A Somewhat Quick Introduction to Predicate Logic

Conor Mayo-Wilson

Pick up a mathematics textbook or journal, and you'll discover dozens of specialized symbols (e.g., $\emptyset$, $\wp$, and ∂). Like most academic books and journals, mathematical texts also contain hundreds of esoteric terms (e.g., diffeomorphism). So it's rather surprising that *every* mathematical statement can be translated into a symbolic language that contains only half a dozen symbols. That language is called the language of **first-order predicate logic.**

In these notes, I'll provide a brief lesson about how to translate introductory mathematical statements - like those you'd find in a high-school geometry or algebra class - into the language of predicate logic. Then I'll discuss how to represent Aristotelian syllogisms in the language of predicate logic. Finally, I'll explain why Aristotelian syllogisms are insufficient for representing the logical inferences made in introductory mathematics.

1 Constants and Predicate Symbols

Very simple English sentences contains two parts: a subject and a predicate. The subject is a noun-phrase, which describes a person, place or thing. The predicate is a verb-phrase, which describes what the subject did, is doing, or will do. In the sentence "Socrates drank hemlock," for example, the subject is "Socrates" and the predicate is "drank hemlock."

Simple sentences in predicate logic likewise contain two parts. The first are **constant symbols**, which behave much like proper names do in English. Constant symbols, like c_1, c_2, and so on, are intended to denote specific objects. For example, c_1 might represent Socrates, c_2 your favorite sweater, and so on. Just as two different English names can refer to the same person (e.g., "Snoop Dogg" and "Cordozar Calvin Broadus, Jr."), so can two different constant symbols (e.g., c_1 and c_2) refer to the same object in predicate logic.

In predicate logic, predicates are represented by uppercase letters P, Q, R, etc. which are called **predicate symbols**. For example, P might represent the phrase "is purple", Q "is quiet", and R "is red."

In predicate logic, we write $P(c)$ to represent the claim "c has property P." For example, if $c =$ John's favorite sweater and $P =$"is purple", then $P(c)$ means "John's favorite sweater is purple." Easy so far, right?

Predicate symbols can also take more than one argument. Such predicate symbols are often called **relation symbols** because they designate a relationship between two or more objects. For example, suppose that L represents the predicate "loves," c_1 represents John, and c_2 represents John's sweater. Then $L(c_1, c_2)$ represents "John loves his sweater," and $L(c_2, c_1)$ represents "John's sweater loves John." This example shows that *order matters* in predicate logic, just as it does in mathematics (e.g., $4/3 \neq 3/4$).

Although it's rarer, some mathematical statements involve predicates that take three or more arguments. For example, we might write $T(a, b, c)$ to represent the claim that points a, b, and c form a triangle. In order to indicate how many arguments a predicate symbol takes, it is common to use **variables**, like x, y, and z, and to say that, for example, $L(x, y)$ represents the claim that "x loves y" and that $T(x, y, z)$ represents the claims that " x, y and z form a triangle." I will talk more about the differences between variables and constant symbols in future sections.

Predicate symbols *cannot* be nested. For instance, suppose $P(x)$ means "x is purple" and $S(x)$ means "x is a sweater." Then to represent the claim that c is a purple sweater we ought to write $P(c)\&S(c)$; it is *incorrect* to write $S(P(c))$.

English Sentence	Correct	Incorrect
c is a purple sweater	$P(c)\&S(c)$	$P(S(c))$

Exercises

Suppose a, b, and c represent the numbers $2, 3$ and 6 respectively. Further, assume $D(x, y)$ represents the claim that "x is divisible by y" and that $L(x, y)$ represents the claim, "x is less than y."

1. What do each of the following sentences of predicate logic represent? Which are true?

   (a) $D(a, c)$

   (b) $D(c, a)$

   (c) $L(a, b)$

   (d) $L(b, a)$

2. Translate the following sentences into predicate logic:
   (a) 3 is divisible by 6.
   (b) 6 is less than 3.
   (c) 3 is less than 6.

