Philosophy of Mathematics: An Introduction

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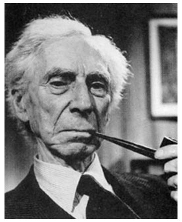
Philosophy and Mathematics



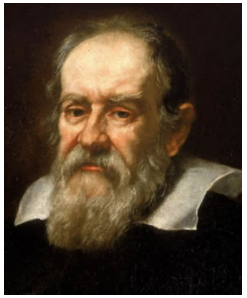
Fact: For over two thousand years, mathematics was been tremendously influential on philosophy and vice versa.

Mathematics and Philosophy

"Let none ignorant of geometry enter here."

Purported Inscription above Plato's Academy

Mathematics and Philosophy



The universe cannot be read until we have learned the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word. Without these, one is wandering about in a dark labyrinth.

Mathematics for the philosopher

Question: Why have philosophers cared so deeply about mathematics?

Answer: Many reasons. Here's one:

- Philosophers are interested in how to acquire knowledge.
- For two millennia, mathematics was seen as the paradigm of knowledge and certainty.
- So philosophers wanted to figure out how mathematics worked.

Philosophy for the mathematician

Question: Why did many mathematicians care so deeply about philosophy?

Answer: Many reasons, including

- Many mathematicians (e.g. Descartes) were philosophers too!
- Not a clear separation between mathematics and philosophy:
  - E.g., Questions about about infinity in mathematics raised questions about we can known about God, how powerful God is, and so on.

Course Goals

Central Goal: Explain how changes in mathematical practice changed philosophical theorizing, and vice versa.

At least in Western Europe prior to 1872 ...

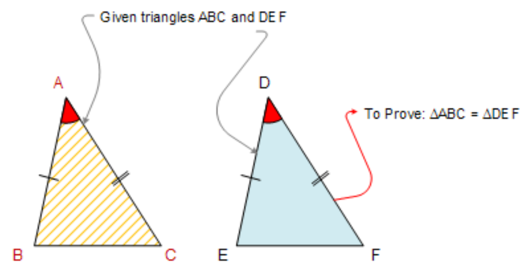
Course Structure

Course Structure: We will focus on two related themes

1. Mathematical generality and abstract ideas, and
2. Geometry vs. Arithmetic, and Empiricism vs. Rationalism

Abstract Ideas

Euclid I.4



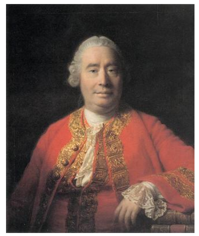
Modern Statement: Side-angle-side entails congruence.

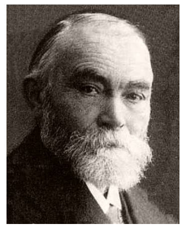
Question: The proof makes use of a particular diagram. How do we know it applies to **all** triangles?

Abstract Ideas

Theorem about **all** triangles = Theorem about abstract idea of a triangle

Critics of Abstract Ideas

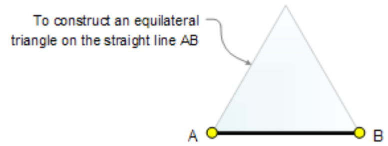




Locke's doctrine is vehemently criticized, but a rigorous, formal logic for addressing Locke's worry is not developed until the end of the nineteenth century.

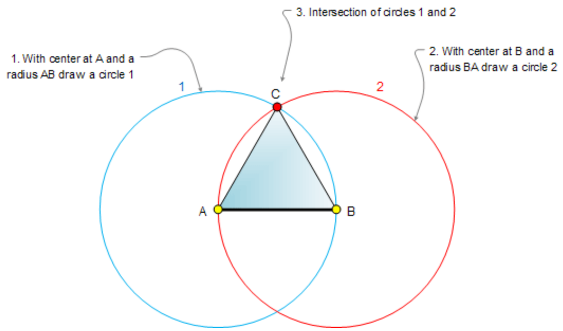
Experience vs. Reason

Euclid I.1



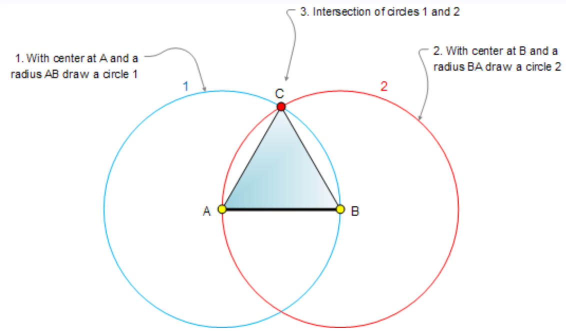
Modern Statement: You are given a line segment AB. Using only a straight-edge and compass, construct an equilateral triangle with AB as one of its sides.

