Eudoxus' Theory of Proportion, Greek Number Theory, and Incommensurability

Conor Mayo-Wilson

University of Washington

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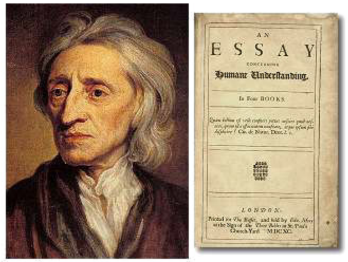
Review



Unit 1: Generality and Abstract Ideas

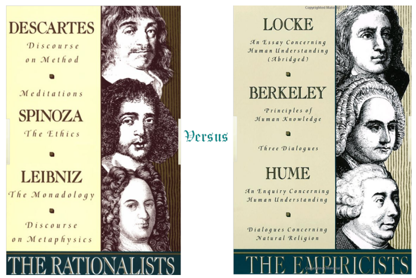
- What makes a proof general?
- Are there "abstract ideas" of triangles, circles, etc. that can explain why a proof is general?

Two New Themes

Unit 2: Empiricism vs. Rationalism, Geometry vs. Arithmetic

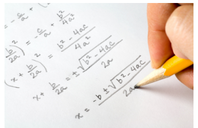
1. How are mathematical concepts acquired? How do mathematical theorems become known?
2. Are geometric magnitudes more conceptually primitive than numbers? Vice versa? Are geometric proofs more rigorous than arithmetic ones? Vice versa? Are proofs in one subject more general?

Empiricism vs. Rationalism



Empiricists = All knowledge comes from the senses
Rationalists = Some knowledge is innate; some comes from "rational intuition."

Empiricism vs. Rationalism



Question: What is the relationship between the two themes?

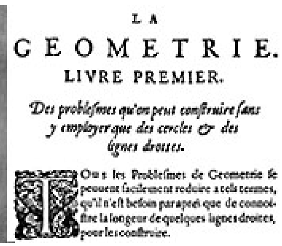
Answer:

- Paradigmatic rationalists (Descartes and Kant) develop and promote algebraic techniques that lead to the dethroning of geometry as the paradigm of knowledge.
- Not coincidentally, algebraic proofs rely less on visual reasoning than do geometric proofs.

Disclaimer

Disclaimer: The story is a bit more complicated.

- The common distinction between rationalists and empiricists blurs together important distinctions among the rationalists (respectively empiricists), it and ignores similarities between certain pairs of empiricists and rationalists.
- No one in the modern period (esp. Descartes) advocated the total elimination of geometric reasoning; it was still a paradigm source of knowledge.



LA
GEOMETRIE.
LIVRE PREMIER.
Des problesmes qu'on peut construire sans y employer que des cercles & des lignes droites.
TOus les Problesmes de Geometrie se peuvent facilement reduire a tels termes, qu'il n'est besoin par apres que de connoistre la longeur de quelques lignes droites, pour les construire.

Today: The division between geometry and arithmetic prior to (and well-after) Descartes

General Lesson: For much of the history of mathematics, geometry and arithmetic were regarded as separate, fairly unrelated disciplines of mathematics:

- Points in the plane were not represented by pairs of numbers,
- Curves in the plane were not represented by equations, etc.

This division had at least three important scientific, mathematical, and philosophical consequences ...

Eudoxus' Scientific Importance

Consequence 1: Scientific applications of geometry (e.g., use of geometric curves to study motion) could not immediately employ arithmetic or algebraic techniques (e.g., descriptions of curves via equations). Nor vice versa.

- Because geometry, rather than arithmetic, was often seen as a paradigm for rigor, this meant that quantification of physical phenomena (e.g., motion) took centuries.

Eudoxus' Mathematical Importance

Consequence 2: Mathematical discoveries in geometry did not translate to discoveries in arithmetic or vice versa.

- In particular, the discovery of the incommensurability of the diagonal did not require the extension of the number concept to include irrational numbers. Geometry could work with irrational quantities, even if arithmetic could not.
- In the third unit, we will see the restriction of the number concept is consonant with Aristotelian attitudes about the "actual infinite."

