Locke's Philosophy of Mathematics

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Review

The Generality Problem: How do we know geometric proofs (esp. Euclid, Apollonius, etc.) apply/work for all triangles, circles, etc., rather than just for the objects in the particular diagrams?

Two Solutions

Question What two solutions to the generality problem have we discussed? Why were the solutions problematic?

Answer:

1. The diagram is unnecessary.
   - Problem: Euclidean diagrams are necessary to eliminate underspecification and draw inferences about intersections points of curves,that are not guaranteed by the
2. The **logic** of the proofs applies to all triangles, circles, etc.
   - Problem: The language of Aristotelian logic is insufficient for formalizing statements of Euclidean geometry.

Review of Aristotle's Logic

Syllogisms

Aristotelian arguments are **syllogisms**, like the following example of Festino.

No fish talks.
Some humans talk.

Some humans are not fish.

Festino

Exercise from Last Class: Translate syllogisms into predicate logic, and compare the formula (e.g., a single premise or the conclusion) with the mathematical formulae that you translated for homework.

No fish talks	$(\forall x)(F(x) \rightarrow \neg T(x))$
Some humans talk	$(\exists x)(H(x) \& T(x))$
Some humans are not fish.	$(\exists x)(H(x) \& \neg F(x))$

Example from last class: $(\forall x)(\forall y)(\forall z)(\exists w)C(x, y, z, w)$

Monadic Predicates

- In modern parlance, Aristotle's logic contains only monadic predicate symbols, i.e., symbols denoting properties that apply to exactly one object.
  - E.g., Aristotle's logic could represent statements "*A* is a point" or "*L* is a line."
- However, many geometric statements are relational, e.g., "*A* lies on line *BC*."

Multiple Quantifiers

- In modern parlance, Aristotle's logic contains sentences with only one quantifier.
  - E.g., Aristotle's logic allows one to say "Everything is awesome."
- However, many geometric statements involve multiple nested quantifiers
  - $(\forall x)\,(P(x) \rightarrow (\exists y)(L(y)\&O(x,y)))$ represents "Every point lies on some line" if the domain of discourse is points and lines, $P(x)$ represents "x is a point," $L(x)$ represents "x is a line," and $O(x,y)$ represents "x lies on y."

No constant symbols

- In modern parlance, Aristotle's syllogisms contains no sentences with constant symbols
  - E.g., Aristotle's logic allows one to say "Not everyone is special" but not "Joey is special."
- However, many geometric statements involve sentential connectives of precisely this sort:
  - K is a circle.

Upshot 1: It is hard to **translate** Euclid's postulates, theorems, etc. into an Aristotelian framework, let alone **prove** the theorems using Aristotle's methods.

Modern proof of Festino

1.	$(\forall x)(F(x) \rightarrow \neg T(x))$	Premise
2.	$(\exists x)(H(x)\&T(x))$	Premise
3.	$H(v)\&T(v)$	Assumption
4.	$H(v)$	$\&EL$ 3
5.	$T(v)$	$\&ER$ 3
6.	$F(v)$	Assumption
7.	$F(v) \rightarrow \neg T(v)$	$\forall E$ 1
8.	$\neg T(v)$	$\rightarrow E$ 6,7
9.	$\perp$	$\perp I$ 5,8
10.	$\neg F(v)$	$\neg I$ 9
11.	$H(v)\&\neg F(v)$	$\&I$ 4, 10
12.	$(\exists x)(H(x)\&\neg F(x))$	$\exists I$ 11
13.	$(\exists x)(H(x)\&\neg F(x))$	$\exists E$ 12

Generality of Mathematics

- The generality of geometric proofs **could have** been a concern for Greek and modern philosophers . . .
- But it wasn't: few doubted geometric proofs were general.
- Until the 19th century, the question was not **if** geometric proofs were general, but rather **why**.

> *If then the perception, that the same ideas will eternally have the same habitudes and relations, be not a sufficient ground of knowledge, there could be no knowledge of general propositions in mathematics; for no mathematical demonstration would be any other than particular: and when a man had demonstrated any proposition concerning one triangle or circle, his knowledge would not reach beyond that particular diagram. If he would extend it further, he must renew his demonstration in another instance, before he could know it to be true in another like triangle, and so on: by which means one could never come to the knowledge of any general propositions.*

Book IV, Chapter 1. Section 9. [Locke, 1975]

Generality and Locke

Moral:

- Locke accepts that geometric proofs involving diagrams are general.
- His goal is to explain **why**.