Euclid I.1



Proof: Draw two circles that contain AB as their radius as shown in the diagram. Connect points A and B to the point C, which lies at the intersection of the two circles. The triangle ABC is equilateral because $AB = AC$ (as both are radii or circle 1) and $AB = BC$ (because both sare radii of circle 2).

Euclid I.1



Important: Euclid's proof uses a diagram and relies on your ability to see and draw particular shapes.

Many mathematical proofs (e.g., in knot theory) rely on diagrams.

Other proofs do not seem to engage your vision in the same way.

The Irrationality of the Square Root of Two

Consider the following theorem, which is stated (in a very different form!) in Aristotle's **Posterior Analytics**.

Modern Statement: The square root of two is irrational.

The Irrationality of Root 2

Theorem

The square root of two is irrational.

Proof: Suppose for the sake of contradiction that $\sqrt{2} = \frac{m}{n}$ for two integers m and n. We may assume that the fraction $\frac{m}{n}$ has been "reduced", so that m and n contain no common divisors. In particular, m and n are not both even (i.e., divisible by two).

Since $\sqrt{2} = \frac{m}{n}$, it follows that $2n^2 = m^2$. So m^2 is even. Now m^2 would be odd if m were odd (because the product of two odds is odd), and so m must be even. So $m = 2d$ for some $d > 0$. It follows that

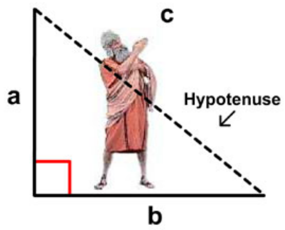
$$2n^2 = m^2 = (2d)^2 = 4d^2$$

and hence $n^2 = 2d^2$. So n^2 is even. Again, this entails that n is even, contradicting the assumption that n and m are not both even.

Mathematics and Sensory Experience

- Unlike the first proof, this proof does not seem to rely on your ability to visualize anything.
- Still other proofs rely on a combination of symbolic and visual reasoning.

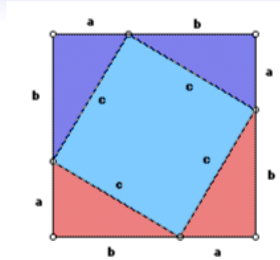
Pythagorean Theorem



Modern Statement: If c is the length of the hypotenuse of a right triangle, and a and b are the lengths of the other two sides, then

$$a^2 + b^2 = c^2$$

Pythagorean Theorem - One Proof



$$\begin{aligned} c^2 &= \text{Area of light blue square} \\ &= \text{Area of big square - Area of Four Triangles} \\ &= (a+b)^2 - 4 \cdot (\frac{1}{2}ab) \\ &= (a^2 + 2ab + b^2) - 2ab \\ &= a^2 + b^2 \end{aligned}$$

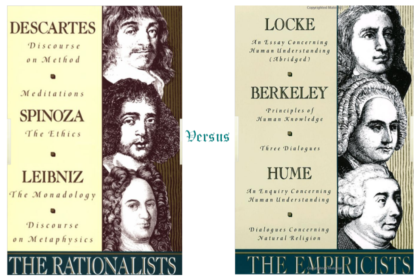
Question: Where does this proof use the assumption abc is a **right** triangle?

Mathematics and Sensory Experience

Central Question: To what extent do mathematical proofs rely on our senses, in particular, our ability to see, draw, and feel particular objects?

To what extent do they depend upon some sort of abstract, non-visual reasoning?

Knowledge and Experience



Empiricists = All knowledge comes from the senses
Rationalists = Some knowledge is innate; some comes from "rational intuition."

Knowledge and Experience

It is not a coincidence that

- **Empiricists**
  - Mostly British
  - Newton preferred geometric proofs to algebraic ones
- **Rationalists**
  - Mostly French and German
  - Descartes and Leibniz's pioneered particular algebraic techniques