Classical Probability and the Principle of Indifference

Philosophy of Probability
April 22nd, 2013

Review:

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- What is an algebra? What is an event?
- In Kolmogorov's axiomatization, what properties does a probability measure satisfy?

Define an **algebra** $\mathcal{E}$ to be a collection of subsets of Ω satisfying the following properties.

- If A is a member of $\mathcal{E}$, then so is its complement A^c.
- If A and B are members of $\mathcal{E}$, then so is their union $A \cup B$.
- The empty set $\emptyset = \{\}$ is a member of $\mathcal{E}$.

The members of an *algebra* will be called **events**.

Suppose $\mathcal{E}$ is an algebra (i.e. a collection of events).

A **probability** measure assigns every event A in $\mathcal{E}$ some number $P(A)$ between 0 and 1 (inclusive) such that

- $P(\emptyset) = 0$.
- If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.

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- About the structure of events to which probability is assigned.

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  - How do we determine or measure probabilities?

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  - Why should probability have particular mathematical properties?
- **Ascertainability:** An interpretation ought to explain how probabilities can be measured
  - How do we determine or measure probabilities?
- **Applicability:** An interpretation ought to explain why probability is so use useful (especially in the sciences).

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- He thinks the applicability requirement is vague.

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    - E.g. When we want to know the probability of cancer among smokers, we try to gather a "random" sample of smokers and determine what percentage develops cancer.
    - Why are probable events more frequent? Why is the frequency of an event a guide to its probability?

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2. **Confidence:** Ideally, an interpretation would answer the question, "In what ways are judgments of probability of events and confidence in an event's occurrence related?"
3. **Argumentative Strength:** An ideal interpretation would answer the question, "How is probability related to assessing the strength of arguments?"

Outline

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1. Review
2. Laplace's Definition
   - Admissibility
   - Ascertainability
   - Applicability

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- One of the first applications of probability occurs in Pascal's famous wager, however . . .
- Most of the first applications of probability concerned games of chance.
- E.g., Rolling a die, flipping a coin, pulling a card from a deck, or taking a ball from the urn.

Equipossibility

- Games of chance normally involve a physical system that is "symmetric" in some way
  - A die has a center of gravity in the middle
  - Balls in an urn are of similar size and weight
  - Coins are symmetrical and have a center of mass in the center

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  - A die has a center of gravity in the middle
  - Balls in an urn are of similar size and weight
  - Coins are symmetrical and have a center of mass in the center
- In these cases, it is intuitive to say there is some set of "equipossible" outcomes:
  - Die: $\Omega = \{1, 2, 3, 4, 5, 6\}$
  - Urn with one red, two black and two yellow balls: $\Omega = \{$Red, Black, Black, Yellow, Yellow $\}$
  - Coin: $\Omega = \{H, T\}$

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- Let Ω be any finite set. Games of chance only typically have finitely many outcomes.
- Let $\mathcal{P}(\Omega)$ be the power set of Ω. So any collection of outcomes is an event.
- For any event A, define $P(A)$ to be (i) the size of A (i.e., the number of elements of A) divided by (ii) the size of Ω. In symbols:

$$P(A) = \frac{|A|}{|\Omega|}$$

- In Laplace's words, the probability of A is the number of outcomes that are "favorable" to A divided by the total number of outcomes.

Evaluating Laplace's Definition

Let's see which of the "criteria of adequacy" Laplace's definition satisfies . . .

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- Since the size of the empty set is zero, $P(\emptyset) = 0$.

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If $A \cap B = \emptyset$, then the size of $A \cup B$ is the sum of the sizes of A and B. So

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So the definition satisfies the probability axioms.

This may seem like a silly exercise, but

For two of the four most popular interpretations of probability, it's not immediately obvious that the interpretation satisfies Kolmogorov's axioms.

Discussion Time

Ascertainability: How do we measure or determine probabilities according to Laplace's definition?

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Difficulties with Equipossibility

How do we determine probabilities according to Laplace?

- Once we know what the what outcomes are "equipossible", we can just count.
- But determining what outcomes are "equipossible" is difficult ...


UNITED STATES OF AMERICA
E PLURIBUS UNUM
QUARTER DOLLAR
E PLURIBUS UNUM
OF AMERICA
DOLLAR

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- So the coin has a center of balance that is clearly towards the "eagle" side.

How should we define the set of equipossible outcomes (i.e. the sample space) Ω?

