my infinite wisdom:

- I sell you the bet $H \cup R$,
- I buy both H and R bets.

Dutch Book

So before the bets are settled:

- You earn $Pr(H) + Pr(R)$ from the bets I bought from you.
- You spend $Pr(H \cup R)$ on the bet you bought from me.

So you are in the hole:

$$c = Pr(H) + Pr(R) - Pr(H \cup R) < 0$$

by assumption.

Dutch Book

Suppose Hillary wins:

- I owe you \$1 because you win the $H \cup R$ bet.
- You owe me \$1 because I win the H bet.
- So **no money exchanges hands**.

And you are in the hole $c < 0$ dollars.

Dutch Book

Suppose Rand wins:

- I owe you \$1 because you win the $H \cup R$ bet.
- You owe me \$1 for winning the R bet.
- So **no money exchanges hands**.

And you are in the hole $c < 0$ dollars.

Dutch Book

Suppose neither wins:

- Neither of us wins any bet.
- So **no money exchanges hands**.

And you are in the hole $c < 0$ dollars.

Dutch Book - Additivity

In other words, if your prices do not add, then betting is strictly dominated by abstaining:

	Hilary	Rand	Neither
Abstain	0	0	0
Bet	c	c	c

where

$$c = Pr(H) + Pr(R) - Pr(H \cup R) < 0$$

Dutch Book - Additivity

An analogous argument holds if $Pr(H \cup R) < Pr(H) + Pr(R)$.

Theorem (Dutch Book Theorem)

Betting is strictly dominated by abstaining if and only if your degrees of belief violate the probability axioms.

What must be the case about our degrees of belief if we want to avoid weakly dominated actions?

Weak Dominance and Regularity

Suppose your fair price (i.e., degree of belief) that aliens will will invade earth tomorrow is zero.

So you are willing to give me a $1 ticket at no cost.

Weak Dominance and Regularity

Then betting is weakly dominated by abstaining.

	Aliens	No Aliens
Abstain	0	0
Bet	-1	0

Theorem (Dutch Book Theorem)

Betting is weakly dominated by abstaining if your degrees of belief are not representable by a regular probability measure. The converse is true if the event space is finite.

Regularity = All events have non-zero probability.

See [Shimony, 1955]. See [Pedersen, 2014] for a discussion of weak-dominance and a representation theorem for non-Archimedean probabilities.

Regularity in Statistics

- If the algebra of events is uncountable, it's impossible for a probability measure (whether finitely or countably additive) to be regular.
- Statistical problems routinely require probability measures to be defined over an uncountable algebra of events.
- For this, and other reasons, weak dominance is often not thought to be a principle of rationality.

Objections to Dutch Book Arguments

Objections:

1. We cannot always be forced to engage in bets, and when we are forced, it's not clear that the prices we offer are indicative of our beliefs [Kyburg, 1978].
2. The bookie cannot be an expected utility maximizer.
3. Avoiding book requires logical omniscience.

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Eliciting Belief

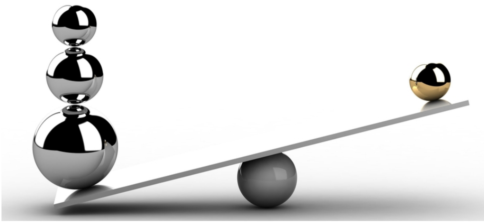
- Eliciting degrees of confidence by bets seems unrealistic.
- Perhaps we can measure strength of belief some other way ...

Measuring Probability

How can degrees of belief be measured?

Elicitation Procedure 2: Ask individuals to **compare** the likelihood of two events.

Comparative Probability



Suppose we ask an individual which of two event is more likely.

Write $A \precsim B$ if B is judged at least as likely as A.

Comparative Probability

Define:

- Write $A \prec B$ if $A \precsim B$ and it's not the case that $B \precsim A$
- Write $A \sim B$ if $A \precsim B$ and $B \precsim A$.

What properties should a rational person's judgments satisfy?

See Fishburn [1986] for a nice summary.

