Kant's Philosophy of Mathematics

Conor Mayo-Wilson

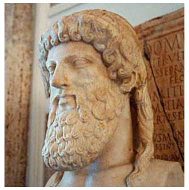
University of Washington

Phil. 373
February 27th, 2017

Review

Last Month: To what extent does justification of mathematical theorems depend upon our senses, pure reason, symbolic calculation, or some combination there of?

Plato on mathematics

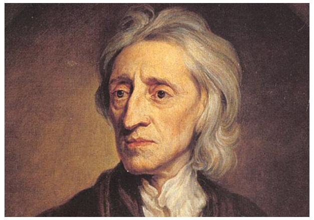


- Mathematical forms are insensible and unchanging ⇒ The senses are misleading sources of mathematical judgments.
- Mathematical concepts (e.g., equality) are **recollected**

Aristotle's mathematics

- Mathematical forms are instantiated in physical objects ⇒ The senses can aid us in learning mathematical theorems

Locke's philosophy of mathematics



- Mathematical ideas are **abstracted** and **compounded** from ideas acquired via senses and reflection.
- Mathematical demonstrations (i.e. proofs) require **intuition**, i.e., the ability to immediately recognize whether some pairs of objects are equal or congruent.
  - It is unclear whether this intuition is visual for geometric ideas.

Leibniz

- Mathematical theorems are **necessary**.
  - Necessary = True in all possible worlds = Negation is contradictory

Leibniz

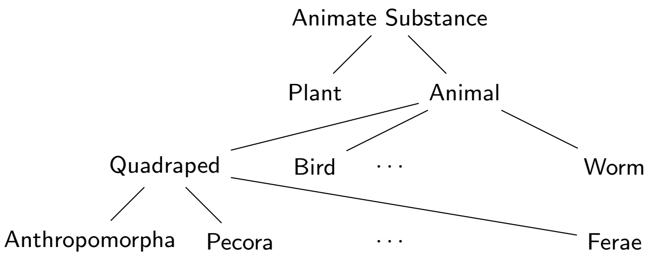
- Recognizing the validity of any given proof requires only the ability to recognize it is an instance of a series of Aristotelian syllogisms.
- **Rational insight** is required to recognize that Aristotle's syllogims have a valid form and that mathematical axioms are necessary; this cannot be learned via the senses.

Leibniz

Leibniz also claimed that mathematical theorems are **analytic**.

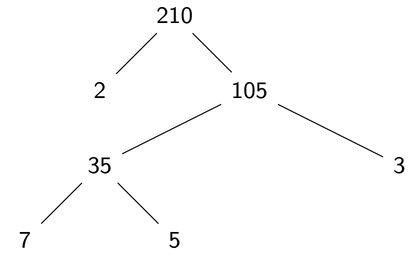
- Analytic = Subject contained in predicate.
- E.g., A swan is a bird.

Leibniz on Analyticity



Concept containment can be visualized by trees.

Prime Factorization and Analyticity



Just as prime factorization can.*

* Tree may not me the right word in factorization

Leibniz's Universal Characteristic

Leibniz's Universal Characteristic: An attempt to assign numbers to concepts and then verify whether a statement is analytic by use of facts about prime factorization.

Leibniz's Motivation

Leibniz's Big Idea:
IF

- Mathematical theorems are necessary truths;
- All necessary truths are analytic and vice versa; and
- All analytic truths can be verified by use of Leibniz's universal characteristic,

THEN

- Mathematical theorems can likewise be verified by use of Leibniz's universal characteristic.

Kant's Intellectual Inspiration

If you thought all of that was confusing, now you get to see it all put together . . .

Kant's main distinction

- **A priori** = Justified by something other than the senses
  - **Note:** You might *come to believe* mathematical truths via the senses (e.g., by having someone tell you), but the justification is independent of the senses.
  - **Analogy:** You might come to believe a suspect is guilty via hearsay, but you have to justify his guilt differently.
- **Analytic** = Subject contained in predicate = Denial is a contradiction

Kant's Philosophy of Mathematics

Idea	**Predecessors**
Mathematics is **a priori**	Plato, Leibniz
Proof requires **construction in intuition**	Descartes
Proof requires immediate, intuitive recognition of equalities and congruence	Locke
Analytic = Subject contained in predicate = Denial is contradiction	Leibniz

Kant's Departure

	Analytic	Synthetic
A Priori	Conceptual truths	Mathematics; Ethics
Empirical		Empirical science

Kant rejects the identification of analytic and *a priori*.

