Keynes' Logical Theory

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Philosophy of Probability
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  - Do propensity theories satisfy the probability axioms in general?
- [Hájek, 2009]: Frequency theories do not match "pre-theoretic" intuitions about probability.
- [Eagle, 2004]: Propensity theories do not match intuitions and fail to possess other theoretical (esp. metaphysical) virtues about theorizing.

Where We're Going

- **Review:** Propensity and frequency theories are objective interpretations of probability: they are about "the world" in some way.
- **In Two Weeks:** We'll discuss subjective interpretations of probability: under such interpretations, probability is an attribute of our beliefs.
- **Today and Next Week:** Logical theories are also objective, but they are about belief, not about physical systems.

Outline

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   - Probability and Validity
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Some arguments are valid: when the premises are true, so are the conclusions.

- Either Jon or Suzy fed the chickens.
- Jon did not feed the chickens.
- Conclusion: Suzy fed the chickens.

Deductive vs. Inductive Arguments

Invalid arguments come in differing "degrees" of strength.

- Good
  - John works at Google.
  - Therefore, John knows a lot about computers.
- Not so good
  - John is a man.
  - Therefore, John knows a lot about computers.

For Keynes, probability is an extension of logic.

- Validity is a relation between premises and conclusions of an argument.
- Probability is similarly such a relation:
    - High probability is a characteristic of strong arguments.
    - Validity is, in a sense, a "limit" of high probability. We'll come back to this.
    - Contradictoriness is, in a sense, a limit of low probability.

There are several important consequences (and motivations) for the claim that probability is a relation between the premises and conclusions of an argument.

Consequence 1: Probability does not quantify uncertainty about perception. It captures the uncertainty of a proposition's truth given some set of assumptions representing our knowledge.

> *Part of our knowledge we obtain direct; and part by argument. The Theory of Probability is concerned with that part which we obtain by argument, and it treats of the different degrees in which the results so obtained are conclusive or inconclusive.*

[Keynes, 2004], pp. 2.

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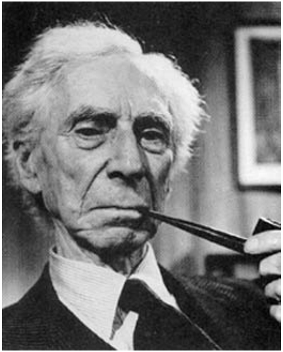
Answer: Bertrand Russell.

Russell [1997] distinguishes between two types of knowledge.

Acquaintance: Sense-data, of which we are absolutely certain. Also, universals.

Description: Propositional Knowledge obtained by "describing" objects that are not sense-data.



> *Part of our knowledge we obtain direct; and part by argument. The Theory of Probability is concerned with that part which we obtain by argument, and it treats of the different degrees in which the results so obtained are conclusive or inconclusive.*

Keynes' "direct" knowledge seems analogous to Russell's knowledge by acquaintance.

We start from things, of various classes, with which we have, what I choose to call without reference to other uses of this term, direct acquaintance. Acquaintance with such things does not in itself constitute knowledge, although knowledge arises out of acquaintance with them. The most important classes of things with which we have direct acquaintance are our own sensations, which we may be said to experience, the ideas or meanings, about which we have thoughts and which we may be said to understand, and facts or characteristics or relations of sense-data or meanings [universals?], which we may be said to perceive.

Keynes [2004], pp. 11. [my emphasis]

Because probability is a relation:

Consequence 2: Probability is always conditional, relating conclusions to some stock of knowledge.

Probability is Relational

> *No proposition is in itself either probable or improbable, just as no place can be intrinsically distant; and the probability of the same statement varies with the evidence presented, which is, as it were, its origin of reference.*

Keynes [2004], pp. 6.

Keynes also seems to think that he is committed to the following view:

Consequence 3: Probability is a relation between known assumptions and some conclusion.

> *In order that we may have rational belief in p of a lower degree of probability than certainty, it is necessary that we* **know** *a set of propositions h, and also* **know** *some secondary proposition q asserting a probability-relation between p and h.*

Keynes [2004], pp. 16.

Consequence 4: Probability is objective: it describes the strength of an argument, not subjective judgments about the argument.

Probability is Relational

> *[G]iven the body of premisses which our subjective powers and circumstances supply to us, and given the kinds of logical relations, upon which arguments can be based and which we have the capacity to perceive, the conclusions, which it is rational for us to draw, stand to these premisses in an objective and wholly logical relation. Our logic is concerned with drawing conclusions by a series of steps of certain specified kinds from a limited body of premisses.*

Keynes [2004], pp. 18. [my emphasis]

A Difficulty

One difficulty that we'll see arise again and again for logical and subjective theories of probability concerns mathematical knowledge.

