Savage's Theory

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Philosophy of Probability
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Review: Three objective Interpretations of Probability

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- Frequentist: Probability is just a relative frequency.
- Propensity: Probability is a tendency towards an outcome.
- Logical: Probability is the measure of the degree to which a set of sentences support a conclusion.

Rest of Class: Subjective (or Personalistic) Theory of Probability

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- Probability is a measure of an individual's strength of belief, or
- It is the strength of a rational individual's degree of belief.

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Purported Virtues of Subjective Probability:

- **Admissibility:** Several representation theorems that indicate that degrees of belief are (or ought to be) represented by probabilities.
- **Ascertainability:** Those same theorems often suggest a way to measure degree of belief.
- **Applicability:** Subjective probability is one component (the other is utility) in the most widely applied theory of rational-decision making: subjective expected utility theory.

Last Week: Two Paths to Subjective Probability

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- Lindley's measure of uncertainty
- Dutch Book Theorems

Last Week: Two Problems for Dutch Book Arguments

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- "Cunning" bookies may be incoherent
- Utility is not linear in money: gambles involve judgments of both value and uncertainty.

Today: According to Savage . . .

- An individual makes judgments about the likelihood of an event in virtue of having preferences over particular actions. That is, "degree of belief" is derivative of (i.e., defined from) preference.
- An individual's likelihood judgments ought to satisfy the probability axioms because her preferences over action ought to satisfy particular axiomatic constraints of rationality.

On face, Savage's representation theorem avoids both of the problems with Dutch book theorems we discussed:

- There is no bookie. Savage's postulates for rational preference are binding for isolated agents who may never engage in interaction.
- Savage's postulates assuming nothing about the shape of individual's utility function.

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On the other hand:

- Savage's postulates are greater in number and less immediately intuitive than avoiding dominated actions.

Outline

1. Review

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2. Criteria for a Theory of Probability
   - On Intuitions and Conceptual Analysis

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3. Probability and Decision
4. Framework and Postulates

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HÁJEK: "INTERPRETING" PROBABILITY

According to [Hajek, 1996]:

> *'Probability', after all, is not just a technical term that one is free to define as one pleases. Rather, it is a concept whose analysis is answerable to our intuitions, a concept that has various associated platitudes (for example: "if X has probability greater than 0, then X can happen"). Thus, it is unlike terms like 'complete metric space' or 'Granger causation' or 'material conditional', for which there are stipulative definitions.*

Savage on "Interpreting" Probability

In contrast, Savage does not care much about intuitions or how the word "probability" is used in speech:

> *[the notion of probability defined here] should be judged by the contribution it makes to the theory of decision, not by the accuracy with which it analyzes ordinary usage.*

[Savage, 1972], pp. 27.

Savage on "Interpreting" Probability

Savage thinks trying to either (i) analyze or (ii) elicit judgments from others concerning the intuitive concept of "more probable than" is problematic for at least two reasons.

1. "Many doubt that the concept "more probable to me than" is an intuitive one." pp. 27
2. Disregards behavioral aspect of concept. Savage writes,

   > *Even if the concept were so completely intuitive . . . what could such interrogation have to do with the behavior of the person in the face of uncertainty, except of course for his verbal behavior under investigation?"*

[Savage, 1972] pp. 27

Outline

As noted, for Savage the central interest in probability is for decision theory:

> *[the notion of probability defined here] should be judged by the contribution it makes to the theory of decision, not by the accuracy with which it analyzes ordinary usage.*

[Savage, 1972], pp. 27.

This is also why he dispenses with other views so quickly . . .

Three Theories of Probability according to Savage

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- **Objective:**
- **Personalist:**
- **Necessary:**

What do the three correspond to in our terminology?

Three Theories of Probability according to Savage

- **Objective:** Frequency
- **Personalist:** Subjective
- **Necessary:** Logical

Why does the requirement that probability be useful in decision-making rule out the objectivistic interpretation?

Objective probability applies only to repeated events:

> *The difficulty in the objectivistic position is this. In any objectivistic view, probabilities can apply fruitfully only to repetitive events, that is, to certain processes.*

[Savage, 1972], pp. 4.

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- Premise 2: Rational decision-making requires evaluating the likelihood of non-repeatable events.
  - E.g., I could be uncertain about whether or not I remembered to lock my front door this morning, and that will influence my decision about whether I will go home during lunch time today.
  - E.g., Less controversial: Decisions depend upon facts about one-time historical events (e.g., moon landings).

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  - E.g., Less controversial: Decisions depend upon facts about one-time historical events (e.g., moon landings).
- Conclusion: The objectivistic view is not sufficient for decision theory.

For better or worse, Savage [1972] dismisses the necessary (i.e. logical) theories of probability for the same reason:

> *It seems to me obvious, however, that what is ultimately wanted is criteria for deciding among possible courses of action, and therefore, generalization of the relation of implication seems at best, a roundabout method of attack.*

Savage [1972], pp. 7.