Suppes claims there's no obvious way to answer this question:

> *suppose we are dealing with a coin that on the basis of considerable experience we have found to have a probability of .55 that in a standard toss a head will come up. It is obvious that there is no way of defining a Laplacean probability space ... that will represent in a natural way the result of tossing such a coin twice.*

Suppes [2002].

- How would Laplace answer?

- How would Laplace answer?
- Recall, for Laplace, probability arises from our ignorance, not from objective feature of the world.

Laplacean Determinism

We ought then to regard the present state of the universe as the effect of its anterior state and as the cause of the one which is to follow. Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it an intelligence sufficiently vast to submit these data to analysis it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes.

[Laplace and Dale, 1995].

Laplacean Determinism

> *Probability is relative, in part to [our] ignorance, in part to our knowledge. We know that of three or a greater number of events a single one ought to occur; but nothing induces us to believe that one of them will occur rather than the others. [my emphasis]*

[Laplace and Dale, 1995].

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- **Informal Idea**: If we have no evidence about which of a collection of events will occur, we ought to assign each event equal probability.

Here's one way of trying to make this formal:

- Formally: Define the sample space Ω and the algebra of possible events $\mathcal{A}$ however you like.

Principle of Indifference

Here's one way of trying to make this formal:

- Formally: Define the sample space Ω and the algebra of possible events $\mathcal{A}$ however you like.
- POI: For any collection of mutually exclusive events $A_1, A_2, \ldots A_n$ (i.e., $A_i \cap A_j = \emptyset$ unless $i = j$), it must be the case that

$$p(A_1) = p(A_2) = p(A_3) = \ldots = p(A_n)$$

- In Chapter 4, [Keynes, 2004] argues that POI, naively stated, entails contradictions.

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- Keynes gives lots of examples; I'll give the simplest one I know.

- Flip a coin twice:
  - $\Omega = \{\langle H, H\rangle, \langle T, T\rangle, \langle H, T\rangle, \langle T, H\rangle\}$
  - $\mathcal{A} = \mathcal{P}(\Omega)$.

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  - $\Omega = \{\langle H, H\rangle, \langle T, T\rangle, \langle H, T\rangle, \langle T, H\rangle\}$
  - $\mathcal{A} = \mathcal{P}(\Omega)$.
- You might have no evidence that any two flips is more likely than the other, in which case:

$$\frac{1}{4} = P(\langle H, H\rangle) = P(\langle H, T\rangle) = \dots$$

- However, you might have no evidence about the number of heads that you will observe. So let

$$
\begin{aligned}
A_0 &= \{\langle T, T\rangle\} \text{ i.e., “No heads”} \\
A_1 &= \{\langle H, T\rangle, \langle T, H\rangle\} \text{ i.e., “One head”} \\
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- Applying POI yields:

$$\frac{1}{3} = P(A_0) = P(A_1) = P(A_2)$$

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$$\begin{aligned}\frac{1}{3} &= P(A_1)\\ &= P(\{\langle H,T\rangle,\langle T,H\rangle\})\\ &= P(\langle H,T\rangle)+P(\langle T,H\rangle) \text{ by additivity}\end{aligned}$$

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Bertrand's Paradox(es): The general form of Bertrand's paradox is as follows:

- There are different ways of dividing (or "carving up") the set of possible outcomes of an experiment.
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- If we divide the sample space in one way and apply POI, the we get one probability measure.
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  - E.g., Any ordered pair specifying the outcome of two flips of the coin
- If we divide the sample space in a different way and apply POI, then we normally obtain a different probability measure that we did the first time.
  - E.g., No heads vs. One head vs. Two heads

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- But the same problem, as you see, emerges in finite spaces and is a bit easier to understand.

- **Response**: Whenever there are multiple ways of "carving up" the sample space, always use the "narrowest" or most specific.
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    - Formally: Use the **finest** partition.
- E.g., The set of ordered pairs specifying the outcome of two flips of the coin is narrower than the "No heads vs. One heads vs. Two heads" carving of the space.

Problems for the response:

- Why the most narrow carving? Why not the widest? Or my favorite one?
- To see another reason this response is incomplete, let's talk about **applicability**.

How does Laplace's definition fair with respect to **applicability**?
(i.e. Is it useful?)

- Many applications in science require assigning probabilities to continuous quantities like space and time.
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- Many applications in science require assigning probabilities to continuous quantities like space and time.
  - E.g. We may want to assign probabilities to the location of an electron, or to the time at which an atom decays.
- Note: Wanting a probability measure on an infinite sample space does not mean that we think that space and time are "**really**" continuous; it may just be mathematically **useful** to imagine such quantities are continuous.