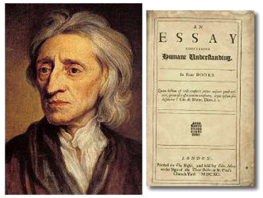
Where We're Going

This Month: Modern philosophers realized Aristotle's logic was insufficient for mathematical reasoning. They provided alternative reasons for the generality of mathematical proofs:

1. Locke's theory of abstract ideas and demonstration
2. Descartes' theory of rational insight
3. Leibniz's extension of logic
4. Kant's forms of pure intuition

Outline

Locke's Philosophy of Mathematics



Locke's explanation of mathematical generality has two parts:

1. A theory of demonstration
2. A theory of abstract ideas

Ideas

For Locke, an idea is any object of thought.

> *Idea is the object of thinking. Every man being conscious to himself that he thinks; and that which his mind is applied about whilst thinking being the ideas that are there, it is past doubt that men have in their minds several ideas, - such as are those expressed by the words whiteness, hardness, sweetness, thinking, motion, man, elephant, army, drunkenness, and others . . . (I, 1, i)*

Ideas

So for Locke, ideas include

- Sensation (e.g., blue patch),
- "Internal" reflections on our own mind (I.1.4) such as thinking, doubting, etc.
- Abstract Concepts (e.g., blueness, thinking, motion, man),
- Etc.

Locke on Knowledge

Locke defines knowledge as follows:

> *Knowledge is the perception of the agreement or disagreement of two ideas*

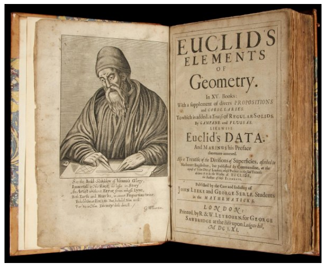
Book IV. Chapter 1. Section 2.

Locke on Mathematical Knowledge

Examples of mathematical knowledge:

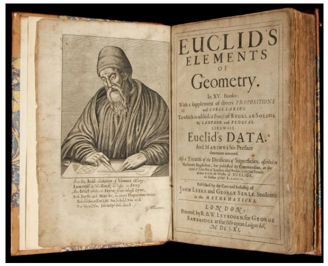
- "Two triangles upon equal bases between two parallels are equal." IV, 1, vii.
- "The three angles of a triangle are equal to two right ones." IV, 1, ix.
- ". . . that a circle is not a triangle, that three are more than two and equal to one and two." IV, 2, i.

Euclid: Two Types of Theorems



Recall, we discussed two types of theorems in Euclid. What were they? Give an example of each.

Euclid: Two Types of Theorems

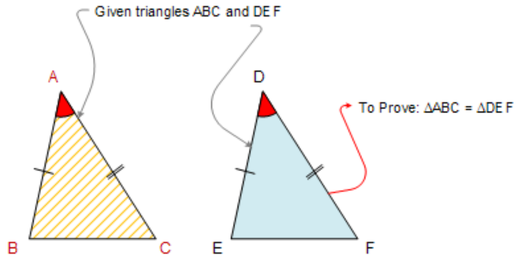


Recall, we discussed two types of theorems in Euclid:

1. Constructions
2. "Equivalences"
   - Theorems about congruity or inequalities of geometric magnitudes (angles, lengths, areas, etc.)

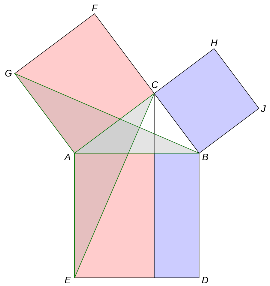
Congruence and equality of area, length, and angles

Euclid I.4



Modern Statement: Side-angle-side entails congruence.

Euclid I.47



Euclid's I.47 = The Pythagorean theorem.

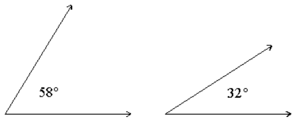
Euclid's Statement: In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

Euclid I.47

Euclid's Statement: In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

- There are no numbers mentioned at all.
- E.g., The length of the sides of the triangle are not numerically represented; nor is the area of the squares.

Euclidean Geometry and Quantification

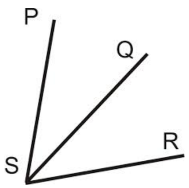


In general, geometric magnitudes (angles, lengths, areas, volumes, etc.) were not numerically quantified in Euclidean geometry, as we do today (e.g., in the above picture).