Comparative Probability

"Uncontroversial" Axioms:

- **Asymmetry:** If $A \prec B$, then $B \nprec A$.
- **Non-Triviality:** $\emptyset \prec \Omega$.
- **Non-Negativity:** $\emptyset \precsim A$ for all A.

Comparative Probability

More "Uncontroversial" Axioms:

- **Monotonicity:** If $A \subseteq B$, then $A \precsim B$.
- **Inclusion Monotonicity:**
  - If $C \prec B$ and $B \subseteq A$, then $C \prec A$.
  - If $C \subseteq B$ and $B \prec A$, then $C \prec A$.

Comparative Probability

"Controversial" Axioms:

- **Transitivity:** If $A \precsim B$ and $B \precsim C$, then $A \precsim C$.
- **Additivity:** If $A \cap C = B \cap C = \emptyset$, then

$$A \prec B \Leftrightarrow (A \cup C) \prec (B \cup C)$$

- **Completeness:** $A \precsim B$ or $B \precsim A$ for all events A, B.

Comparative Probability

Goal: Show there is a probability function P on events such that

$$A \precsim B \Leftrightarrow P(A) \leq P(B)$$

Unfortunately, this is false [Kraft et al., 1959].

Comparative Probability

Two common additional axioms . . .

Uniform Partitions

- Let A be any event, e.g., "Hillary is the next president."
- Take a coin that you believe to be fair.
- So you think that, if I flip the coin, heads and tails are equally likely.
- Intuitively you should judge the event "Hilary is president **and** the coin lands heads" to be equally likely as "Hilary is president **and** the coin lands tails."

Uniform Partitions

- Let A be any event, e.g., "Hillary becomes the next US president."
- Take a coin that you believe to be fair.
- So you think that, if I flip the coin five times, any sequence of heads and tails are equally likely, e.g.,

$$\langle H, H, H, H, H \rangle \sim \langle H, T, H, H, H \rangle \sim \langle T, T, H, H, T \rangle$$

- Intuitively you should judge the event "Hilary becomes president **and** the sequence of coin flips will be the left-most sequence" to be equally likely as "Hilary becomes president **and** the sequence of coin flips will be the right-most sequence."

Partitions

A partition of Ω is a collection of events $\{E_1, E_2, \dots E_n\}$ such that

- $\bigcup_{k \leq n} E_k = \Omega$, and
- $E_j \cap E_k = \emptyset$ if $j \neq k$.

Uniform Partitions

A partition $\{E_1, E_2, \ldots E_n\}$ of Ω is called $\precsim$-almost uniform if when given

- A union C of r many elements of $\{E_1, E_2, \ldots E_n\}$, and
- A union D of $r+1$ many elements of $\{E_1, E_2, \ldots E_n\}$,
- It follows that $C \precsim D$.

Fine Partitions

Assumption †: For each natural number n, there is a natural number $m \geq n$ and an almost uniform partition of Ω of size m.

This is an axiom of Savage [1972]

Representation for Comparative Probability

A subset of the assumptions above and † are sufficient for the existence of a probability function P on events such that

$$A \precsim B \Leftrightarrow P(A) \leq P(B)$$

Infinite Spaces

But these assumptions require us to compare infinitely many events!

Is there a postulate that works for comparisons of only finitely many events?

Scott's Axiom

Together with a technical axiom called Scott's Axiom, the above assumptions are necessary and sufficient for the existence of a probability function P on events such that

$$A \precsim B \Leftrightarrow P(A) \leq P(B)$$

Scott's axiom works in finite spaces. De Finetti [1937] discusses axiom in the reading.

What justifies these assumptions?

All of "uncontroversial" axioms can be motivated by "money pump" arguments.

Justifying the Axioms

Two types of arguments:

- All of "uncontroversial" axioms can be motivated by "money pump" arguments.
- Embarrassment/regret arguments.

Money Pump Arguments

What is a "money pump"?

E.g. Suppose your judgments violate asymmetry: you judge $A \prec B$ and $B \prec A$.