2 Connectives

2.1 Binary Connectives

In many natural languages (e.g., English, Spanish, and Turkish) one can combine two or more sentences into a larger sentence using words like "and," "after," "because," and more. For instance, we can combine the sentences, "Socrates drank the hemlock" and "The jury sentenced Socrates to death" in the following ways:

- The jury sentenced Socrates to death, and Socrates drank the hemlock.
- Socrates drank the hemlock because the jury sentenced Socrates to death.
- After the jury sentence Socrates to death, Socrates drank the hemlock.

Similarly, in predicate logic, one can combine two shorter formula into longer ones using **binary connectives**. The language of predicate logic contains three binary connectives, and their English equivalents are listed in the following table:

Symbol	**Name of connective**	**English Equivalent**
& (sometimes $\wedge$ is used instead)	Conjunction	and
$\vee$	Disjunction	or
$\rightarrow$	Conditional	if-then

For example, suppose a, b, and c represent the numbers $2, 3$ and 6 respectively. Further, assume $D(x, y)$ represents the claim that "x is divisible by y" and that $L(x, y)$ represents the claim, "x is less than y." Then the formula $D(c, a) \& L(a, c)$ represents the claim that '6 is divisible by 2 *and* 2 is less than 6." That's true.

Similarly, the formula $D(c, a) \vee L(a, c)$ represents the claim that "6 is divisible by 2 *or* 2 is less than 6." This is true because six is divisible by two. Notice that the word "or" is interpreted **inclusively**, which means that the a sentence of the form $\varphi \vee \psi$ is true if either φ is true, or ψ is true, or both are true.

Lastly, the formula $D(c, a) \to L(a, c)$ represents the claim that "*If* 6 is divisible by 2, *then* 2 is less than 6." This claim is also true, but it's important to say something about how conditionals are understood in predicate logic. A conditional $\varphi \to \psi$ in predicate logic is assumed to be true if either of the following holds: (1) φ is false, or (2) both φ and ψ are true. That is, there is precise one circumstance in which a conditional is false, namely, if φ is true and ψ is false. I won't explain the reasons for this convention, but I will note that it does have some unintuitive consequences. According to this convention, the formula $D(a, b) \to L(c, a)$, which asserts that "If two is divisible by three, then six is less than two," is true.

If you have difficulty remembering this convention, I would recommend thinking of the conditional $\varphi \to \psi$ as the following equivalent disjunction: $\neg\varphi \vee \psi$. This disjunction is true if either (1) φ is false or (2) both φ and ψ are true, just as I said the conditional is.

2.2 Unary Connectives

English also contains ways of changing the meaning of a *single* sentence by inserting a phrase at the outset. For example, consider the phrases "It's not the case that" and "It's surprising that." Then we can take a single sentence, like "Socrates drank hemlock" and insert those phrases at the outset to obtain:

- It's not the case that Socrates drank hemlock.
- It's surprising that Socrates drank hemlock.

The language of predicate logic likewise contains a single **unary connective** that allows one to *negate* a sentence.

Symbol	**Name of connective**	**English Equivalent**
$\neg$	Negation	"It's not the case that"

If $P(x)$ represents "x is purple," $L(x, y)$ represents "x loves y," c_1 represents John, and c_2 represents "John's sweater," then $\neg P(c_1)$ represents the claim " John is not purple" and $\neg P(c_2) \to \neg L(c_1, c_2)$ represents the claim that "If John's sweater is not purple, then John doesn't love it."

2.3 More Complicated Connectives

For reasons that we'll not explore, it turns out that the connectives $\neg, \vee, \&$ and $\rightarrow$ are all we'll need, in the sense that for any other connective,[1] we can find expressions that are equivalent to that connective using the four that we've already discussed.

To see why, it's helpful to first note that we can use more than one connective in a sentence. For example, suppose $P(x)$ represents "x is purple," $L(x, y)$ represents "x loves y," c_1 represents John, and c_2 represents "John's sweater." Then $(L(c_1, c_2)\&P(c_2)) \vee P(c_1)$ represents the claim that "John loves his sweater, and his sweater is purple, OR John is purple." This sentence is true if either (i) John has a purple sweater that he loves, or (ii) John is purple, or both (i) and (ii) hold.