Outline

Quiz: Explain the following terms:

- State (of the world)
- Consequence
- Act
- Event

Formally:

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- An event is a set of states.
- An act is a function from states to consequences.
    - E.g., The act f of "answering true" on a true/false quiz question is the function such that
        - $f(s_{true}) =$ right answer ☺
        - $f(s_{false}) =$ wrong answer ☹

Consequence: The definitions entail that, when deciding among actions, individuals know the consequence of each action in each state of the world.

That sounds pretty unintuitive and unhelpful as a model of decision. Consider US foreign policy a few years back.

- Suppose there are two acts: (1) Invade Iraq and (2) Don't.
- Suppose there are two states of the world: Successful and not.

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- Suppose there are two acts: (1) Invade Iraq and (2) Don't.
- Suppose there are two states of the world: Successful and not.
- What are the consequences of the two actions in these two states?

Question: What is Savage's solution?

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Answer: Specify the states of the world in greater detail.

The argument might be raised that the formal description of decision that has thus been erected seems inadequate because a person may not know the consequences of the acts open to him in each state of the world. He might be so ignorant, for example, as not to be sure whether on rotten egg will spoil a six-egg omelet. But in that case nothing could be simpler than to admit that there are fourst states in the world corresponding to the states of the egg [i.e. rotten and good] and the two conceivable answers to the culinary question whether one bad egg will spoil a six-egg omelet. It seems to me obvious that this solution works in the greatest generality ...

[Savage, 1972], pp. 15.

There's a second interesting feature of Savage's definitions ...

Savage [1972] claims that his definitions entail that the decision-maker is logically omniscient: she must believe all the logical consequences of her beliefs.

> *In so far as 'rational' means logical, there is no live question . . . In particular, such a person cannot be uncertain about decidable mathematical propositions.*

[Savage, 1972]

Logical Omniscience

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- Actions are functions from states of the world to consequences.
- A \$1 bet on any tautology (e.g. $p \vee \neg p$) is the same action as betting \$1 on a different tautology no matter how complex (e.g. that set theory entails the fundamental theorem of calculus).

Moral: In Savage's framework, actions are the same, regardless of how they are described.

- In Philosophical Jargon: Equivalence among acts is extensional, not intensional.

Outline

From Preferences to Probabilities?

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From Preferences to Probabilities?

Why is this discussion of actions, etc. relevant to subjective probability?

- Individuals have all sorts of preferences among actions. E.g., I prefer eating brussell sprouts to mushrooms.
- Write $f \preceq g$ to mean that the decision-maker finds doing g at least as preferable as doing f.
- Savage will use preferences among actions to define a (qualitative) probability relation $\leq$ among events:
  - Given events E and F, the claim that the agent considers A more likely to B will be represented by $E \leq F$.

From Preferences to Probabilities?

Importantly, Savage's theory is normative.

The claim he aims to prove is:

- If an agent has rational preferences among actions, then
- She can be modeled as if she assigns probabilities to events.

So what axioms ought a rational agent's preferences satisfy?

P1: The set of all actions ought to be simply ordered.

- **Reflexivity:** $f \preceq f$
- **Transitivity:** $f \preceq g$ and $g \preceq h$ entails that $f \preceq h$.
- **Totality:** For any pair of actions f and g, either $f \preceq g$ or $g \preceq f$.

Having preferences among actions produces preferences among consequences. How?

Let c be any consequence. Define $\widetilde{c}$ to be the constant act (i.e. function) such that

$$\widetilde{c}(s) = c$$

for all states of the world s.

Define an ordering $\trianglelefteq$ among consequences as follows:

$$c \trianglelefteq d \Leftrightarrow \widetilde{c} \preceq \widetilde{d}$$

Here's the intuitive idea of Savage's definition of probability.

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- Take two "prizes" (i.e. consequences) that you have preferences among.
- For example, let c be eat celery for breakfast and d be eat a donut.
- Clearly, $c \prec d$.

From Preferences to Probabilities

- Let E and F bet two events. E.g.
  - E is the event the Euro is worth less than the dollar next year.
  - F is the event that Americans get fatter (on average) next year.
- Clearly, you think E is less likely than F.

- Consider two acts:
  - Act 1: You eat a donut d if E occurs and eat celery c otherwise.
  - Act 2: You eat a donut d if F occurs and eat celery c otherwise.
- Which act do you prefer?

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- Since you prefer donuts to celery, and
- You think the Euro becoming less valuable than the dollar is less likely than increased American obesity,
- You increase your chances at the delicious prize if you choose the second act.

Rational Preferences

To be more specific, I'll need to use the blackboard.

Notes are available on the course website.

References I

Hajek, A. (1996). "Mises redux" – redux: Fifteen arguments against finite frequentism. *Erkenntnis*, 45(2-3):209–227.

Savage, L. J. (1972). *The foundation of statistics*. Dover publications.