Rationality and Decision Theory: Problem Set 3

Directions: For each exercise below, show all your work, and explain what you've done *in complete English sentences*. Further directions for how to typeset your problem sets are available on the Canvas website. If you prefer to write your problem set by hand, you must submit a paper copy of your problem set, as explained on the syllabus. See the syllabus for more details about submitting problem sets.

Important Notes: Remember that a Bayesian decision-maker maximizes subjective expected utility. In the absence of information, you should *not* assume that a Bayesian decision-maker regards all states of the world as equiprobable.

Exercise 1: Suppose that I am about to roll a six-sided die, where each side is labelled with a number between one and six. I offer you the following fair prices on bets for/against various events. For example, I am willing either (i) to pay five cents for a ticket that is worth one dollar if a six is rolled (and is worthless otherwise), or (ii) sell you a ticket for five cents that is worth one dollar if a six is rolled (and is worthless otherwise). You can purchase and/or sell at most $2 worth of bets (e.g., it you sell me $1.60 worth of bets, you can purchase at most $.40 worth of bets).

Event	**My Fair Price**
$\{1\}$	4¢
$\{2\}$	80¢
$\{3\}$	1¢
$\{6\}$	5¢
$\{1,3,6\}$	10¢
$\{1,3,4,5,6\}$	15¢

A. Construct (what I call) a *minimal* Dutch Book. Recall, a Dutch Book is series of bets that you can make against me such that you are *guaranteed* to make money, regardless of how the die lands. I call a Dutch Book *minimal* if you cannot eliminate any bets from your Dutch book without it ceasing to be a Dutch book.

B. Suppose you think the die is fair (i.e., you regard each side is equally probable). Find a sequence of bets that (1) is *not* a Dutch Book and yet (2) has higher expected earnings than the Dutch Book that you constructed in Part A.

Exercise 2: Ada Lovelace, one of the first philosophers of computing, suffered from chronic illness. So when she planned her day, she might have chosen activities without knowing whether she would feel ill that afternoon. Suppose Ada maximized expected utility and that, on a particular day, her preferences were summarized in the following decision matrix.

	No Sickness	Sick
Program Analytical Engine	3	0
Write a letter	2	2
Solve Equations	1	5

A. Show that Ada did not write a letter, regardless of her subjective degrees of belief. To do so, let $x = P(\text{Sick})$ be the probability that Ada assigned to the possibility she assigned to becoming ill in the afternoon. Using algebra only, show there is no value of x for which writing a letter maximizes Ada's expected utility.

B. For each of the three actions, graph Ada's expected utility as a function of $x = P(\text{Sick})$. Graph on all three functions on the same set of axes. (Hint: All three functions are linear!) Why does your graph indicate that Ada did not write a letter, regardless of her subjective degrees of belief?

Exercise 3: Given a decision matrix and a decision rule R (e.g., maximin), we call R-**EU-rationalizable** in a decision problem if there is some way of assigning probabilities to the states such that expected utiility maximization will recommend the exact same set of actions as R. For instance, in Figure 1, the rule maximin recommends actions 1 and 2 (but not 3). Maximin is EU-rationalizable in this case because if we assign probability 1/2 to state 1 and probability 1/2 to state 3, then only acts 1 and act 2 maximize expected utility; their expected utility is 1/2 in this case, whereas the expected utility of Act 3 is zero.

	State 1	State 2	State 3
Act 1	0	1	1
Act 2	1	1	0
Act 3	-1	1	1

Figure 1

Decision rules are not always rationalizable, however. In this exercise, you will prove that maximin, minimax regret, and the optimism-pessimism rule all fail to be EU rationalizable in some circumstances. Consider the decision matrix in Figure 2 below.

	State 1	State 2	State 3
Act 1	1	0	0
Act 2	0	1	0
Act 3	0	0	1
Act 4	0	1	1
Act 5	1	0	1
Act 6	1	1	0

Figure 2

A. In Figure 2, which actions are choiceworthy according to maximin? According to minimax regret? According to the optimism-pessimism rule? Note: I have not told you the value α to use in the optimistim pessimism rule, but the answer to my question is the same no matter the value of α.

B. Prove that there is no way of assigning probabilities to States 1, 2 and 3 in Figure 2 such that all six actions maximize expected utility. *Hint:* First explain why at least one state must have probability less than or equal to 1/3. Then explain why, for any choice of $n \in \{1, 2, 3\}$, if state n has probability less than or equal to 1/3, then Act n does not maximize expected utility.