Comparing geometric magnitudes

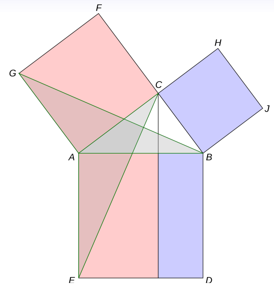
Question: How can we check whether one angle, area, or length is bigger than another geometric object of the same kind?

Euclid's Common Notions



- Euclid's **Common Notions** tell us how to compare angles (or lengths, or areas) when they are *touching or contained within one another*.
- **Common Notion 5:** The whole is greater than the part.
- In these cases, we can just see whether one angle (or segment, or circle) is contained in another.

Euclid's Comparing Geometric Quantities



- Euclid's **Common Notions** tell us how to compare angles (or lengths, or areas) when they are *contained within one another*.
- Question: How do we compare the sizes of areas, lengths, etc. that are not contained within one another (nor can they be moved to fit within one another), as the Pythagorean theorem requires?

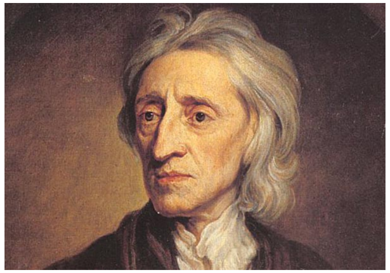
Euclid's Proofs of Equivalences

Question: How do we compare the sizes of areas, lengths, etc. that are not contained within one another (nor can they be moved to fit within one another), as the Pythagorean theorem requires?

Answer: To compare two objects O_1 and O_2 (e.g., squares), we construct a sequence of intermediate objects $C_1, C_2, \ldots C_n$ such that

- C_i is contained in or touching C_{i+1}, and therefore comparable by the Common Notions and our visual capacities,
- Repeatedly applying transitivity of equality (Common Notion 1)

Lockean Demonstration



Locke's Theory of Demonstration

Locke on Knowledge

Locke defines knowledge as follows:

> *Knowledge is the perception of the agreement or disagreement of two ideas (IV, 1, ii.)*

Important: Locke's definition seems to indicate that his primary goal is to explain theorems involving **equivalence**.

Locke on Mathematical Knowledge

Moreover, Locke's examples of mathematical knowledge are all "equivalence" statements:

- "Two triangles upon equal bases between two parallels are equal." IV, 1, vii.
- "The three angles of a triangle are equal to two right ones." IV, 1, ix.
- "… that a circle is not a triangle, that three are more than two and equal to one and two." IV, 2, i.

Intuition vs. Demonstration

When discussing knowledge, Locke distinguishes between two types:

1. Intuition
2. Demonstration

Intuitive Knowledge

Locke defines intuitive knowledge as follows:

> *For if we will reflect on our own ways of thinking, we will find, that sometimes the mind perceives the agreement or disagreement of two ideas immediately by themselves, without the intervention of any other: and this I think we may call intuitive knowledge (IV, 2, i).*

Lockean Intuition

His examples include

- "that white is not black, that a circle is not a triangle, that three are more than two and equal to one and two." IV, 2, i.

Important: With the exception of arithmetic knowledge (more on this in minute, though), the examples Locke gives involves (perhaps internally imagined) visual judgment of identity of shapes and/or colors.

Lockean Intuition: My interpretation

My interpretation: Intuitive judgments justify the application of a common notion

- Example 1: We intuitively recognize that one angle (or line segment, etc.) is contained another.
- Example 2: We intuitively recognize when two figures share a common part (e.g., line segment).

Lockean Demonstration

Locke distinguishes intuition from demonstration:

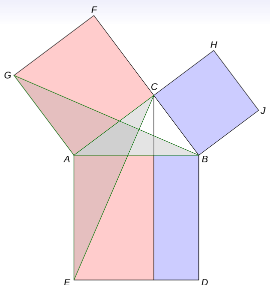
> *The next degree of knowledge is, where the mind perceives the agreement or disagreement of any ideas, but not immediately (IV, 2, vii).*

Intermediate Steps in Lockean Derivations

What is a demonstration, according to Locke?

