Generality and Euclidean Geometry

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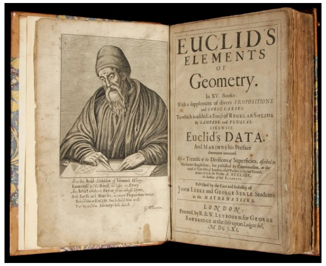
Why The Elements?



Why did I have you read sections of Euclid's **Elements**?

- To appreciate how different mathematics is during different time periods, and
- To know what philosophers prior to the 20th century are talking about when they discuss mathematics!

Two Types of Theorems and Postulates

There are two types of theorems and postulates in Euclid's **Elements**:

- Constructions
- "Equivalences"

Constructions:

- For now, think of a construction postulate or theorem as giving you an **idealized** drawing ability.
- No one can draw a perfect circle or produce an infinitely thin straight line with no area, but Euclid's postulates allow you to do so.

Philosophy for the mathematician

Construction Postulates

- Postulate 2: To produce a finite straight line continuously in a straight line.
  - I.e., If a give you a line segment AB, you can draw a line (or line segment) extending AB to any length.
- Postulate 3: To describe a circle with any center and radius.
  - I.e., If I give you a point A and line segment AC, then you can draw a circle

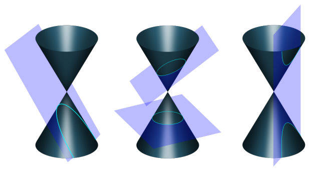
Note: Sometimes, Euclid implicitly assumes the constructed object in Postulates 1 and 2 is unique.

Idealization

These postulates are idealizations of your ability to use particular **instruments**

- Postulates 1 and 2: Idealize you ability to use a straight edge
- Postulate 3: Idealizes your ability to use a compass.

Other Constructions



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- Constructions involving the idealized use of a straight-edge and compass are the most important in the history of mathematics for reasons we'll discuss in the second unit.
- However, most Greek mathematicians did use other construction techniques, e.g., the use of **conic sections** and **moving** instruments. We'll also discuss these later.

Outline

The Generality Problem

- Euclid's use of diagrams raises a central question for mathematicians and philosophers:
- **The Generality Problem:** How do we know the proof applies/works for all triangles, circles, etc.? [Mumma, 2010]

Solving the generality problem

Potential Solution: The diagram is unnecessary.

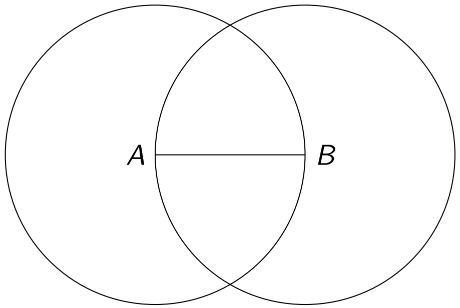
Necessity of the diagram

There are at least three reasons why diagrams are necessary in Euclidean Geometry:

1. "Underspecified" objects [Netz, 1999]
2. Existence of objects is not guaranteed by the postulates alone
   - This is a common criticism of Euclid in the 19th and 20th centuries.
3. Case distinctions

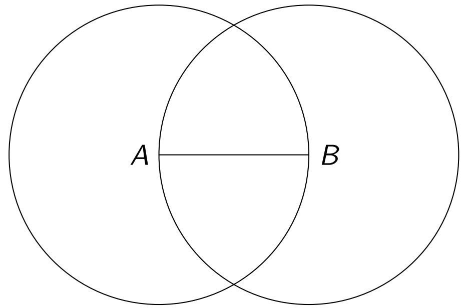
Underspecification

Specification



An object is **underspecified** if it occurs in a construction imperative, but is not uniquely determined by the construction.
E.g., Construct a circle with radius AB.

Specification



Example: Proposition 1 says defines C as "**the** point . . . in which the circles cut one another." But looking at the diagram, there are in fact two such intersection points.

Underspecified objects

Exercise: Follow Euclid's instructions to draw the following diagram. Do not look at Euclid's text.

Completely unspecified objects

Proposition 2: To place at a given point (as an extremity) a straight line equal to a given straight line.

Proof: Let A be the the given point, and BC the given straight line . . . From the point A, to the point B, let the straight line AB be joined [Post. 1]. and on it, let the equilateral triangle DAB be constructed [I.1]. Let the straight lines AE, BF be produced in a straight line with DA and DB [Post. 2]; with center B and distance BC, let the circle CGH be described . . .

Completely unspecified objects

Questions:

- Is the line segment *DE* in your picture longer than the given line segment *BC*? What about *DF*?
- Does the point *G* in your picture lie on the line segment *DF*?

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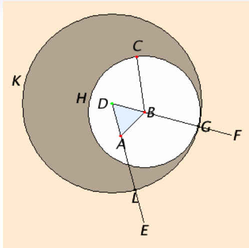
Underspecified objects



- In Euclid's diagram, both *DE* and *DF* are longer than *BC*.
- In Euclid's diagram, the point *G* lies on the segment *DF*.
- The points *E*, *F* and *G* are **underspecified**. Why? They occur in construction imperatives and are not unique because
  - One could continue the line segments *DA* and *DB* for as long or as short as one would like given only the directions in the text.
  - *G* could lie anywhere on the circle with center *B* and radius *BC*.