Mathematics is synthetic

A priori vs. Analyticity

- Like Plato and Leibniz, Kant thinks mathematical truths are universal (i.e., exception-less) and necessary (in a way to be explained).
- Since we can't justify universal, necessary truths by appeal to finitely many observations, mathematical theorems must be *a priori*, i.e., their justification requires something other than sense data.
- But why are they **synthetic?**

Kant on the concept of a triangle

Give the concept of a triangle to a philosopher and have him find out, in his manner, how the sum of its angles may be related to the right angle. He has then nothing but the concept of a figure that is enclosed in three straight lines, and the concept of the same number of angles in that figure. Let him now contemplate this concept for as long as he wants, he will ascertain nothing new. He can analyze and clarify the concept of a straight line, or of an angle, or of the number three, but he cannot come upon any other properties, which simply are not to be found in these concepts.

[Kant, 1999, A717/B745]

Kant on the concept of a sum

One might well at first think: that the proposition 7 + 5 = 12 is a purely analytic proposition that follows from the concept of a sum of seven and five according to the principle of contradiction. However, upon closer inspection, one finds that the concept of the sum of 7 and 5 contains nothing further than the unification of the two numbers into one, through which by no means is thought what this single number may be that combines the two. The concept of twelve is in no way already thought because I merely think to myself this unification of seven and five, and I may analyze my concept of such a possible sum for as long as may be, still I will not meet with twelve therein.

[Kant, 2004, §4:269]

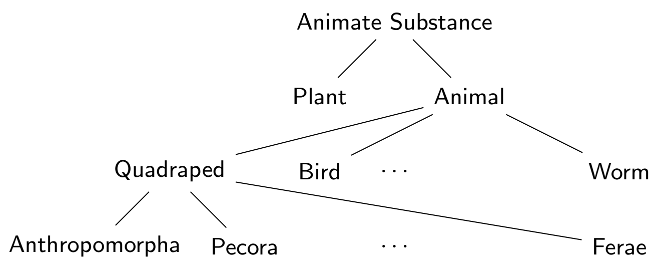
Kant on the concept of a sum

- Interpretation of Kant's views on arithmetic remains an active area of research.
- Here is one interpretation ...

Kant and Leibniz

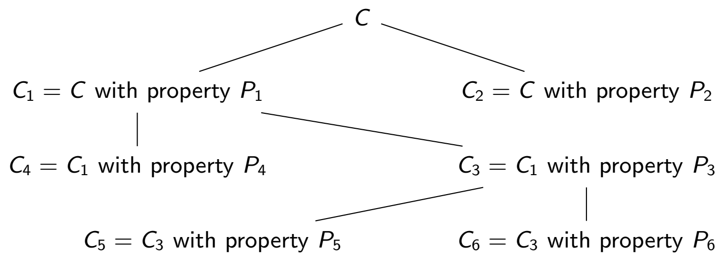
- Kant borrows from Leibniz's ideas on concept containment.
- Concept containment is indicated via trees, which are constructed in a particular way.

Concept Containment



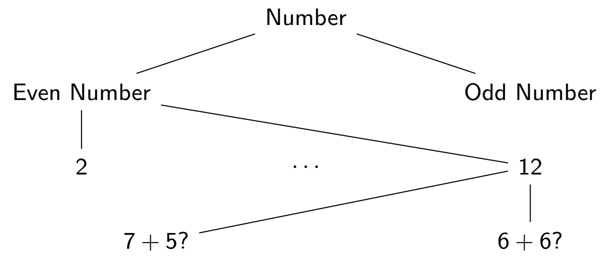
Notice that as you go up the three, the objects denoted by the concepts are nested (i.e. subsets of one another).

Concept Containment



Notice the "tree" is an actual tree; no branches interesect down the tree after splitting.

Kant on the concept of a sum



- Attempts to place equivalent representations of the same number in the tree create problems [Anderson, 2004]

How are proofs possible?

