Logic and Topology for Knowledge, Knowability, and Belief

Reference Handout

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Axioms and axiom systems

Given unary modalities $\star_1, \ldots, \star_k$, let $\mathcal{L}_{\star_1,\ldots,\star_k}$ denote the propositional language given by

$$\varphi ::= p \,|\, \neg\varphi \,|\, \varphi \wedge \psi \,|\, \star_i\, \varphi,$$

where $p \in$ PROP, the (countable) set of *primitive propositions*, and $1 \leq i \leq k$.

$(\mathrm{K}_\star)$	$\vdash \star(\varphi \to \psi) \to (\star\varphi \to \star\psi)$	Distribution
$(\mathrm{D}_\star)$	$\vdash \star\varphi \to \neg \star \neg\varphi$	Consistency
$(\mathrm{T}_\star)$	$\vdash \star\varphi \to \varphi$	Factivity
$(4_\star)$	$\vdash \star\varphi \to \star\star\varphi$	Positive introspection
$(.2_\star)$	$\vdash \neg\star\neg\star\varphi \to \star\neg\star\neg\varphi$	Directedness
$(5_\star)$	$\vdash \neg\star\varphi \to \star\neg\star\varphi$	Negative introspection
$(\mathrm{Nec}_\star)$	from $\vdash \varphi$ infer $\vdash \star\varphi$	Necessitation

Let CPL denote an axiomatization of classical propositional logic.

$$\begin{aligned}
\mathsf{K}_\star &= \mathsf{CPL} + (\mathrm{K}_\star) + (\mathrm{Nec}_\star) \\
\mathsf{S4}_\star &= \mathsf{K}_\star + (\mathrm{T}_\star) + (4_\star) \\
\mathsf{S4.2}_\star &= \mathsf{S4}_\star + (.2_\star) \\
\mathsf{S5}_\star &= \mathsf{S4}_\star + (5_\star) \\
\mathsf{KD45}_\star &= \mathsf{K}_\star + (\mathrm{D}_\star) + (4_\star) + (5_\star).
\end{aligned}$$

Stalnaker works with the language $\mathcal{L}_{K,B}$, augmenting the logic $\mathsf{S4}_K$ with the following:

(D_B)	$\vdash B\varphi \to \neg B \neg\varphi$	Consistency of belief
(sPI)	$\vdash B\varphi \to KB\varphi$	Strong positive introspection
(sNI)	$\vdash \neg B\varphi \to K\neg B\varphi$	Strong negative introspection
(KB)	$\vdash K\varphi \to B\varphi$	Knowledge implies belief
(FB)	$\vdash B\varphi \to BK\varphi$	Full belief

Let $\mathsf{EL}_{K,\Box} = \mathsf{S5}_K + \mathsf{S4}_\Box + (K\varphi \to \Box\varphi)$. We strengthen $\mathsf{EL}_{K,\Box}$ with the following:

(K_B)	$\vdash B(\varphi \to \psi) \to (B\varphi \to B\psi)$	Distribution of belief
(sPI)	$\vdash B\varphi \to KB\varphi$	Strong positive introspection
(KB)	$\vdash K\varphi \to B\varphi$	Knowledge implies belief
(RB)	$\vdash B\varphi \to B\Box\varphi$	Responsible belief
(wF)	$\vdash B\varphi \to \Diamond\varphi$	Weak factivity
(CB)	$\vdash B(\Box\varphi \vee \Box\neg\Box\varphi)$	Confident belief

Topology and subset spaces

A **topological space** is a pair $(X, \mathcal{T})$ where X is a nonempty set and $\mathcal{T} \subseteq 2^X$ is a collection of subsets of X that covers X and is closed under finite intersections and arbitrary unions. The collection $\mathcal{T}$ is called a *topology on* X and elements of $\mathcal{T}$ are called *open* sets. For $A \subseteq X$, say that x lies in the **interior** of A if there is some $U \in \mathcal{T}$ such that $x \in U \subseteq A$. The set of all points in the interior of A is denoted $int(A)$; it is easy to see that $int(A)$ is the largest open set contained in A.

A **topological subset model** is a topological space $(X, \mathcal{T})$ together with a function $v : \textsc{prop} \to 2^X$ specifying, for each primitive proposition $p \in \textsc{prop}$, its *extension* $v(p)$.

Formulas of $\mathcal{L}_{K,\Box}$ are interpreted with respect to pairs of the form (x, U), where $x \in U \in \mathcal{T}$:

$$
\begin{array}{lll}
(x,U) \models p & \text{iff} & x \in v(p) \\
(x,U) \models \neg\varphi & \text{iff} & (x,U) \not\models \varphi \\
(x,U) \models \varphi \wedge \psi & \text{iff} & (x,U) \models \varphi \text{ and } (x,U) \models \psi \\
(x,U) \models K\varphi & \text{iff} & U \subseteq [\![\varphi]\!]^U \\
(x,U) \models \Box\varphi & \text{iff} & x \in int([\![\varphi]\!]^U),
\end{array}
$$

where $[\![\varphi]\!]^U = \{x \in U \,:\, (x,U) \models \varphi\}$.

Thus, knowledge corresponds to truth in all epistemically possible worlds (i.e., all worlds in U), as in the standard semantics for knowledge using relational models. Furthermore, given $p \in \textsc{prop}$, we have $x \in int([\![p]\!]^U)$ precisely when there is some open set (piece of evidence) $V \in \mathcal{T}$ that is true at x and that entails p. Accordingly, we think of $\Box$ as representing knowability. The dual modality correspondingly satisfies

$$(x,U) \models \Diamond\varphi \quad \text{iff} \quad x \in cl([\![\varphi]\!]^U),$$

where cl denotes the topological closure operator: $x \in cl(A)$ iff $(\forall V \in \mathcal{T})(x \in V \Rightarrow V \cap A \neq \emptyset)$.