Decision Theory: Problem Set 1

Directions: For each exercise below, show all your work, and explain what you've done *in complete English sentences.* Further directions for how to typeset your problem sets are available on the Canvas website. If you prefer to write your problem set by hand, you must submit a paper copy of your problem set, as explained on the syllabus. See the syllabus for more details about submitting problem sets.

Exercise 1: Find a *single* decision problem in which the following decision rules deem different sets of actions choice-worthy: maximin, optimism-pessimism (with your choice of parameter), and minimax-regret. To do so, first describe the problem informally, and then describe it using a decision matrix with numerical utilities. In other words, tell a bit of a story (e.g., a paragraph) about a decision problem a person might encounter and pick numbers that might represent how much different outcomes would be valued by such a person.

Finally, specify a plausible parameter for the optimism-pessimism rule and show that (1) the set of maximin actions is not identical to the set of minimax regret actions, (2) the set of maximin actions differs from the set of actions permissible according to the optimism-pessimism rule, and so on.

Exercise 2: Some decision theorists defend a principle that is called *Chernoff's condition* or *Sen's* α. The principle is best illustrated by an example. Suppose you are choosing dessert at a restaurant to share with a friend. The menu says that you may choose chocolate cake, lemon cake, or a scoop of ice cream. You think the chocolate cake sounds best, and your friend agrees to order it.

Now suppose the waiter comes to the table and tells you the ice cream is unavailable. Your friend says, "Well, we had settled on getting the chocolate cake anyway." You interrupt your friend, however, and say "No. Now that ice cream is no longer an option, only the lemon cake is acceptable." Your friend thinks you're behaving irrationally. You didn't want the ice cream. So how did the elimination of an "irrelevant" option change which cakes were choice-worthy? Chernoff's condition captures your friend's judgment: it entails that your behavior is irrational.

More generally, given two set of options/actions S and T, let $C(S)$ and $C(T)$ respectively denote the options that you deem choiceworthy (or acceptable) when choosing from options/actions in S and when choosing from options in T. Then Chernoff's condition asserts that, for any option x, IF (i) S is a subset of T (i.e., $S \subseteq T$), (ii) $x \in S$, and (iii) $x \in C(T)$, THEN it ought to be the case that $x \in C(S)$. In the above example, your set of options was $T = \{\text{Chocolate cake, Lemon cake, Ice cream}\}$ and you deemed the set of desserts $C(T) = \{\text{Chocolate cake}\}$ choiceworthy. Next, let $x =$"chocolate cake" and let $S = \{\text{Chocolate cake, Lemon cake}\}$ be the set of options you had upon the waiter's return. Then $S \subseteq T$, as the waiter did not notify you any new desserts were available, and moreover $x \in S$, as you learned only the ice cream was unavailable. However, your preferences violate Chernoff's condition because $x \notin C(S) = \{\text{Lemon cake}\}$, as you deemed only lemon cake choiceworthy upon learning only the cakes were available.

Show that, when choosing among actions, the minimax regret principle violates Chernoff's condition. To do so, find numerical utilities $u_1, u_2, \ldots u_6$ that can be plugged into the following matrices so that (i) if one uses the minimax regret rule to choose between Action 1 one and Action 2, then only Action 1 is choiceworthy but (ii) if one uses the minimax regret rule to choose between Actions 1, 2, and 3, then only Action 2 is choiceworthy. Make sure to show your work by including the regret matrices in your answer. Briefly explain why your example violates Chernoff's action by saying which actions belong to the sets $S, T, C(S)$, and $C(T)$ in your example and what the option x is. Finally, tell a plausible story about what the actions and states might represent, and explain why the actions might plausibly have the numerical utilities you have chosen in each state of the world. Your story should *not* involve choices of cakes at a restaurant.

