Decision Theory: Problem Set 4
Risk-Weighted Expected Utility

Directions: For each exercise below, show all your work, and explain what you've done *in complete English sentences.* Further directions for how to typeset your problem sets are available on the Canvas website. If you prefer to write your problem set by hand, you must submit a paper copy of your problem set, as explained on the syllabus. See the syllabus for more details about submitting problem sets.

Exercise 1: Let L_1 denote the lottery in which you earn (i) \$0 with probability 1/4, (ii) \$5 with probability 1/2, and (iii) \$10 with probability 1/4. Let L_2 denote the lottery in which you earn (i) \$2 with probability 1/8, (ii) \$5 with probability 5/8, and (iii) \$10 with probability 1/4.

A. Show that L_2 stochastically dominates L_1 if more money is better.

B. Suppose that you assign a utility of x to \$$x$ and that your risk function is given by the equation $r(p) = p^2$. Calculate the risk-weighted expected utilities of L_1 and L_2 to check that the risk-weighted expected utility of L_2 is greater than that of L_1.

Bonus C: Show that the risk-weighted expected utility of L_2 is greater than that of L_1 assuming only that (i) your utility is a strictly increasing function of money and (ii) your risk function is strictly increasing.

Exercise 2:

A. On previous problem sets, you explored which decision rules satisfy *Chernoff's condition.* Chernoff's condition entails that, that if a rational agent chooses an action A when the alternatives B and C are available, then she would also choose act A if B were the only alternative. On problem set 1, you showed that minimax regret decision-makers violate Chernoff's condition, and on problem sets 2 and 3, you showed that expected utility maximizers satisfy Chernoff's condition. Explain why Buchak's risk-weighted expected utility theory likewise satisfies Chernoff's condition.

B. On previous problem sets, you explored a condition that is sometimes called "Column Linearity." You showed that maximin reasoning violates Column Linearity, whereas expected utility maximization satisfies it. In this exercise, you will show that *risk-weighted* expected utility theory violates the condition.

To understand Column Linearity, consider the two (abstract) decision matrices below, where the outcomes are assigned numerical utilities u_1, u_2, u_3 and u_4. Column linearity entails that, if the two decision matrices contain numerical utilities and if c is any number, then a rational decision-maker ought to choose the same actions in both decision problems.

Problem 1

	State 1	State 2
Action 1	u_1	u_2
Action 2	u_3	u_4

Problem 2

	State 1	State 2
Action 1	$u_1 + c$	u_2
Action 2	$u_3 + c$	u_4

Show that the risk weighted expected utility maximizer chooses Action 2 in Problem 1 but Action 1 in Problem 2 if the following holds: (1) The probability of state 2 is 3/4, (2) $r(p) = p^2$ is the risk function, and (3) $u_1 = 0, u_2 = 4, u_3 = 8$, $u_4 = 2$, and $c = 4$.

Motivation for This Problem: The following text explains why this exercise is important; you do not need to incorporate any of the following into your answer.

Column Linearity is often thought to be an intuitive/desirable feature of decision rules. To see why, consider two possible decision problems you might encounter, which are instances of the types of problems represented by the decision matrices Problems 1 and 2 above.

In Problem 1, suppose you're making a decision about whether to go on a hike (Action 1) or go to a movie (Action 2). It might rain (State 1) or not (State 2). When it's raining, going to a movie is better than hiking, but conversely, if it's nice outside, hiking is much more pleasant than watching a movie. No matter what, you'd prefer it not to rain because getting home from a hike or from the movies is a bit unpleasant if it's raining. Suppose that, given your beliefs about the possibility of rain, you decide to go to the movie.

But now suppose your decision problem changes in the following way. Imagine you remember that you made a bet with a friend and that, if it rains today, you'll win \$20. You win that money regardless of whether you hike or go to the movie. The utility of that money is represented by the value c added to the utilities in State 1. Intuitively, the extra money should not cause you to switch your decision from going to the movie to hiking, but if a decision rule violates Column Linearity, then such a monetary bonus can have that effect. For this reason, many decision theorists believe that rational decision-makers act in a way in according with Column Linearity.

Exercise 3: In previous classes and problem sets, we considered the following two (hypothetical) choices:

- Lottery 1: \$3000 with certainty,
- Lottery 2: \$4000 with probability 80%.

And

- Lottery 3: \$3000 with probability 25%.
- Lottery 4: \$4000 with probability 20%.

Most subjects prefer Lottery 1 to Lottery 2 and Lottery 4 to Lottery 3. Those two preferences, as we've seen, are in tension with expected utility theory as a descriptive theory of human decision-making.

In this exercise, you will show these preferences are consistent with risk-weighted expected utility theory. To do so, let:

- $u_0 = u(\$0) = 0$,
- $u_1 = u(\$3000) = 7$,
- $u_2 = u(\$4000) = 10$,
- $c = r(1/5) = 5/12$,
- $b = r(1/4) = 1/2$, and
- $a = r(4/5) = 2/3$

A. Express the risk weighted expected utilities of $L_1 - L_4$ in terms of u_0, u_1, u_2, a, b, and c. In other words, your answer should contain four symbolic equations involving those variables (Note: not every equation will contain all six variables).

B. Calculate the numberical risk weighted expected utilities of $L_1 - L_4$. Confirm the risk weighted expected utility of L_1 is greater than that of L_2, and that the the risk weighted expected utility of L_4 is greater than that of L_3.

Bonus C: Find a strictly increasing function r such that $r(0) = 0, r(1) = 1, r(1/5) = 5/12, r(1/4) = 1/2$ and $r(4/5) = 2/3$. Is your function convex? Concave? Both? Neither?

Bonus D: Find a *differentiable* risk function satisfying the conditions in C.