Question: But how can we **prove** a theorem, if its negation does not entail a contradiction (or if its negation is possible in some sense)?

Answer: By construction in intuition . . .

Construction in intuition

Construction in geometry

Give the concept of a triangle to a philosopher and . . . Let him now contemplate this concept for as long as he wants, he will ascertain nothing new. . . . But let the geometer take up this question. He begins forthwith to construct a triangle. Because he knows that two right angles taken together amount to exactly as much as all the adjacent angles taken together that can be erected from a point on a straight line, he therefore extends one side of his triangle and gets two adjacent angles, which are equal to two right angles taken together. Of these two angles, he now divides the exterior one by erecting a line parallel to the opposite side of the triangle, and he sees that here an exterior adjacent angle is produced that is equal to an interior angle, and so on. In this way, through a chain of inferences, always led by intuition, he arrives at a fully evident and (at the same time) universal solution to the question.

[Kant, 1999, A717/B745]

Construction in intuition

- In geometry, Kant's discussion of "construction" describes the act of mental drawing.
- What about arithmetic?

Kant on the concept of a sum

. . . one finds that the concept of the sum of 7 and 5 contains nothing further than the unification of the two numbers into one, through which by no means is thought what this single number may be that combines the two. . . . One must go beyond these concepts, in making use of the intuition that corresponds to one of the two, such as ones five fingers, or (like Segner in his arithmetic) five points, and in that manner adding the units of the five given in intuition step by step to the concept of seven. One therefore truly amplifies ones concept through this proposition 7 + 5 = 12 and adds to the first concept a new one that was not thought in it; that is, an arithmetical proposition is always synthetic, which can be seen all the more plainly in the case of somewhat larger numbers, for it is then clearly evident that, though we may turn and twist our concept as we like, we could never find the sum through the mere analysis of our concepts, without making use of intuition.

[Kant, 2004, §4:269]

Kant and the unity of a number

(• • • • • • •)(• • • • •)

7 5

• • • • • • • • • • • •

12

- Construction involves the repetition of some image, like counted fingers, dots, or dashes.

Certainty

Certainty?

Question: If the denial of a mathematical theorem is not a contradiction, then how can we be **certain** in mathematical truths? How can they be **necessary**?

Answer: They are necessary truths about objects **as we experience them**, not about objects as they are.

Appearances vs. Things-in-themselves

- For Kant, theorems about triangles, circles, etc. may not be true (or even approximately true) of physical objects ("objects in-themselves").
- Rather, mathematical theorems are about how we experience/sense them. They are **necessary** for beings like us, in virtue of the way our senses work.

Kant on preconditions of experience

Why are mathematical theorems necessary for the way we experience world?

- We can't sense objects, except as being in space in time.
  - Try to imagine a triangle that is nowhere in space. Can't do it?
  - Try to recall a pitch that is nowhere in time? Can't do it?
- In other words, *regardless of which sense you employ*, the object of your sensation is somehow spatio-temporally situated.
  - Notice similarity to previous views on common sense and primary qualities.
- In proofs, we use only features of space and time, and so we can be sure the results hold for all objects that we might experience.

A prioricity revisited

. . . propositions which relate merely to this form of sensory intuition will be possible and valid for objects of the senses; also, conversely, that intuitions which are possible a priori can never relate to things other than objects of our senses.

[Kant, 2004, §9, 4:282]

Mathematics and different beings

Question: Would beings unlike us (e.g., aliens) have different mathematics?

Answer: Possibly. God's concept of a triangle is complete, and he requires no intuition justify theorems.

Up Next: Kant's legacy and the foundations for calculus

Response Question

Response Question: Give your own examples of truths that are *a priori* and analytic, synthetic *a priori*, and synthetic *a posteriori*. Explain in what way we can be certain of mathematical truths according to Kant.

References I

Anderson, R. L. (2004). It Adds Up After All: Kant's Philosophy of Arithmetic in Light of the Traditional Logic. *Philosophy and Phenomenological Research*, 69(3):501–540.

Kant, I. (1999). *Critique of pure reason*. Cambridge University Press.

Kant, I. (2004). *Prolegomena to any future metaphysics: with selections from the Critique of pure reason*. Cambridge University Press, Cambridge.