Decision Theory: Problem Set 2

Directions: For each exercise below, show all your work, and explain what you've done *in complete English sentences.* Further directions for how to typeset your problem sets are available on the Canvas website. If you prefer to write your problem set by hand, you must submit a paper copy of your problem set, as explained on the syllabus. See the syllabus for more details about submitting problem sets.

Exercise 1: Suppose Ada prefers her coffee sweet and that she is asked to rank 100 cups of coffee – which we will call $C_1, \ldots C_{100}$ – in terms of how much she prefers them. We will use the symbol C_n to denote the cup that contains exactly n many grains of sugar, e.g., C_1 contains one grain, and C_{37} contains 37 grains of sugar, and so on. Ada prefers one cup of coffee to another if it the latter is noticeably sweeter than the other. However, Ada cannot tell the difference between two cups of coffee unless they differ by at least five grains of sugar. Thus, Ada prefers cup C_n to cup C_m if and only if $n \geq m + 5$. Let's use the symbol $\succ$ to represent Ada's strict preference relation. Thus, $C_n \succ C_m$ if and only if $n \geq m + 5$.

A. Is Ada's strict preference relation $\succ$

   i. Transitive?

      – Recall, a relation R is called *transitive* if whenever xRy and yRz, it follows that xRz. The relation "taller than" is transitive: if Ada is taller than Boris and Boris is taller than Carole, then Ada is taller than Carole. To determine whether Ada's preference relation is transitive, let C_n, C_m and C_k denote three cups of coffee. Suppose $C_n \succ C_m$ and $C_m \succ C_k$, does it follow that $C_n \succ C_k$? Justify your answer.

   ii. Asymmetric?

      – Recall, a relation R is called asymmetric if whenever xRy holds, it is not the case that yRx holds. The relation "taller than" is asymmetric: if Ada is taller than Boris, then Boris is *not* taller than Ada. To determine whether Ada's preference relation is asymmetric, let C_n and C_m denote two arbitrary cups of coffee. Suppose $C_n \succ C_m$. Does it follow that $C_m \not\succ C_n$ (i.e., Ada does *not* prefer C_m to C_n)? Justify your answer.

   iii. Negatively transitive?

      – A relation R is called negatively transitive if whenever neither xRy nor yRz

holds, it is likewise *not* the case that xRz holds..[1] The relation "taller than" is negatively transitive. To see why, suppose that Ada is *not* taller than Boris and that Boris is *not* taller than Carole. Then Ada cannot be *not* taller than Carole.[2] To determine whether Ada's preference relation is transitive, let C_n, C_m and C_k denote three cups of coffee. Suppose $C_n \not\succ C_m$ and $C_m \not\succ C_k$, does it follow that $C_n \not\succ C_k$? Justify your answer.

B. Suppose Ada is indifferent between two cups of coffee if she prefers neither to the other. Let $\sim$ represent Ada's indifference relation. So $C_n \sim C_m$ if and only if $C_n \not\succ C_k$ and $C_k \not\succ C_n$. Is Ada's indifference relation $\sim$ transitive? Justify your answer.

Exercise 2: In this exercise, use the notation ApB to denote the lottery that yields the prize A with probability p and the prize B with probability $1 - p$.[3] Assume that your preferences over various lotteries satisfy von Neumann and Morgenstern's axioms so that it's as if you assigned numerical utilities to various prizes and lotteries. Use the lower-case letter u to represent your utility function so that, for example, $u(A)$ is the utility you assign to some option A.

A. Suppose that, when asked which of several lotteries you prefer, you always name *all* the lotteries that you must prefer. For example, if I ask which of lotteries L_1 and L_2 you prefer, then you will say

- "L_1" if you strictly prefer L_1 to L_2,
- "L_2" if you strictly prefer L_2 to L_1,
- "Both L_1 and L_2 are acceptable!" if you are indifferent between L_1 and L_2.

Now suppose I ask you two questions. I first ask, "Which of the following two lotteries do you prefer: L_1 or L_2?" You say, L_1. I then ask, "Which of the following lotteries do you prefer: L_1, L_2, or L_3?" What possible answers might you give to this question if you are an expected utility maximizer as I assume? What answers can I be certain that you will *not* give? Explain your answer.

