Geometrical Exactness

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Knowing how vs. Knowing that

In English, the word "knows" can be followed by

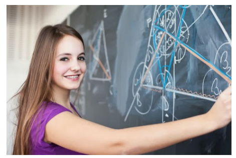
- Persons, Places and Things
- How to φ (e.g., ride a bicycle)
- That φ (e.g., that the bus arrives at 6:18AM)

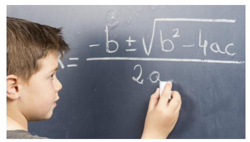
Knowing in mathematics

Exercise: Discuss whether there are similarly different uses of the word "know" in mathematics.

- Talk to your neighbors, and then we'll brainstorm collectively.

Knowing how vs. Knowing that





In mathematics, the word "knows" can be followed by

- Disciplines, objects, algorithms (e.g., calculus, $\sqrt{2}$, the Euclidean algorithm, the definition of a Turing Machine, etc.)
- How to φ (e.g., construct an equilateral triangle, solve cubic equations, differentiate, etc.)
- That φ (e.g., that $\sqrt{2}$ is irrational)

Knowing that in mathematics

Today: There is a widely-accepted view about the conditions under which we know **that** some mathematical theorem is true.

Knowing that in mathematics

- **Question:** Why do we know that $\sqrt{2}$ is irrational?
- Common Answer: Because we can prove so!

Norms of Proof

- **Puzzle:** But not every mathematical argument is a proof.
- **Question:** What makes an argument a “good” proof?
- Answer: The argument should be **rigorous**.
- Okay, but what does “rigorous” mean?

Rigor

Thanks to 20th century logic, we have a relatively good characterization of **rigor**.

Here is a standard story.

Arguments

An **argument** is a series of assertions $\varphi_1, \varphi_2, \ldots, \varphi_n$ where each φ_i is either an

- Assumption or
- Purported consequence of the assertions $\varphi_1, \ldots \varphi_{i-1}$.

Formal Derivation

∗54·43. $\vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54\cdot26 . \supset \vdash :. \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

$[*51\cdot231] \qquad \equiv . \iota`x \cap \iota`y = \Lambda .$

$[*13\cdot12] \qquad \equiv . \alpha \cap \beta = \Lambda \qquad (1)$

$\vdash . (1) . *11\cdot11\cdot35 . \supset$

$\vdash :. (\exists x, y) . \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda \qquad (2)$

$\vdash . (2) . *11\cdot54 . *52\cdot1 . \supset \vdash .$ Prop

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

Roughly, a **formal derivation** is argument where each φ_i is either an

- Axiom or
- A (syntactic) consequence of previous assertions obtainable via a rule of inference.

Formal Derivation

Discussion: Find at least one neighbor who has taken a logic course. How did derivations in your logic course differ from mathematical proofs in textbooks?

Formal Derivation

∗54·43. $\vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

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A **formal derivation** is argument where each φ_i is either an

- Axiom or
- A (syntactic) consequence of previous assertions obtainable via a mechanical rule of inference.
  - So even a machine can determine whether one assertion follows from the previous ones!
  - Most mathematical proofs in textbooks and journals cannot be so easily checked by a machine.

Knowing that in mathematics

- A proof is **rigorous** to the extent that it "indicates" the existence of a formal derivation [**?**]. So we've argued
- **Definition:** To **know that** a mathematical theorem is true = To have a rigorous proof = To have an argument indicating the existence of a mechanically-checkable formal derivation.

Knowing that in mathematics

Disclaimer: No one, I think, would endorse the view I've just sketched so crudely stated, but pieces of it appear everywhere.

Knowing objects and procedures

Roughly, **?** argues that debates about "exactness" in geometry in the modern period are analogous to debates about the "rigor" of proofs in the 19th and 20th centuries.

Knowledge	**Means**	**Norm**	**Ideal**
That φ	Proof	Rigor	Mechanically -checkable proof
How to make $\mathcal{O}$, or of $\mathcal{O}$	Construction	Exactness	Up-for-debate . . .

Knowing that in mathematics

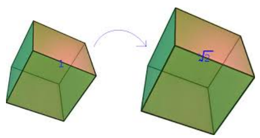
Question: According to Bos, how did the importance of "rigor" in proofs change in the 17th and 18th centuries? What about the importance of exactness?

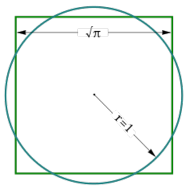
- Talk to your neighbors.

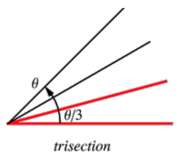
Answer: Rigorous proofs became less important. Exactness, however, remained fundamental.

Classical Construction Problems

"Impossible" Constructions



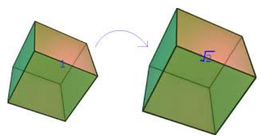


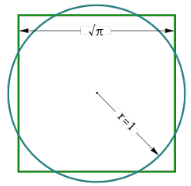


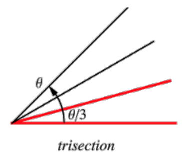
You may have previously "learned" that the Greeks posed several problems that were impossible to solve, namely,

- Double a cube - Given a cube, construct another that is twice the volume of the first.
- Square a circle - Given a circle, construct a square of the same area.
- Trisect an angle - Given an angle, divide it into three equal parts.

"Impossible" Constructions







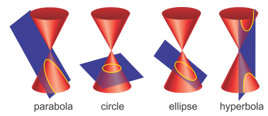
These problems are impossible using **straight-edge and compass** constructions.

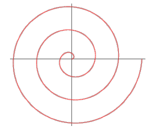
But the Greeks allowed other types of constructions . . .

Types of Constructions

You tell me! What were three types of constructions (and problems) described by Pappus?

Types of Constructions





- Planar - Requiring circles and lines (i.e., straight edge and compass)
- Solid - Requiring parabolas and hyperbolas (i.e. conic sections)
- Line-like - Requiring spirals, quadtrixes, etc. (curves producible only with movable instruments)

Types of Constructions

- Some constructions that are impossible using circles and lines are possible using conic sections.
- **Question:** Which construction methods are exact?

Exactness

Justifying Exactness

?, §1.6 describes four ways in which concepts of exactness were justified.

Discussion: What were the four methods? Try to think of an example of each type of justification, and if you can't think of a historical example, try to make up a realistic one.

Justifying Exactness

?, §1.6 describes four ways in which concepts of exactness were justified:

- Appeal to authority (e.g., Pappus)
- Idealization of practical tools (e.g., straight-edge, compass, cones)
- Philosophical analyses of geometrical intuition
- Appreciating the resulting mathematics

You tell me! If a geometer were to appeal to Pappus' authority for geometric constructions, what types of constructions would she think were exact?

Pappus on Exactness

Answer: All three construction methods (planar, solid, and line-like) were acceptable if **they matched the problem**.

> *Among geometers it is in a way considered to be a considerable sin when somebody finds a plane problem by conics or line-like curves and when, to put it briefly, the solution of the problem is of an inappropriate kind.*

Pappus on Exactness

Moral?

- This passage was interpreted in many ways, in part, because it wasn't clear whether Pappus was following it.
- Common interpretation: If planar methods suffice, use them. If conic methods suffice and planar ones don't, use them. Otherwise, use line-like constructions.

Up Next

Where We're Going

Leibniz on intuition

Today's Response Question

Response Question: Distinguish three types of constructions. What is "exactness"? How did different mathematicians and philosophers justify particular constructions as exact?