Eudoxus' Philosophical Importance

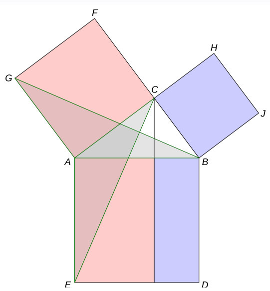
Consequence 3: Philosophical theories of knowledge, justification, and perceptual evidence had to explain both geometric and arithmetic knowledge; understanding one was not obviously sufficient for understanding the other.

Arithmetic and Geometry

Today: Why were arithmetic and geometry regarded as different in Greek mathematics and in Europe after words?

1. The discovery of incommensurability
2. Eudoxus' theory of proportion

Euclid I.47



Euclid's I.47 = The Pythagorean theorem.

Euclid's Statement: In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

Euclid I.47

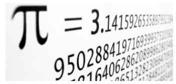
Euclid's Statement: In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

- There are no numbers mentioned at all.
- E.g., The length of the sides of the triangle are not numerically represented; nor is the area of the squares.

Geometric Magnitudes vs. Numbers

- In Greek mathematics, geometric magnitudes were not numerically quantified.
- Today, however, the Greeks are credited with a number of outstanding discoveries that seem to require numerically quantifying geometric magnitudes.

Geometric Magnitudes vs. Numbers



- For example, π is the ratio of the circumference of a circle to its diameter, i.e., a ratio of two geometric magnitudes.
- Archimedes is said to have shown $\pi < 22/7$.
- The Greeks are also credited with the formula for the volume of cones, cylinders, etc.

Question: How did the Greeks avoid numerical representation of geometric magnitudes, and yet calculate digits of π, volumes of cones, cylinders, etc.?

Answer: Eudoxus' theory of proportion.

Eudoxus' Theory of Proportions

Ratios

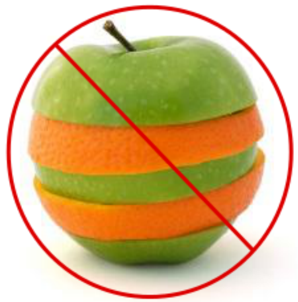
Definition 3: A ratio is a sort of relation in respect of size between two magnitudes **of the same kind**.

Same type ratios

Examples:

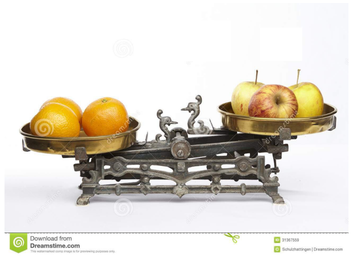
- Line segment: Line segment
- Area: Area
- Number: Number

Same type ratios



In Eudoxus' theory, a line segment **cannot** stand in a ratio to an area or number.

Comparing Ratios



However, in Eudoxus' theory, ratios can be **compared**.

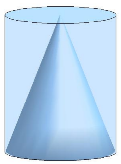
- E.g. The ratio between two line segments can be compared to the ratio between two numbers.
- E.g. The ratio between two line segments can be compared to the ratio between two areas.

Comparing Ratios

To permit comparisons, Eudoxus defines when

- Two ratios are equal, and
- One ratio is greater than another.

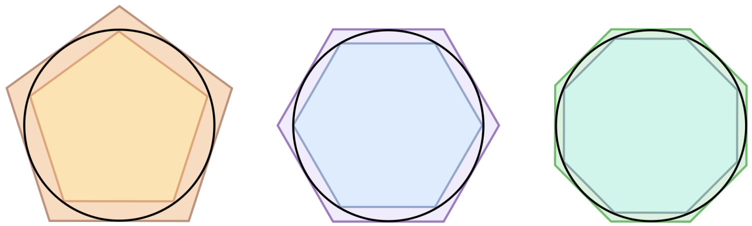
Comparing Ratios



Eudoxus' definitions allow him to state (and prove) the following theorem:

- Let C be a cone and D be a cylinder with the same base and height.
- Volume(C): Volume(D) $= 1 : 3$
- Note: The left-hand side is a ratio of volumes, and the right-hand side is a ratio of numbers.