[1] It's typically easier to understand and apply the contrapositive of the definition of negatively transitive, which states the following. A relation is negatively transitive if whenever xRz holds, either xRy or yRz (or both) hold.

[2] It's easier to understand and verify that "taller than" satisfies contrapositive of the definition of negatively transitive. Suppose Ada is taller than Carole. Then either Ada is taller than Boris, or Boris is taller than Carole (or both). For if Ada is no taller than Boris, and Boris is no latter than Carole, then Ada is likewise no latter than Carole.

[3] For example, suppose that A represents an apple and B represents a banana. Further, suppose that $p = 1/3$. Then ApB would represent the lottery/gamble in which you first get to roll a fair die; then, you are awarded an apple if the die lands on 1 or 2, and you're awarded a banana if it lands on $3 - 6$.

B. Suppose there are six prizes A, B, B^*, C, D, and D^* that you might win in some upcoming gambles. Further, suppose r is some number such that $u(B^*) = u(B) + r$ and $u(D^*) = u(D) + r$. In other words, the difference between that utilities that you assign to B^* and B is the same as the difference utilities that you assign to D^* and D. Suppose that I know that you prefer the lottery ApB to CpD and that, again, I assume you are an expected utility maximizer. Can I determine which of the following two lotteries you prefer: (1) ApB^* and (2) CpD^*? Justify your answer. Hint: A short, algebraic proof/calculation is necessary here.

C. Suppose that, in Part A of this question, I had instead assumed you are a minimax-regret decision-maker, not an expected utility maximizer. What answers might you give in response to the two questions in Part A of this problem? Hint 1: Consider exercise 2 from Problem Set 1. Hint 2: To show that it is possible for a minimax regret decision-maker to give a particular answer (e.g., L_1 in response to the first question and L_1 in response to the second), you must produce a pair of decision matrices with the same underlying states. The first matrix should represent your choice between Lottery L_1 and L_2, and the second matrix should represent your choice among L_1, L_2, and L_3. The second matrix must contain the same states and same payoffs for L_1 and L_2 as the first matrix. If you believe multiple pairs of answers are possible, then you'll need to produce multiple pairs of decision matrices.

D. Suppose that, in Part B of this question, I had instead assumed you are a maximin decision-maker, not an expected utility maximizer. Can I determine which of the following two lotteries you prefer: (1) ApB^* and (2) CpD^*? Justify your answer. Hint 1: Consider exercise 3 from Problem Set 1.

Exercise 3: Consider the two choices below. In lab experiments, most subjects choose Lottery 2 when confronted with Choice 1 and Lottery 3 when confronted with Choice 2. Explain why these preferences are incompatible with von Neumann and Morgenstern's axioms preferences among lotteries. *Carefully name all of the axioms that are used in your argument.*

Choice 1:

Lottery 1: A fair coin will be flipped, and if it lands heads (which has 50% chance of occurring), you win an all-expenses paid *three*-week tour of England, France, Italy.

Lottery 2: You win (with 100% certainty) an all-expenses paid *one*-week tour of England.

Choice 2:

> Lottery 3: A random number between 1 and 20 will be generated, and if the number is 7 (which has 5% chance of occurring), you win an all-expenses paid *three*-week tour of England, France, Italy.
>
> Lottery 4: A random number between 1 and 20 will be generated, and if the number is 7 **or** 10 (which has 10% chance of occurring), you win an all-expenses paid *one*-week tour of England.

Hints for Exercise 3:

- Let A denote a three-week tour of England, France, and Italy; let B denote a one-week tour of England, and let C denote the situation in which no prize is awarded. For each of Lottery 1-4, use an expression of the form XpY to abbreviate the lottery, where p represents the probability of getting X and $(1-p)$ is the probability of Y. For example, L_1 can be abbreviated ApC if $p = .5$.

- Let p be any probability that is strictly greater that zero. If a person prefers L_1 to L_2, what can you infer about her preferences between L_1pC and L_2pC if her preferences satisfy von Neumann and Morgenstern's axioms? Which axioms are relevant?

- Now consider if $p = .1$. Let L_5 be the lottery L_1pC. If a person's preferences satisfy von Neumann and Morgenstern's axioms, what do you know about how person views the relationship between L_5 and the four lotteries in the choices below? Which axioms are relevant? Ask yourself the same question about the lottery L_6 that is abbreviated L_2pC.