> *By which it is plain that every step in reasoning that produces knowledge, has intuitive certainty; which when the mind perceives, there is no more required but to remember it, to make the agreement or disagreement of the ideas concerning which we inquire visible and certain. So that to make anything a demonstration, it is necessary to perceive the immediate agreement of the intervening ideas, whereby the agreement or disagreement of the two ideas under examination (whereof the one is always the first, and the other the last in the account) is found (IV, 2, vii).*

Euclid's Demonstrations



Compare with Euclid's proofs: To compare two objects O_1 and O_2 (e.g., squares), we construct a sequence of intermediate objects $C_1, C_2, \ldots C_n$ such that

- C_i is contained in or touching C_{i+1}, and therefore comparable by the Common Notions and our visual capacities,
- Repeatedly applying transitivity of equality (Common Notion 1)

Lockean geometric demonstrations

Locke's examples of demonstrations also seem to fit this model:

> *Thus, the mind being willing to know the agreement or disagreement in bigness between the three angles of a triangle and two right ones, cannot by an immediate view and comparing them do it: because the three angles of a triangle cannot be brought at once, and be compared with any other one, or two, angles; and so of this the mind has no immediate, no intuitive knowledge. In this case the mind is fain to find out some other angles, to which the three angles of a triangle have an equality; and, finding those equal to two right ones, comes to know their equality to two right ones.*

Locke on Demonstration

My interpretation: There is an uncanny analogy between

- Locke's discussion of intuition and Euclid's common notions,
- Locke's theory of demonstration, and Euclid's method for demonstrating geometric equalities.

Problem: We still haven't explained why geometric proofs are general . . .

Abstract Ideas

Locke on Origin of Ideas

Locke believes all ideas originate from sensation (of the world or ourself):

> *The steps by which the mind attains several truths. The senses at first let in particular ideas, and furnish the yet empty cabinet, and the mind by degrees growing familiar with some of them, they are lodged in the memory, and names got to them . . .*

Operations on Ideas

Locke hypothesizes that we acquire new ideas by performing operations on existing ones.

Question: For our purposes, the two most important operations are what? Give examples of each.

1. Compounding
2. Abstraction

Compounding Ideas

Compounding: Locke argues we can acquire new ideas by

> *Combining several simple ideas into one compound one; and thus all complex ideas are made (II, 7, i).*

Compounding Ideas

Example: Abstract ideas of numbers are formed by compounding:

> *For the idea of two is as distinct from that of one, as blueness from heat, or either of them from any number: and yet it is made up only of that simple idea of an unit repeated; and repetitions of this kind joined together make those distinct simple modes, of a dozen, a gross, a million (II, 13, i).*

Abstraction

Locke describes the process of abstraction as follows:

> *if every particular idea that we take in should have a distinct name, names must be endless. To prevent this, the mind makes the particu1ar ideas received from particular objects to become general; which is done by considering them as they are in the mind such appearances, - separate from all other existences, and the circumstances of real existence, as time, place, or any other concomitant ideas. This is called abstraction, whereby ideas taken from particular beings become general representatives of all of the same kind (II, 11, ix).*

Abstraction an Example

Example: Blueness

- I saw the sky this morning (!), I saw my (blue) Nalgene bottle.
- I recognize the sky, my bottle, etc.. share a color.
- I can form an abstract/general idea of blueness, by **removing** particular features of my perception of my bottle (e.g., its shape, texture, etc.)

Abstraction and representation

Example: Blueness

- My abstract/general idea of blueness **represents** all of my particular ideas, and whatever I learn about the abstract idea blueness (e.g., it is darker than whiteness) must be true of all particular instances of the abstract idea.

Abstract ideas of triangles

Abstract ideas have no properties that are not shared by all instances:

> *For example, does it not require some pains and skill to form the general idea of a triangle, (which is yet none of the most abstract, comprehensive, and difficult,) for it must be neither oblique nor rectangle, neither equilateral, equicrural, nor scalenon; but all and none of these at once. In effect, it is something imperfect, that cannot exist; an idea wherein some parts of several different and inconsistent ideas are put together (IV, 7, ix.)*

Combing abstraction and compounding

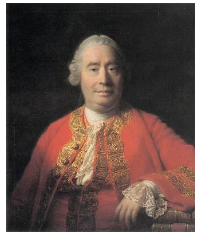
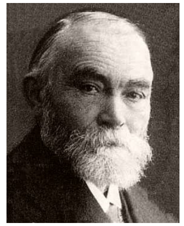
Most of our mathematical ideas arise from a combination of abstraction and compounding:

- The idea of a **unit** is abstracted from particular objects (eliminate its color, shape, spatial location, etc.)
- The idea of a **number** is a combination of units.

Compare with Euclid's definition next week ...

Up Next

Where We're Going

Why Locke's solution is/was unpopular

Today's Response Question

Response Question: Explain how one might acquire the general idea of a triangle via abstraction.

References I

Locke, J. (1975). *An essay concerning human understanding*. Clarendon Press, Oxford.