- Because you judge $A \prec B$, you would pay a sufficiently small fee to trade a bet on A for one on B.
- Because you judge $B \prec A$, you would pay a sufficiently small fee to trade a bet on B for one on A.

Similar arguments work for the other "uncontroversial" axioms (some under the assumption $\precsim$ is complete).

I am unaware of similar "money pump" arguments for additivity.

Justifying the Axioms

Two types of arguments:

- All of "uncontroversial" axioms can be motivated by "money pump" arguments.
- Embarrassment/regret arguments.

Regret

> *Suppose someone says to me, 'I am a rational person, that is to say, I seldom, if ever make mistakes in logic. But I behave in flagrant disagreement with your postulates, because they violate my personal taste, and it seems to me more sensible to cater to my taste than to a theory arbitrarily concocted by you.' I don't see how I could really controvert him, but I would be inclined his introspection with some of my own. I would, in particular, tell him that, when it is explicitly brought to my attention [that my preferences are intransitive] I feel uncomfortable in much the same way that I do when it is brought to my attention that some of my beliefs are logically contradictory ...*

Savage [1972], page 21.

Objections to Comparative Representations

Do individuals regret violating these comparative axioms?

Objection: Not really. Rational individuals' comparative judgments of likelihood need not satisfy the axioms.

Transitivity?

Marco Rubio

John Kerry

Bobby Rush

Objection to Transitivity

Suppose you make the following judgments about the likelihood that the next US president will possess certain characteristics:

- Hispanic $\prec$ Black $\prec$ White
- Independent $\prec$ Democrat $\prec$ Republican
- Northeast $\prec$ South $\prec$ Midwest

Objection to Transitivity

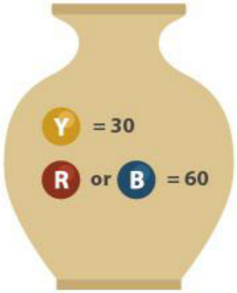
You also employ the following heuristic:

Heuristic: If A is more likely than B on a **majority** of dimensions, then I judge A to be more likely than B.

Objection to Transitivity

Marco Rubio	Republican	South	Hispanic
	$\succ$	$\succ$	
John Kerry	Democrat	Northeast	White
	$\succ$		$\succ$
Bobby Rush	Independent	Midwest	Black
		$\succ$	$\succ$
Marco Rubio	Republican	South	Hispanic

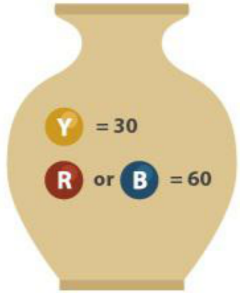
Ellsberg



Suppose an urn contains ninety balls.

- 30 are yellow
- 60 are red or black: you don't know the proportions.

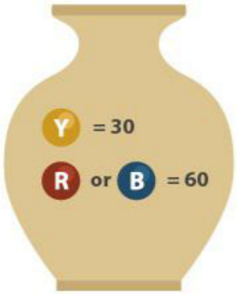
Ellsberg



Which do you prefer?

- Bet on yellow.
- Bet on red.

Ellsberg



Which do you prefer?

- Bet on yellow or black.
- Bet on red or black.

Objections to Additivity

If you are like most subjects, you said the following:

- $Y \succ R$
- $Y \cup B \prec R \cup B$.

If your preferences indicate how likely you think the draws are, then your judgments about likelihood violate additivity.

Preview

Upcoming Weeks:

- The arguments today tried to divorce judgments of likelihood from those of value.
- Two weeks: We'll see arguments that axiomatize judgments of value and likelihood simultaneously.

Structure of the Course

Upcoming Weeks:

- More on the subjective theory: theorems that employ utilities
- Discussion of updating one's degrees of belief by conditionalization
- The propensity interpretation

Structure of the Course

Course Mechanics:

- Website: Printable Slides, Paper Topics Available
- Download updated syllabus (class meeting canceled because of "Whit Tuesday")

References I

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