Combining connectives gives one lots of expressive power. For example, we said a formula of predicate logic of the form $\varphi \vee \psi$ is true if φ is true, or ψ is true, or if both are true. How can we represent the claim that "Either φ or ψ is true, but both are not true" or in other words, that "Exactly one of φ or ψ is true?"

Well, first we'd like to say that one of the two sentences must be true, and we can represent that claim by the formula $\varphi \vee \psi$. But we also need to say "It's not the case that *both* φ and ψ are true." That second claim can be represented by the formula, $\neg(\varphi\&\psi)$. Putting them together, we get $(\varphi \vee \psi)\&\neg(\varphi\&\psi)$. That formula represents the claim we want, namely, that "Either φ or ψ is true, but both are not true."

Let's consider one more example. Suppose we want to represent the claim "Neither φ nor ψ is true." To put it another way, we'd like to represent the sentence, "It's not the case that either φ or ψ is true." Then we can write $\neg(\varphi \vee \psi)$.

Exercises

Suppose a, b, and c represent the numbers $2, 3$ and 6 respectively. Further, assume $D(x, y)$ represents the claim that "x is divisible by y" and that $L(x, y)$ represents the claim, "x is less than y."

1. What do each of the following sentences of predicate logic represent? Which are true?

   (a) $\neg D(a, c) \vee D(c, a)$

   (b) $L(c, a) \rightarrow (D(a, c)\&D(b, c))$

[1] Technically, this is true only for every *truth-functional* connective.

(c) $L(a, b)\&(L(c, a) \vee \neg D(a, b))$

2. Translate the following sentences into predicate logic. Which sentences are true?

   (a) If 3 is divisible by 6, then 3 is not less than 6.

   (b) If 6 is divisible by 2 or 3, then either 2 is less than 6 or 3 is.

   (c) It's not the case that both 2 is less than 3 and 2 is divisible by 3.

   (d) If 6 is divisible by neither 2 or 3, then 3 is less than two.

2.4 Identity and Function Symbols

Mathematics is full of equations. In predicate logic, there is a special symbol - the **identity symbol** $=$ - that is used to represent the claim that two objects are the same. For example, if c_1 represents "Clark Kent" and c_2 represents "Superman," then $c_1 = c_2$ represents the claim that "Clark Kent is Superman." In the same way that we can negate other formula of predicate logic, we can negate equalities by placing $\neg$ in fron. For example, using the above translation key, $\neg(c_1 = c_2)$ represents the claim, "Clark Kent is not Superman." Sometimes I'll write "$c_1 \neq c_2$" instead of $\neg(c_1 = c_2)$.

Equations of predicate logic are most useful when combined with **function symbols**, which will be denote by the lower-case letters f, g and h (sometimes with subscripts). You likely learned about functions in a high-school mathematics class when you were asked to graph numerical functions like $f(x) = x^2$. In mathematics more generally, you can think of a function as a type of machine with specified inputs and outputs.[2] For instance, some functions might take the name of a person as input and output the name of the person's mother. Suppose, for instance, f is that "mother function" just described, that c_1 represents "Michelle Obama" and "Sasha Obama." Then the equation $f(c_2) = c_1$ represents the claim "Michelle Obama is the mother of Sasha Obama."

Like predicate symbols, function symbols can take more than one argument. In fact, the most familiar mathematical functions take at least two arguments. For example, suppose that $f(x, y)$ represents "x plus y" and assume that c_1, c_2 and c_3 respectively represent $2, 3$ and 6. Then the formula $\neg f(c_1, c_2) = c_3$ represents the claim that "2 plus 3 does not equal 6."

[2] This is a bit misleading because some mathematical functions are *uncomputable*, in the sense that there is no machine/computer that can calculate their values, even if given an infinite amount of time and memory. Take Phil 471 if you want to learn more about uncomputable functions.

Unlike predicate symbols, function symbols *can be nested*. Suppose, for instance, f is that "mother function" described above and that c_1 represents "Michelle Obama" and "Sasha Obama." Then $f(f(c_2))$ represents the mother of the mother of Sasha Obama, i.e., it represents Sasha Obama's maternal grandmother. Thus, the equation $f(f(c_2)) = f(c_1)$ represents the claim "Sasha Obama's maternal grandmother is Michelle Obama's mother."