How can we extend Laplace's definition to infinite sample spaces?

Let's consider an example. Suppose John will arrive in the next five minutes, but we don't know when.

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- Let $P(r)$ represent the probability that John arrives after exactly r many minutes.
- The "narrowest" carving of our space consists of every real number between 0 and 5.

- If we use the "narrowest" carving of our space and apply the POI, then it must be the case that $P(r) = 0$ for each real number r between 0 and 5. Why?

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- If $P(r) = \epsilon > 0$, then let n be the result of rounding up $\frac{1}{\epsilon}$.
- So $n + 1 > n \geq \frac{1}{\epsilon}$.
- Take $n + 1$ many real numbers $r_1, \ldots r_{n+1}$. By finite additivity,

$$P(\{r_1, \ldots r_{n+1}\}) = (n+1) \cdot \epsilon > 1$$

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- The problem arises that there are infinitely many probability measures that assign probability zero to each number between 0 and 5.

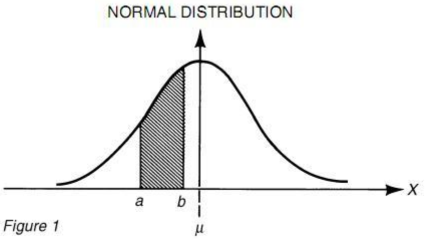
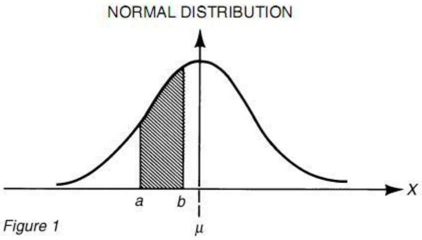
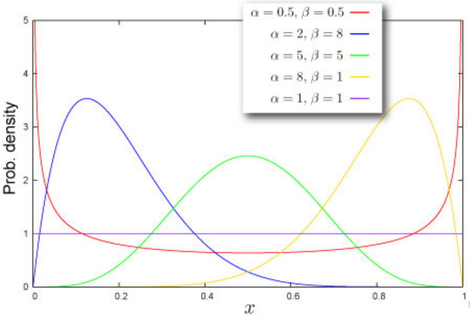


Figure 1


NORMAL DISTRIBUTION
X
a
b
μ
Figure 1


α = 0.5, β = 0.5
α = 2, β = 8
α = 5, β = 5
α = 8, β = 1
α = 1, β = 1
Prob. density
x
0
0.2
0.4
0.6
0.8
1
1
2
3
4
5

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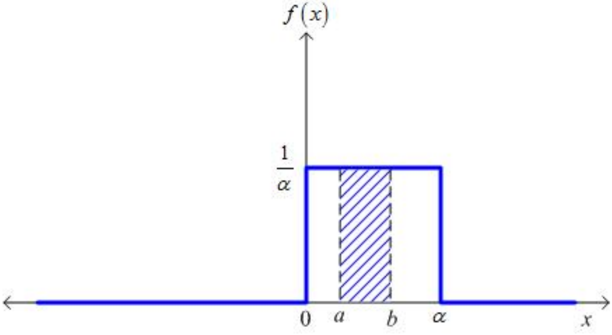
  - This is why Bertrand's paradoxes are often discussed when the sample space is infinite.
  - Namely, doing so prevents one from offering the response above.
  - However, the idea of the paradoxes are the same: carving up the sample space in different ways leads to different probability measures if one applies POI. How so?

Just as in the finite case, there is a uniform distribution in infinite spaces.



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- So Jenny thinks there is $\frac{\sqrt{2}}{\sqrt{5}} = \sqrt{\frac{2}{5}}$ probability that John arrives in the next $\sqrt{2}$ minutes squared, i.e., that John arrives in the next two minutes.
- Because Jenny and I have chosen different units of measurement, what we consider to be a uniform distribution differs.

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Applicability Probability

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What are his examples?

One is "Normalized length"

- Consider for a moment the property $L(x)$ which is interpreted as "length of an object x that is at most a meter long."
- If we interpret $\cup$ as placing two objects x and y end to end, then $L(x \cup y) = L(x) + L(y)$
- And clearly $0 \leq L(x) \leq 1$.

But what about the event structure?

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- Length is useful, but why is $L(x)$ useful?