Comparing Ratios



Similarly, Eudoxus' definitions allow Archimedes to state (and prove) that $223/71 < \pi < 22/7$

- Note: π is a ratio of two curves (a circumference and radius), whereas $22/7$ is a ratio of two numbers.

Equal Ratios

Definition 5: Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

Equality of Ratios

In modern notation, write $w : x = y : z$ if for every pair of natural numbers n, m

$$\begin{aligned} nw > mx &\Rightarrow ny > mz \\ nw = mx &\Rightarrow ny = mz, \text{ and} \\ nw < mx &\Rightarrow ny < mz. \end{aligned}$$

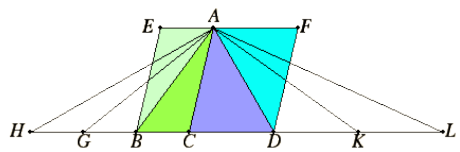
Notes:

- The (in)equalities in the defieniens concern magnitudes themselves, not ratios. So the definition is not circular.
- The antecedent (and consequent) of each conditional concerns comparing magnitudes **of the same type**.

Inequality of Ratios

Definition 7: When, of the equimultiples, the multiple of the first magnitude exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said to have a greater ratio to the second than the third has to the fourth.

Equal Ratios



In an ideal world, I would show you how these definitions are used. We don't have time. See VI.1 for the simplest use of them.

We'll return to these definitions at least twice more in the course.

Number Theory

Units and Numbers

1	|	6	𝍸|
2	||	7	𝍸||
3	|||	8	𝍸|||
4	||||	9	𝍸||||
5	𝍸	10	𝍸𝍸

Definition: A unit is that by virtue of which each of the things that exist is called one.

Definition: A number is a multitude composed of units.

Book VII - Number Theory

Question: I asked you to read the definitions of Book VII, which concerns number theory. Why didn't I ask you to read the postulates (i.e., axioms) like I did in Book I?

Answer: There aren't any! Notice there are no postulates and/or common notions beyond Book I.

Book VII - Number Theory

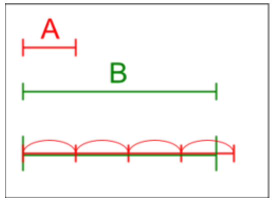
- Numbers are one type of magnitude.
- So they satisfy the same common notions that all magnitudes satisfy
- The definitions that apply to all magnitudes (e.g., concerning ratios and proportions) also apply to numbers.

Book VII - Number Theory

Two important consequences:

- Some properties of numbers are implicit in definitions.
- However, the Elements does not contain an axiomatization of the natural numbers in the same way the postulates axiomatize geometry.

Archimedean Property



Definition 4: Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another.

Book VII - Euclidean Algorithm

Proposition VII.1: Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unit is left, the original numbers will be prime to one another

Book VII -Euclidean Algorithm

Euclid is describing the following algorithm for finding the greatest common divisor of two numbers a and b.

The algorithm is now called the **Euclidean algorithm**.

Book VII -Euclidean Algorithm

Take any two numbers, say $a = 81$ and $b = 35$. Then form the following table:

		Remainder
81	33	15
33	15	3
15	3	0

Algorithm:

- The right-hand column is the remainder left when dividing the left-column by the middle one.
- The left-hand column in row $n+1$ is the middle column from row n.
- The middle column in row $n+1$ is the right-hand column from row n

Book VII -Euclidean Algorithm

		Remainder
81	33	15
33	15	3
15	3	0

Notice: 3 is the last number in the table greater than zero, and it is also the greatest-common divisor of 81 and 33.

Book VII -Euclidean Algorithm

Student Exercise: Fill in the following table.

		Remainder
96	36	24
36	24	12
24	12	0

Notice: 12 is the greatest-common divisor of 96 and 36.

Book VII - Euclidean Algorithm

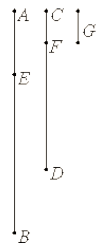
Proposition VII.1: Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unit is left, the original numbers will be prime to one another.