Binary functions can also be nested. For instance, suppose that $f(x, y)$ represents "x plus y" and assume that c_1, c_2 and c_3 respectively represent $2, 3$ and 6. Then $f(c_1, f(c_1, c_2))$ represents $2 + (2 + 3)$, and the equation $f(c_1, f(c_1, c_1)) = c_3$ represents the claim that $2 + (2 + 2) = 6$.

Finally, function symbols can also be nested in predicates but *not* vice versa. For example, suppose that $P(x)$ represents the claim "x is a woman," that c_1 represents Sasha Obama, and that f is the mother function above. Then $P(f(c_1))$ represents the claim that "Sasha Obama's mother is a woman." It is *incorrect* to try to nest predicate symbols in function symbols. For instance, $f(P(c_1))$ is ill-formed in predicate logic; it is meaningless.

Exercises

Suppose a, b, and c represent the numbers $2, 3$ and 6 respectively. Further, assume $D(x, y)$ represents the claim that "x is divisible by y" and that the function symbol $f(x, y)$ represents "x times y."

1. What do each of the following sentences of predicate logic represent? Which are true?

   (a) $f(c_1, c_2) = c_3 \rightarrow D(c_3, c_1)$.

   (b) $f(c_1, c_2) = f(c_2, c_1)$.

   (c) $f(f(c_1, c_2), f(c_1, c_2)) = f(c_3, c_3)$.

   (d) $D(c_3, f(c_1, c_1))$

2. Translate the following sentences into predicate logic.

   (a) If two times three is six, then three times two is six as well.

   (b) If two squared is not six, and three squared is not six, and two times three is not six, then six is not divisible by two or three.

   (c) Six is divisible by the product of two and three.

   (d) Three times two cubed is equal to six times two squared.

3 Quantifiers and Variables

In natural languages, we often want to say how many objects of a particular type there are. For example, we might want to say "Most people are kind," or "There are 27 students in the class," or "Everyone has a mother," or "Something stinks," or "Nothing is impossible." Words like "most," "all," "none, " and "some" are called **quantifiers** because they tell us how many objects (i.e. the *quantity* of objects) have some property.

Predicate logic has exactly two quantifiers: "all" and "at least one." These two quantifiers are represented using the symbols $\forall$ and $\exists$ respectively, as shown in the following table.

Symbol	**Name of Quantifier**	**English Equivalent**
$\forall$	Universal quantifier	All
$\exists$	Existential quantifier	At least one

Before giving examples of formula using the symbols $\forall$ and $\exists$, one last remark is necessary. In English, we typically do not make assertions about every object in existence, even if we use the word "every." Rather, in a normal conversation, we might talk about "every student in the class" or "every grocery store in Seattle." In a particular conversational context, the set of objects that one is discussing is called the **domain of discourse**. Because some domains of discourse are so common, we have special words that combine "every" with specific domains of discourse. For example, we say "everything" or "everyone" or "everywhere" to signal when we're taking about things, people, and places respectively. Similar remarks apply to the quantifier "some:" we say "someone," "something," and "somewhere."

Confusion can arise quickly if two people are using different domains of discourse in some conversation. If you use the word "everyone" to refer to hip kids between the ages of 18 and 25, and I think interpret "everyone" to refer to all people, then I will be very confused if you tell me that "Everyone uses Twitter."

When translating a sentence into predicate logic, you must also be careful to specify your domain of discourse. For example, suppose our domain of discourse is all natural numbers (i.e., the numbers $0, 1, 2,$ and so on) and that $P(x)$ represents the claim that "x is prime." Then $(\forall x)P(x)$ represents the claim "All natural numbers are prime" and $(\exists x)P(x)$ represents the claim "At least one natural number is prime." If I said the domain of discourse is all *even* natural numbers, then $(\forall x)P(x)$ and $(\exists x)P(x)$ would respectively represents the claims "All even natural numbers are prime" and "At least one even natural number is prime."

