Philosophy of Probability: Problem Sets

To learn the philosophy of probability, one must know a bit of the mathematics of probability as well. Thus, on May 27th, there will be a short quiz on basic probability theory. The quiz will contain fewer than ten questions and should take you no more than 40 minutes to complete. You are required to know both (i) the material from lecture and (ii) the entirety of Chapter 1 and Sections 2.1 and 2.2 of of DeGroot's *Probability and Statistics* (Second Edition). Specifically, you will be asked to define some of the terms introduced in these sections (e.g., independence, conditional probability), to perform one or two calculations, and to prove one or two simple facts. The questions on the quiz will be very similar to the problems below.

As noted above, all of the assigned readings and exercises below are from DeGroot's *Probability and Statistics*, Second Edition, unless otherwise noted. For weeks 1 -4, there are also additional exercises in the ensuing pages that you ought to complete.

Week	**Readings**	**Exercises**
22/4	Sections 1.1 - 1.5	Section 1.4: 1, 3, 5a,b,d Section 1.5: 3, 6-8
29/4	Chapter 1.6 - 1.7	Section 1.6: 1-4, 7 Section 1.7: 1-7
6/5	Chapter 1.8 - 1.11	Section 1.8: 2,3,5, 15. Section 1.11: 1-3; 7-8
13/5	Chapter 2.1 - 2.2	Section 2.1: 1-3, 6. Section 2.2: 1,2, 4, 10

1 Additional Exercises

1.1 Week 1

Exercise 1 *Let P be a probability measure and suppose that $A \subseteq B$. Show that $P(B \setminus A) = P(B) - P(A)$, where $B \setminus A = \{x \in B : x \notin A\}$.*

In class, we defined an **algebra** as follows. Let Ω be a set that is called a **sample space**. Then an **algebra** $\mathcal{A}$ on Ω is a collection of subsets of Ω such that

- $\emptyset \in \mathcal{A}$,
- If $S \in \mathcal{A}$, then $S^c \in \mathcal{A}$, and
- If $S, T \in \mathcal{A}$, then $S \cup T \in \mathcal{A}$.

Exercise 2 *Let $\mathcal{A}$ be an algebra. Prove the following:*

- *Suppose that $S, T \in \mathcal{A}$. Then $S \cap T \in \mathcal{A}$.*
- *Show that if $S_1, \ldots, S_n \in \mathcal{A}$ is some finite number of sets, then the union $S_1 \cup S_2 \cup \ldots \cup S_n$ is also a member of $\mathcal{A}$.*

Exercise 3 *Which of the following sets are algebras? If the set $\mathcal{A}$ as defined is an algebra, prove it. If not, explain which of the defining conditions of an algebra the set does not satisfy and find some algebra $\mathcal{B}$ that contains $\mathcal{A}$.*

- *Let Ω be the set of natural numbers $\{1, 2, 3, \ldots\}$. Let $E = \{2, 4, 6, 8, \ldots\}$ be the set of even numbers, and $O = \{1, 3, 5 \ldots\}$ be the set of odd numbers. Let $\mathcal{A} = \{\Omega, \emptyset, E, O\}$.*
- *Let Ω be a set representing three flips of a coin. So Ω contains sequences like $\langle H, T, T\rangle$ and $\langle H, T, H\rangle$, which respectively represent the outcomes in which (i) one observed a heads and then two tails, and (ii) one observes a heads, followed by tails, followed by heads. When $i = 0, 1, 2, 3$, let E_i represent the set of coin tosses in which exactly i many heads are observed. For example, E_1 contains the sequences $\langle H, T, T\rangle, \langle T, H, T\rangle$, and $\langle T, T, H\rangle$. Let $\mathcal{A} = \{E_0, E_1, E_2, E_3\}$.*
- *Let Ω be the set of natural numbers $\{1, 2, 3, \ldots\}$. A set $S \subseteq \Omega$ is called* ***cofinite*** *if its complement is finite. For example, the set of natural numbers greater than 10 is cofinite, as its complement $\{1, 2, 3 \ldots, 10\}$ is finite. Let $\mathcal{A}$ be the set of all finite and cofinite subsets of Ω. (Note: The empty set is considered to be finite)*

- *Let Ω be any set. A* ***partition*** *of a set Ω is a collection of subsets $\Pi = \{E_i\}_{i \in I}$ such that $E_i \cap E_j = \emptyset$ if $i \neq j$ and $\cup_{i \in I} E_i = \Omega$.*

  Let $\Pi = \{E_i\}_{i \in I}$ be a partition of Ω, and for any subset $J \subseteq I$, define $\mathcal{E}(J) = \cup_{i \in J} E_i$. Let $\mathcal{A} = \{\mathcal{E}(J) : J \subseteq I\}$.

