Mathematical Methods for Philosophy:
Problem Set 7

Due Date: This problem set is due January 14th, 2013 at the start of class.

Exercise 1 *Prove the following:*

- $a \preceq \mathcal{P}(a)$.
- *If* $b \neq \emptyset$, *then* $a \preceq a^b$.
- *If* $a \approx b$, *then* $b \approx a$.
- *If* $a \approx b$, *then* $\mathcal{P}(a) \approx \mathcal{P}(b)$.

Exercise 2 *Say that a set* a *is* ***infinite*** *if* $\mathbb{N} \preceq a$. *A set* a *is called* ***Dedekind infinite*** *if it has a proper subset* $b \subset a$ *such that* $a \approx b$. *Show that if* a *is infinite, then* a *is Dedekind infinite.*

- *Hint: As* $\mathbb{N} \preceq a$, *there is an injective function* $f : \mathbb{N} \to a$. *Let* $x_0 = f(0), x_1 = f(1)$, *and in general,* $x_n = f(n)$ *be the values of the function* f. *Consider the proper subset* $b = a \setminus \{x_0\}$ *consisting of all elements of* a *other than* x_0. *Find a bijection from* a *to* b.

Exercise 3 *A set* a *is called* ***countable*** *if* $a \preceq \mathbb{N}$; *it is called* ***countably infinite*** *if* $a \approx \mathbb{N}$.

- *Show by induction on* k *that* $\underbrace{\mathbb{N} \times \mathbb{N} \times \ldots \times \mathbb{N}}_{k-times}$ *is countably infinite.*
  Hint: In the inductive step, you can use the proof from class that $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$.

- *Suppose* $a_0, a_1, \ldots$ *is a sequence of countably infinite sets that are* ***pairwise disjoint***, *which means that if* $n \neq m$, *then* $a_n \cap a_m = \emptyset$. *Show that* $\bigcup\{a_n : n \in \mathbb{N}\} = a_0 \cup a_1 \cup a_2 \ldots$ *is countably infinite.*

  - *Hint: Because each* a_n *is countably infinite, it follows that there is a bijective function* $f_n : \mathbb{N} \to a_n$ *for each* n. *Let* $x_{n,m} = f_n(m) \in a_n$ *be the* m^{th} *element of the set* a_n *when it is enumerated by* f. *Now try to use a "picture proof" like the one we did in class.*

- *Suppose a formal language contains countably many sentential variables* $p_1, p_2,$, *and so on. Using the previous two exercises, show the set of* WFF *of sentential logic is countable, i.e., that* WFF $\preceq \mathbb{N}$. *Hint: Think of the formula* $((p_1 \& p_2) \to p_3)$ *as an ordered tuple* $\langle (, p_1, \&, p_2,), \to, p_3,) \rangle$.

Exercise 4 *The* ***axiom of choice*** *says that for every set* a*, there is a function* $g_a : \mathcal{P}(a) \setminus \{\emptyset\} \to a$ *such that* $g_a(b) \in b$ *for all* $b \in \mathcal{P}(a) \setminus \{\emptyset\}$*. The idea is that* g_a *"chooses" exactly one element of every non-empty subset* b *of* a*. The following exercises use the axiom of choice.*

- *Show that if there is a surjective function* $f : b \to a$*, then* $a \preceq b$*. Conclude that* $b \not\preceq a$ *entails that* $a \prec b$*.*
  - *Hint: Let* $h : a \to \mathcal{P}(b)$ *be the function defined by* $h(x) = f^{-1}(x) = \{y \in b : f(x) = y\}$*. Consider* $g_b \circ h$*.*
- *Show that if* a *is Dedekind infinite, then* a *is infinite.*
  - *Hint: Because* a *is Dedekind infinite, there exists a proper subset* b *of* a *and a bijective function* $f : a \to b$*. Recursively, define a sequence of sets:*

$$\begin{aligned} b_0 &= a \\ b_{n+1} &= f[b_n] = \{f(x) : x \in b_n\} \end{aligned}$$

    Notice that $b_1 = b$*. Show by induction that* $b_{n+1} \subseteq b_n$ *and* $b_n \setminus b_{n+1}$ *is non- empty for all natural numbers* n*. Then apply a choice function to select an element of* $b_n \setminus b_{n+1}$ *for all natural numbers* n*.*