Importantly, quantifiers always range over **variables**, not constant symbols. I will use the lower-case letters x, y, and z (with subscripts) as variables, whereas the lower-case letters a, b, and c (with subscripts) will be used as constant symbols. So $(\forall x)P(x)$ is a formula of predicate logic, but $(\forall c)P(c)$ is not. The difference between a variable and a constant is *in some ways* similar to (but not exactly like) the difference between a pronoun and a proper name in English. Although 'Kurt Gödel" always refers to the logician, the pronoun "he" can refer to different people depending upon context.

Quantifiers can range over more formula with connectives, equality symbols, and function symbols too. Let's first consider two common formula we'll encounter over and over again:

$$\text{Type A: } (\forall x)(P(x) \to Q(x))$$
$$\text{Type B: } (\exists x)(P(x) \& Q(x))$$

Type A formulae say that every object in our domain of discourse that has property P also has property Q. Type B formulae say that at least one object in our domain of discourse has both property P and property Q.

For example, suppose our domain of discourse is natural numbers. Suppose $P(x)$ represents "x is prime" and $Q(x)$ represents "is odd." Then $(\forall x)(P(x) \to Q(x))$ represents the claim that every prime number is odd, and $(\exists x)(P(x) \& Q(x))$ represents the claim that at least one prime number is odd.

Alternatively, suppose our domain of discourse is all things on earth, and suppose $P(x)$ represents "x is a person" and $Q(x)$ represents "x is mortal." Then $(\forall x)(P(x) \to Q(x))$ represents the claim that all people are mortal, whereas $(\exists x)(P(x) \& Q(x))$ represents the claim that at least one person is mortal.

Caution:

1. Quantifiers *never* occur within a predicate symbol. For example, although $(\forall x)P(x)$ is a formula of predicate logic, $P(\forall x)$ is not.

2. The formulae $(\forall x)(P(x) \to Q(x))$ and $(\forall x)P(x) \to (\forall x)Q(x)$ are *not* equivalent. For example, again assume our domain of discourse is all things natural numbers, that $P(x)$ represents "x is a prime" and that $Q(x)$ represents "x is odd." Then $(\forall x)(P(x) \to Q(x))$ represents the claim that all prime numbers are odd. That's false, as two is a prime number. In contrast, the formula $(\forall x)P(x) \to (\forall x)Q(x)$ represents the claim "If all natural numbers are prime, then all natural numbers are

odd." Given our (strange) convention about conditionals, that formula is true: since not all natural numbers are prime, the conditional has a false antecedent. In general, the formula $(\forall x)(P(x) \to Q(x))$ entails $(\forall x)P(x) \to (\forall x)Q(x)$, but not vice versa.

3. Similarly, the formula $(\exists x)(P(x)\&Q(x))$ is not equivalent to the formula $(\exists x)P(x)\&(\exists x)Q(x)$. For instance, suppose the domain of discourse is all things on earth. Suppose $P(x)$ and $Q(x)$ respectively represent "x is a pear" and "x is a quince." Then the formula $(\exists x)(P(x)\&Q(x))$ says, "At least one object on Earth is both a pear and a quince." That's obviously false. In contrast, the second formula $(\exists x)P(x)\&(\exists x)Q(x)$ says "At least one object on Earth is a pear, and at least one object on Earth is a quince." That's true: both pears and quince exist. In general, $(\exists x)(P(x)\&Q(x))$ entails $(\exists x)P(x)\&(\exists x)Q(x)$, but not vice versa.

Exercises

Suppose the domain of discourse is all natural numbers. Let $P(x)$ represent the claim "x is odd" and $Q(x)$ represent the claim "x is even." Translate the following sentences from predicate logic back into English. Which of the formula are true?

1. $(\forall x)(P(x) \vee Q(x))$
2. $(\forall x)P(x) \vee (\forall x)Q(x)$
3. $(\exists x)(\neg P(x) \vee Q(x))$.
4. $(\exists x)P(x) \to (\exists x)Q(x)$
5. $(\exists x)(P(x) \to Q(x))$. [Hint: Many students struggle with understanding what existentially quantified conditionals say. Recall that a conditional is equivalent to a particular disjunction. Think of this formula as a quantified disjunction.]