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Specification

An object **completely unspecified** if it appears in a declarative statement without being defined anywhere in the text or mentioned in some construction statement. See [Netz, 1999] for an example.

The difference between "underspecified" and "completely unspecified" is not very important for us right now.

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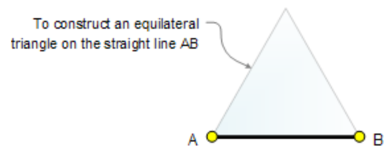
Specification and the Generality Problem

Question: Why does underspecification lead to the generality problem?

Answer: By definition, underspecified objects are not determined uniquely. So sometimes it's not clear whether choosing different points (e.g., chosen different points *E*, *F* and *G* in I.2) undermines the proof, or whether Euclid just needed to be a bit more specific about how to choose the objects.

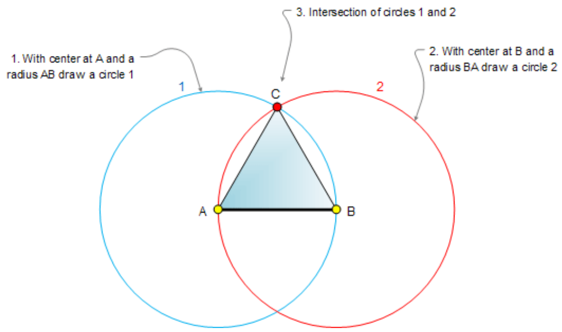
Geometric Existence

Euclid I.1



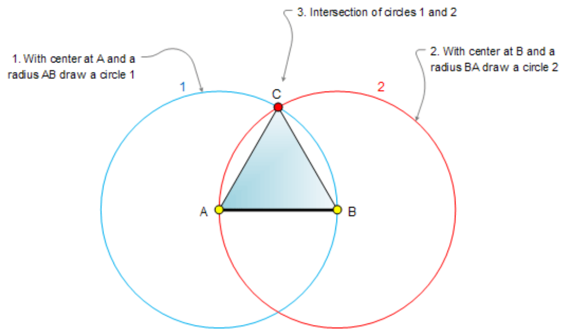
Modern Statement: You are given a line segment AB. Using only a straight-edge and compass, construct an equilateral triangle with AB as one of its sides.

Euclid I.1



Proof: Draw two circles that contain AB as their radius as shown in the diagram. Connect points A and B to the point C, which lies at the intersection of the two circles. The triangle ABC is equilateral because $AB = AC$ (as both are radii or circle 1) and $AB = BC$ (because both sare radii of circle 2).

Euclid I.1



Problem: None of Euclid's postulates or common notions seem to guarantee the existence of the intersection point C. It seems just to be "read off the diagram."

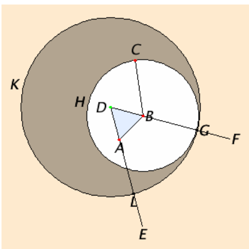
Specification and the Generality Problem

Question: Why does the existence problem lead to the generality problem?

Answer: It's not clear whether, had we drawn the diagram differently, the constructed objects (e.g., posited intersection points) will exist.

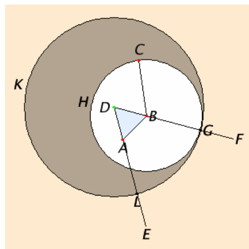
Unstated Cases

Euclid I.2



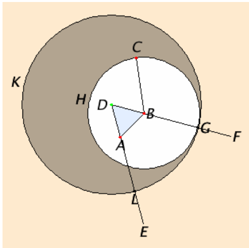
Proof: Let A be the the given point, and BC the given straight line . . .

Euclid I.2



Question: What does Euclid's diagram assume about the relationship between the point A and the line segment BC?

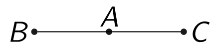
Euclid I.2



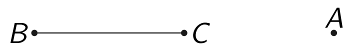
Answer: A does not lie on the line segment BC, nor is A collinear with BC.

Unstated Cases

Case 1



Case 2



Cases and the Generality Problem

Question: Why do unstated cases lead to the generality problem?

Answer: We don't know whether the proof works in the unstated cases!

Was Euclid just sloppy?

Question: Should we just conclude that Euclid was a sloppy, careless mathematician?

Answer: Absolutely not.

- Cases of underspecification can be eliminated.
  - In I.2, Euclid could have first constructed the circle with center B and radius BC; then he could have required E and F to lie outside that circle; finally, he could have required that G lie on DF.
- Euclid followed particular rules for what could be "read off" the diagram (e.g., which points exist, and which don't) [Manders, 2008].
- It was (and still is) standard mathematical practice to leave certain cases in proofs to readers.