1.2 Week 2

Recall from class, a σ-algebra is algebra that is closed under **countable unions**. That is, if $S_1, S_2, \ldots$ is a countable sequence of sets and $S_n \in \mathcal{A}$ for all natural numbers n, then $\cup_{n \in \mathbb{N}} S_n \in \mathcal{A}$.

Exercise 4 *For each of the sets $\mathcal{A}$ in Exercise 3, determine whether or not $\mathcal{A}$ is a σ-algebra. If it is, prove it. If it is not, then provide a counterexample.*

Exercise 5 (Optional) *Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are algebras on Ω. Does it follow that $\mathcal{A}_1 \cap \mathcal{A}_2$ is also an algebra? If so, prove it. If not, find a counterexample.*

Exercise 6 (Optional) *Suppose $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq F_3 \ldots$ is a sequence of algebras on Ω. Does it follow that $\cup_{n \in \mathbb{N}} \mathcal{A}_n$ is also an algebra? If so, prove it. If not, find a counterexample.*

1.3 Week 3

Recall DeGroot defines a probability measure to be **countably additive**. That is, if if $S_1, S_2, \ldots$ is a countable sequence of pairwise disjoint sets (i.e., $S_i \cap S_j = \emptyset$ whenever $i \neq j$), then

$$P(\cup_{n \in \mathbb{N}} S_n) = \sum_{n \in \mathbb{N}} P(S_n).$$

In class, we discussed Kolmogorov's axiomatization, which only requires a probability function to be **finitely additive**. That is, if $S_1, S_2, \ldots S_n$ are pairwise disjoint, then

$$P(\cup_{j \leq n} S_j) = \sum_{j \leq n} P(S_j).$$

Exercise 7 *Let $\mathcal{A}$ be the set of all finite and cofinite subsets of the natural numbers. Define $P(E)$ to be zero if E is finite and $P(E) = 1$ if E is infinite. Show that P is a finitely additive probability measure on $\mathcal{A}$. Show that it is not countably additive.*

Exercise 8 (Optional) *Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are σ algebras on Ω. Does it follow that $\mathcal{A}_1 \cap \mathcal{A}_2$ is also a σ-algebra? If so, prove it. If not, find a counterexample.*

Exercise 9 (Optional) *Suppose $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_3 \ldots$ is a sequence of σ-algebras on Ω. Does it follow that $\cup_{n \in \mathbb{N}} \mathcal{A}_n$ is also a σ-algebra? If so, prove it. If not, find a counterexample.*

Exercise 10 (Optional and Advanced) *Show that there are no countably infinite σ-algebras. That is, every σ-algebra is either finite or uncountable.*

1.4 Week 4

Exercise 11 (Base Rate Fallacies) *Let T be the event that a randomly selected person at an airport is a terrorist. Very few people are terrorists, and so suppose $P(T) = \frac{1}{10^8}$. Let A be the event that a randomly selected person at an airport is of Arabic descent. About one in twenty-five people in the world are Arab, and so suppose $P(A) = \frac{1}{25}$. Finally, suppose that, given a person is a terrorist, the probability that they are of Arabic descent is high. For instance, suppose that $P(A|T) = .99$. This is what many people who advocate racial profiling claim is true. Now suppose that airport security stops a person because he or she is of Arabic descent. What is the probability the person is a terrorist? I.e. What is $P(T|A)$?*

Exercise 12 *Read the introduction and section on Bayes' theorem in the wikipedia entry on the "Monty Hall Problem." In a few paragraphs, explain the solution and how Bayes' theorem is employed in the solution.*