3.1 Quantifiers and Relation Symbols

We can also use quantifiers with predicate symbols that take more than one argument. Again, suppose our domain of discourse is natural numbers. Suppose $P(x)$ represents "x is prime", $Q(x)$ represents "is odd," $L(x, y)$ represents the claim that "x is less than y," and suppose c and d respectively represents the numbers 0 and 2. Then $(\exists x)(P(x)\&L(d,x))$ represents the claim that at least one natural number is prime and greater than two. Now take a moment to think about what the formula $(\forall x)((L(d,x)\&\neg Q(x)) \rightarrow \neg P(x))$ represents. Ready for the answer? It represents the claim that every natural number x that is greater than two and is not odd is also not prime.

Quantified statements become very tricky very quickly. Learning how quantifiers work in mathematics takes considerable practice, and you should not feel discouraged if you struggle at first. So below are a few examples.

Example: As above, suppose our domain of discourse is natural numbers. Suppose $P(x)$ represents "x is prime", $Q(x)$ represents "is odd," $L(x,y)$ represents the claim that "x is less than y," and suppose c and d respectively represents the numbers 0 and 2.

English	**Predicate Formula**	**Truth-Value**
All natural numbers are greater than zero.	$(\forall x)L(c,x)$	False. 0 is not greater than 0.
All natural numbers that are not equal to zero are greater than zero	$(\forall x)(\neg x = c \rightarrow L(c,x))$	True.
There is a prime number that is not odd and greater than two.	$(\exists x)(P(x)\&\neg Q(x)\&L(d,x))$	False.
Every prime number is either odd or equal to two.	$(\forall x)(P(x) \rightarrow (Q(x) \vee x = 2))$	True.
It's not the case that every number is prime.	$\neg(\forall x)P(x)$	True.
There is no prime number that is greater than two but not odd.	$\neg(\exists x)(P(x)\&(L(d,x)\&\neg Q(x)))$	True.
If every number is prime, then it's not the case that every prime number is odd.	$(\forall x)P(x) \rightarrow \neg(\forall x)(P(x) \rightarrow Q(x))$	True.

Exercises

Suppose the domain of discourse is all UW students. Let c represent you, d represent your best friend at UW, and $L(x,y)$ represent the claim "x loves y." Translate the following sentences into predicate logic.

1. Every UW student loves your best friend at UW.
2. There's a UW student who loves your best friend at UW.
3. Your best friend at UW does not love himself/herself.
4. If your best friend at UW does not love himself/herself, then some student at UW does not love him or herself.
5. Your best friend at UW loves you, and every student at UW who loves you is your best friend [Hint: Use the identity symbol].

3.2 Quantifier Order

As I said, quantifiers are hard. And they're about to get a lot harder. So I want to emphasize that mathematicians take an enormous amount of time training students to reason with quantifiers, and even some of the most famous mathematicians have made fallacious inferences because they switched the order of two quantifiers. So if you struggle at first and make lots of errors at first, keep trying.

Perhaps the trickiest thing about quantifiers is that *order matters*. Rather than explain to you how order matters, I'll start by giving examples. Trying to explain how quantifier order works to novices is like trying to explain German grammar to someone who doesn't speak any German at all: it's impossible. You need to first learn some examples, and then we can discuss the general principles.

Example 1: Suppose our domain of discourse is all people and that $M(x, y)$ represents the claim "y is the biological mother of x."

Formula	**English**	**Truth-Value**
$(\forall x)(\exists y)M(x,y)$	Everyone has a mother.	True.
$(\exists y)(\forall x)M(x,y)$	Everyone has the *same* mother, i.e., there is at least one person who is everyone's mother.	False. My mother is not your mother.
$(\forall y)(\exists x)M(x,y)$	Everyone is someone's mother.	False. I'm not a mother.
$(\exists x)(\forall y)M(x,y)$	Someone is everyone's child.	False. No one is his or her own child, for instance.
$(\exists x)(\exists y)M(x,y)$	At least one person has at least one mother.	True.
$(\forall x)(\forall y)M(x,y)$	Everyone is everyone's biological mother.	False

Example 2: Suppose our domain of discourse is all natural. Assume $L(x,y)$ represents the claim "x is less than y."