Generality

Nonetheless, the given proofs raise the generality problem because one can ask

- Is there a method for eliminating underspecification?
- What information can be inferred from the diagram?
- Which are cases left for the reader?

Readers unfamiliar with the cultural and mathematical context in which the **Elements** was written may not know the answers to these questions, and so they present genuine philosophical concerns.

Two Types of Theorems and Postulates

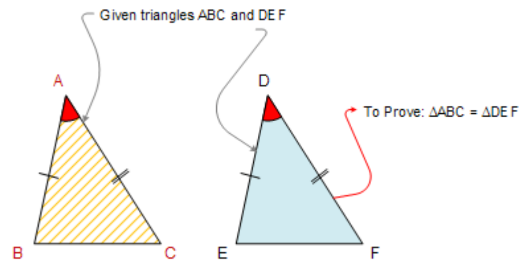
There are two types of theorems and postulates in Euclid's **Elements**:

- Constructions
- "Equivalences"

Let's discuss "equivalence" theorems now.

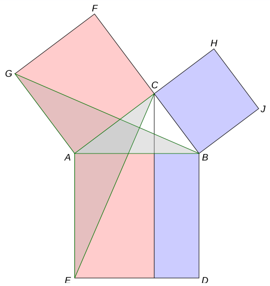
Congruence and equality of area, length, and angles

Euclid I.4



Modern Statement: Side-angle-side entails congruence.

Euclid I.47



Euclid's I.47 = The Pythagorean theorem.

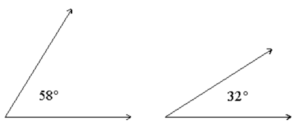
Euclid's Statement: In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

Euclid I.47

Euclid's Statement: In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

- There are no numbers mentioned at all.
- E.g., The length of the sides of the triangle are not numerically represented; nor is the area of the squares.

Euclidean Geometry and Quantification

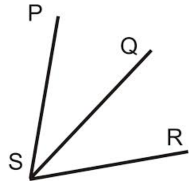


In general, geometric magnitudes (angles, lengths, areas, volumes, etc.) were not numerically quantified in Euclidean geometry, as we do today (e.g., in the above picture).

Comparing geometric magnitudes

Question: How can we check whether one angle, area, or length is bigger than another geometric object of the same kind?

Euclid's Common Notions

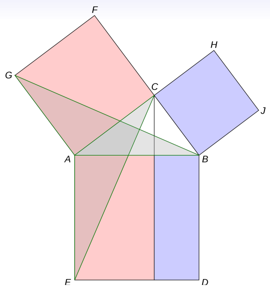


- Euclid's **Common Notions** tell us how to compare angles (or lengths, or areas) when they are *touching or contained within one another.*
- **Common Notion 5:** The whole is greater than the part.
- In these cases, we can just see whether one angle (or segment, or circle) is contained in another.

Euclid's Common Notions

- These notions are "common" because they apply to all geometric magnitudes.
- E.g. **Common Notion 2** asserts, "If equals are added to equals, then the wholes are equal." This entails:
  - If two line segments *AB* and *CD* are equal in length, and two line segments *BF* and *DG* are equal (where *F* and *G* do not lie on *AB* and *CD* respectively), then the line segment *AF* and *CG* are equal in length.
  - Two squares, equal in area, when added to two other (disjoint) squares of equal area, will produce geometric figures of the same total area.

Euclid's Common Notions



- Euclid's **Common Notions** tell us how to compare angles (or lengths, or areas) when they are *contained within one another.*
- Question: How do we compare the sizes of areas, lengths, etc. that are not contained within one another (nor can they be moved to fit within one another), as the Pythagorean theorem requires?

Euclid's Common Notions

Question: How do we compare the sizes of areas, lengths, etc. that are not contained within one another (nor can they be moved to fit within one another), as the Pythagorean theorem requires?

Answer: To compare two objects O_1 and O_2 (e.g., squares), we construct a sequence of intermediate objects $C_1, C_2, \ldots C_n$ such that

- C_i is contained in or touching C_{i+1}, and therefore comparable by the Common Notions and our visual capacities,
- Repeatedly applying transitivity of equality (Common Notion 1)

Where We're Going

- Solution to generality problem? Be more careful with logic.
  - **Friday:** Aristotle's logic is insufficient for reconstructing the proof of Euclid I.1.
- Locke's Solution: abstract ideas.
  - You'll also see how Locke's theory of demonstration is based upon the above ideas about showing equality of geometric magnitudes.

Today's Response Question

Response Question: Explain what the generality problem is and two reasons that Euclid's diagrammatic proofs raised this problem.

References I

Manders, K. (2008). The euclidean diagram. In Mancosu, P., editor, *The Philosophy of Mathematical Practice*, pages 80–133. Oxford University Press.

Mumma, J. (2010). Proofs, pictures, and euclid. *Synthese*, 175(2):255–287.

Netz, R. (1999). *The shaping of deduction in Greek mathematics: A study in cognitive history*. Cambridge University Press Cambridge.