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- **Gut Check:** Can a mathematical theorem (say, whether P=NP) have non-trivial probability? That is, can the probability be something other than zero or one?
- **Question:** To what "degree" does P=NP follow from the axioms of set theory?
    - Assuming no funny business, either P=NP is a theorem or there is a counterexample. So it cannot have non-trivial probability with respect to the axioms of set theory.

What is Keynes' solution? (It's in Chapter 11 for the eager readers among you).

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> *While the relation of certainty exists between the fundamental axioms and every mathematical hypothesis (or its contradictory), there are other data in relation to which these hypotheses possess intermediate degrees of probability. If we are unable through lack of skill to discover the relation of probability which an hypothesis does in fact bear towards one set of data, this set is practically useless, and we must fix our attention on some other set in relation to which the probability is not unknown.*

[Keynes, 2004] pp. 140.

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> *When Newton held that the binomial theorem possessed for empirical reasons sufficient probability to warrant a further investigation of it, it was not in relation to the axioms of mathematics, whether he knew them or not, that the probability existed, but in relation to his empirical evidence combined, perhaps, with some of the axioms. There is, in short, an exception to the rule that we must always consider the probability of any conclusion in relation to the whole of the data in our possession*

[Keynes, 2004] pp. 140.

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Question 1: When Keynes' writes "a/h", what are a and h?
Answer: a is a **sentence** in some formal language. h is a group of sentences.

What is a "group"?

In modern parlance, a group is a theory. It is a set of sentences Γ such that

- Γ is closed under logical consequence: If $\Gamma \models \varphi$, then $\varphi \in \Gamma$.
- Γ is consistent: there is no sentence φ such that $\Gamma \models \varphi$ and $\Gamma \models \neg\varphi$.

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Important Note: Keynes also uses the symbols $+$ and $\cdot$ for addition and multiplication of probabilities, i.e. the symbols are used twice in different senses.

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So the set of sentences looks like an algebra. What's missing?

- Need a privileged symbol $\perp$ (or $\top$) for the contradiction (or tautology).

Question 3: Is a/h a real number in Keynes' theory?

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Answer: Not necessarily.

> *Addition and Multiplication: If we were to assume that probabilities are numbers or ratios, these operations could be given their usual arithmetical signification. In adding or multiplying probabilities we should be simply adding or multiplying numbers. But in the absence of such an assumption, it is necessary to give a meaning by definition to these processes. I shall define the addition and multiplication of relations of probabilities only for certain types of such relations. But it will be shown later that the limitation thus placed on our operations is not of practical importance.*

[Keynes, 2004], pp. 148-149.

Keynes' assumes that probability is unique, but it may not be numerical.

> *Provided that a and h are propositions or conjunctions of propositions or disjunctions of propositions, and that h is not an inconsistent conjunction, there exists one and only one relation of probability P between a as conclusion and h as premiss. Thus any conclusion a bears to any consistent premiss h one and only one relation of probability.*

[Keynes, 2004], pp. 149.

Question 4: How can Keynes write equations involving 0's and 1's then?

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Answer: Think of probabilities as belonging to some set $\mathcal{P}$.

- In set theory, one can say when two sets are equal. So it makes sense to write $P = Q$ for probabilities $P, Q \in \mathcal{P}$.
- The elements 0 and 1 are special members of $\mathcal{P}$, and Keynes' axioms specify what properties these two elements have.

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Answer: All of them (morally).

- Keynes' proves the addition theorem.
- One does need to introduce $\perp$ and prove that $\perp/h = 0$.

Question for Investigation: In what senses, if any, are Keynes' axioms stronger than Kolmogorov's?

Discussion Questions: Why assume these axioms? Are arguments that one might make for logical axioms similar to those one might make for Keynes' logical theory of probability? Why or why not?

Question 6: Does it follow from Keynes' axioms that one can assign each probability a real number?

Question 6: Does it follow from Keynes' axioms that one can assign each probability a real number?

Answer: I'm not sure, but I think not. Here's the philosophical reason, which translates into a formal observation.

Does Keynes think all probabilities are "measurable?"

Does Keynes think all probabilities are "measurable?"

No, but one should be careful about interpreting what he means by "measurable."

By saying that not all probabilities are measurable, I mean that it is not possible to say of every pair of conclusions, about which we have some knowledge, that the degree of our rational belief in one bears any numerical relation to the degree of our rational belief in the other; and by saying that not all probabilities are comparable in respect of more and less, I mean that it is not always possible to say that the degree of our rational belief in one conclusion is either equal to, greater than, or less than the degree of our belief in another.

Formal Observation: Keynes' axioms do not specify that the set of probabilities is **linearly ordered**:

- It is not required that $P \leq Q$ or $Q \leq P$ for all probabilities P and Q (linearity).
- Similarly, it is not required that $P \leq Q$ and $Q \leq P$ entails $P = Q$ (anti-symmetry).
- Keynes' axioms require everything is more probable than 0 and that probabilities add in a particular way, but that's it.

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