Formula	**English**	**Truth-Value**
$(\forall x)(\exists y)L(x,y)$	Everyone number x is less than some number y.	True.
$(\exists y)(\forall x)L(x,y)$	Every number is less than the *same* number, i.e.,there is some number that is greater than all numbers.	False.
$(\forall y)(\exists x)L(x,y)$	Every number y is greater than some number x.	False. Zero is greater than nothing.
$(\forall y)(\neg y = 0 \rightarrow (\exists x)L(x,y))$	Every non-zero number y is greater than some number x.	True.
$(\exists x)(\forall y)L(x,y)$	There's some number x that is less than all natural numbers y.	False. Zero is not less than itself.
$(\exists x)(\forall y)(\neg y = 0 \rightarrow L(x,y))$	There's some number x that is less than all non-zero, natural numbers.	True.
$(\exists x)(\exists y)L(x,y)$	At least one number x is less than some number y.	True. $1 < 2$.
$(\forall x)(\forall y)L(x,y)$	Everyone number is less than every number.	False

Some students ask *why* $(\forall x)(\exists y)M(x, y)$ means "Everyone has a mother" whereas $(\exists y)(\forall x)M(x, y)$ means "Everyone has the *same* mother." This is akin to asking why the sentence "I ate because I was hungry" means something different from "I was hungry because I ate." In any language, whether natural or symbolic, there are rules/conventions for how word order determines the meaning of sentences. In predicate logic, the string $(\forall x)(\exists y)$ just means something different than $(\exists y)(\forall x)$. It's just a brute fact about the language, and we could have adopted a different convention if we chose to do so.

Exercises

Suppose the domain of discourse is all things and events. Let $P(x, y)$ represent "x is the cause of y" and c represent God. Translate the following sentences into predicate logic.

1. Everything has a cause.
2. Everything has the same cause.
3. If God is the cause of everything, then everything has the same cause.
4. Everything has a cause, but not everything has the same cause.

3.3 Function Symbols and Quantifiers

Let's now use function symbols in quantified statements. Suppose the domain of discourse is all natural numbers. Suppose that $f(x, y)$ represents "x times y" and assume that c_1, c_2 and c_3 respectively represent $2, 3$ and 6. Finally, suppose $D(x, y)$ means "x is divisible by y."

Formula	**English**	**Truth-Value**
$(\forall x)(\forall y)f(x,y) = f(y,x)$	For any numbers x and y, $xy = yx$.	True.
$(\forall x)(\exists y)f(x,x) = y$	For any number x, there is some number y such that $x^2 = y$.	True.
$(\exists y)(\forall x)f(x,x) = y$	There is some number y such that $x^2 = y$ for all x.	False. This would entail that $1^2 = 2^2$.
$(\forall x)(\exists y)f(c_1,y) = x$	For any number x, there is a number y such that $2y = x$.	False. Not every number is even.
$(\forall x)(D(x,c_3) \to (\exists y)f(c_2,y) = x)$	For any number x, if x is divisible by 6, then there is a number y such that $3y = x$.	True.

3.4 Exercises

Suppose the domain of discourse is all natural numbers. Suppose that $f(x,y)$ represents "x times y" and assume that c_1, c_2 and c_3 respectively represent $2, 3$ and 6. Finally, suppose $g(x,y)$ represents "x plus y." Translate the following into the language of predicate logic:

1. Associativity of multiplication: $x(yz) = (xy)z$ for all natural numbers x, y, and z.

2. Distributive law: $x \cdot (y + z) = xy + xz$ for all natural numbers x, y, and z.

3. If $6 = 2 + (2 + 2)$, then there is some y such that $6 = 2y$.

3.5 The Upshot and Alternating Quantifiers

Although you may not realize it, you now have the tools to translate basically every mathematical statement into a formula of predicate logic. For example, it turns out that many of the most common definitions and theorems in mathematics involve statements that contain three alternating sets of universal and existential quantifiers. Such multiply quantified statements already pop up in introductory calculus textbooks. Here are some "real" examples of definitions from those textbooks:

- The definition of limit: A number y is said to be the *limit* of a sequence of real numbers $x_0, x_1, x_2 \ldots$ if for all $\epsilon > 0$, there is some natural number n such that $|x_m - y| < \epsilon$ for all natural numbers $m \geq n$.