Modern Statement: If the last positive number in the algorithm above is 1, then a and b are relatively prime, i.e., their GCD is 1.

Question: Are there any features of Proposition VII.1 and/or its proof that you find strange?

Here are three features that are important for our purposes.

Book VII - Euclid's Geometric Representation



Feature 1: Propositions VII.1 and VII.2 are about numbers, but lines are pictured!

- In Greek number theory, lines often represent numbers.
- The reverse (using numbers to represent lines) is problematic.

Symbolic Representation

Feature 2: The Euclidean algorithm is only carried out two steps!

- This seems to have been a limit of the symbolic representation at the time, which lacked a way of representing functions, sequences, and iterated processes generally.
- Other limitations of Greek algebraic symbolism:
  - No separate numeral symbols ($\alpha = 1, \beta = 2$, etc.)
  - Before Diophantus, there was no way to represent algebraic expressions like $3x^2 + 2x + 7$ succinctly,
  - No notation for coefficients

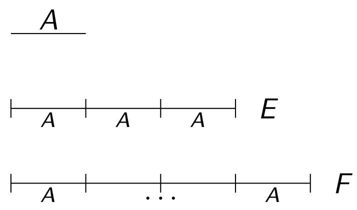
Book VII - Axioms for number theory?

Feature 3: Euclid assumes the algorithm stops.

- This is equivalent to assuming that one number cannot be subtracted from another an infinite number of times.
  - Given the definition of magnitudes "in ratio to one another", this is similar to assuming that all numbers are in ratio to one another.
- The failure to employ any postulate or definition to just this move is indicative of a lack of an axiomatic basis for arithmetic in Greek mathematics.

Incommensurability

Incommensurability Defined



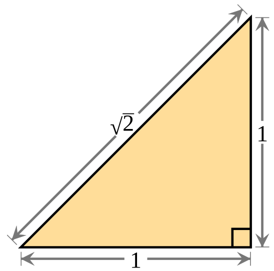
Definition 1: Those magnitudes are said to be commensurable which are measured by the same measure, and those incommensurable which cannot have any common measure.

Example: Commensurable line segments are pictured above.

Incommensurability and Numbers

Propositions X.5 and X.6: Two magnitudes are commensurable if and only if their ratio to one another is equal to a **ratio of two numbers**.

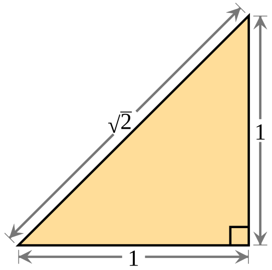
Incommensurability and Irrationality



Propositions X.9: . . . [S]quares which do not have to one another the ratio which a square number has to a square number also do not have their sides commensurable in length either.

Modern Statement: If $\sqrt{n}$ is not an integer, then it is irrational.

Incommensurable Line Segments



Propositions X.9: . . . [S]quares which do not have to one another the ratio which a square number has to a square number also do not have their sides commensurable in length either.

Greek Takeaway: Pairs of incommensurable line segments exist.

Incommensurability, Arithmetic, and Geometry

- In Eudoxus' theory, incommensurable geometric magnitudes (e.g., lines) can bear ratios to one another.
- Those ratios are not equal to ratios of numbers.
- Moral 1: So there are geometric ratios that are not numerically representable.
  - I.e., The Pythagorean discovery of incommensurability created a rift between arithmetic and geometry.

Incommensurability, Arithmetic, and Geometry

Moral 2: Eudoxus' theory allows one to use numerical ratios to study geometric ratios, but it does not require the concept of an irrational number.

- I.e., Eudoxus' theory sustained the rift between geometry and arithmetic.

Preview

Where We're Going

Next Month: We'll explore various rationalist and empiricist philosophies of mathematics, paying close attention to developments in the relationship between geometry and arithmetic at the time.

Up Next: Plato's rationalism and Aristotle's empiricism

Today's Response Question

Response Question: Explain some reasons why arithmetic and geometry were regarded as separate disciplines, and why they could be studied independently of one another.