Problem 1

	State 1	State 2
Action 1	u_1	u_2
Action 2	u_3	u_4

Problem 2

	State 1	State 2
Action 1	u_1	u_2
Action 2	u_3	u_4
Action 3	u_5	u_6

Exercise 3: Another plausible principle for rational choice is sometimes called *column linearity*. Consider the two (abstract) decision matrices below, where the outcomes are assigned numerical utilities u_1, u_2, u_3 and u_4. Column linearity entails that, if the two decision matrices contain numerical utilities and if c is any number, then a rational decision-maker ought to choose the same actions in both decision problems. Show that maximin violates column linearity by finding appropriate values for u_1, u_2, u_3, u_4 and c. Tell a plausible story about what the actions and states might represent in the two decision problems.

Problem 1

	State 1	State 2
Action 1	u_1	u_2
Action 2	u_3	u_4

Problem 2

	State 1	State 2*
Action 1	u_1	$u_2 + c$
Action 2	u_3	$u_4 + c$

Exercise 4:

A. Consider the two decision matrices below. Determine what actions are choice-worthy according to the following decision rules: maximin, minimax regret, principle of insufficient reason, and the optimisim pessimism rule (with $\alpha = 1/2$).

B. Notice the utilities in the second decision matrix below are related to the corresponding entries in the first matrix via the function $f(x) = x^2 + 1$. For instance, Act 2 yields a utility of 3 in State 2 in the first decision matrix, and it yields a utility of $10 = 3^2 + 1$ in the second matrix in State 2. As a second example, Act 3 yields a utility of 2 in State 3 in the first matrix, and it yields a utility of $5 = 2^2 + 1$ in the second matrix. Show the function $f(x) = x^2 + 1$ is *strictly monotonically increasing* if $x \geq 0$: that means that if $0 \leq x < y$, then $f(x) < f(y)$. [A graph suffices here, but if you learned other methods for showing a function is monotonic in a calculus or algebra class, feel free to use them]. Then discuss what this implies about what a decision theorist needs to know about the scale of a subject's utility function if the theorist wants to predict the subject's behavior and believes the subject uses one of the above decision rules.

Problem 1

	State 1	State 2	State 3
Action 1	0	1	0
Action 2	1	3	0
Action 3	0	2	2

Problem 2

	State 1	State 2	State 3
Action 1	1	2	1
Action 2	2	10	1
Action 3	1	5	5

1 Bonus/Challenge/Extra-Credit Exercises

Bonus/Challenge Exercise 1: Which other decision rules (if any) that we have discussed violate column linearity? Justify your answer.

Bonus/Challenge Exercise 2: Which other decision rules (if any) that we have discussed violate Chernoff's Condition. Justify your answer.

Bonus/Challenge Exercise 3: Consider the following two decision problems:

Problem 1: Imagine you're a fan of Pittsburgh sport teams, and suppose you prefer watching football to baseball. You're at a local Seattle sportsbar and you're considering asking the bartender to put on a Steelers game. However, there might be an annoying Seahawks fan present who will harass you incessantly about the 2006 Superbowl if he learns that you cheer for a Pittsburgh team. If you don't watch a game, you'll just read a book at the bar, like a hipster.

	Annoying Fan Present	No Annoying Fan
Request Steelers game	r_1	r_2
Read a book	s_1	s_2

Problem 2: This problem is nearly identical to the last, but this time, you're considering asking the bartender to put on a Pittsburgh Pirates game.

	Annoying Fan Present	No Annoying Fan
Request Pirates game	u_1	u_2
Read a book	s_1	s_2

Suppose that the outcomes above are numerical and that $r_1 > u_1$ and that $r_2 > u_2$. Which decision rules (if any) would deem only choosing to read choice-worthy in the first problem and only requesting the Pirates game choice-worthy in the second problem? Explain your answer. Here, you should consider the following rules: maximin, minimax regret, optimism pessimism, and the principle of insufficient reason. When answering this question for the optimism-pessimism rule, you should assume that the decision-maker uses the same parameter for the optimism-pessimism rule in the two problems.

Importantly, you should make no other assumptions about the numerical relationships other than that $r_1 > u_1$ and that $r_2 > u_2$. For example, you should *not* assume $u_2 > u_1$. The stories above are intended to make the decision problems vivid, and this exercise can be solved without considering the stories at all.