- The definition of "continuous:" A real-valued function f is said to be *continuous* if for all x and all $\epsilon > 0$, there is some $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

- The definition of "uniform continuity:" A function f is said to be *uniformly continuous* if for all $\epsilon > 0$, there is some $\delta > 0$ such that for all numbers x, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Translating "real" definitions of mathematics into predicate logic requires considerable practice. So in a set theory or advanced logic class, you would practice representing more and more statements in predicate logic. You won't be required to do so in this course. But hopefully, with a little bit of effort, you could translate those sentences into predicate logic if I asked you to do so. Nonetheless, I provide the examples above to convince you that, in principle, predicate logic is powerful to represent fairly complex mathematical definitions and theorems. You can see the alternating quantifiers in the above statements, the use of connectives like "if-then", and the use of function symbols and identities. A central discovery of the early 20th century is that the tools of predicate logic were sufficient for all of math.

4 Aristotelian Logic

It's now time to compare the expressive power of predicate logic with that of Aristotelian logic. To do so, it suffices to consider two syllogistic forms as examples: Barbara and Celarent.

Consider the following example of Barbara first:

- All mammals are warm-blooded.

- All warm-blooded creatures can maintain body temperatures higher than the environment.

- Therefore, all mammals can maintain body temperatures higher than the environment.

We might represent this syllogism in predicate logic as follows. Suppose the domain of discourse is all animals. Let $M(x)$ represent "x is a mammal," $W(x)$ represent "x is warm blooded," and $H(x)$ represent "x can maintain body temperatures higher than the environment." Then the three sentences above can be translated as follows:

- $(\forall x)(M(x) \to W(x))$
- $(\forall x)(W(x) \to H(x))$
- $(\forall x)(M(x) \to H(x))$

Here's a second example. Consider the Aristotelian syllogism Celarent, as in the following argument:

- No reptiles have fur.
- All snakes are reptiles.
- Therefore, no snakes have fur.

We might represent this syllogism in predicate logic as follows. Suppose the domain of discourse is all animals. Let $R(x)$ represent "x is a reptile," $S(x)$ represents "x is a snake," and $T(x)$ represents "x has fur." Then the three sentences above can be translated as follows:

- $\neg(\exists x)(R(x) \& T(x))$
- $(\forall x)(S(x) \to R(x))$
- $\neg(\exists x)(S(x) \& T(x))$

You should try translating other Aristotelian syllogisms into predicate logic. Upon doing so, you'll notice that all of the formula in Aristotelian logic are fairly simple in the following ways:

- The predicates in each formula are always **unary**, i.e., they take only one argument. None takes more than one argument.
- The formulae never involve more than one quantifier. Thus, they never contain alternating quantifiers.
- The formulae never contain constant symbols. Surprisingly, one of the most common examples of a syllogism - namely, "All men are mortal. Socrates is a man. Therefore, Socrates is mortal." - is not one of the forms of Aristotelian syllogism. "Socrates" is best represented by a constant symbol, which none of the Aristotelian syllogisms contain.
- The formulae never contain identities and/or function symbols.

As we saw above, even the most basic mathematical formulae will contain relation symbols, alternating quantifiers, constant symbols, and function symbols. The moral: Aristotle's logic is insufficient for representing basic mathematical arguments. This may seem like a simple observation, but no one said this explicitly for 2000 years. Much of philosophy of mathematics, we shall see, is implicitly motivated by the limitations of Aristotelian logic. When we study Kant, for example, it becomes clear that Kant's project is at least partially motivated by the realization that logic, as understood during his time, was insufficient for proving basic geometrical and arithmetic theorems. This is why he introduces a different source for mathematical knowledge, which he calls *intuition*.