Peirce and Brouwer

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Abstract

Although C.S. Peirce's logic has been studied extensively, few have noticed the remarkable resemblance between his ideas on continuity and those of L.E.J. Brouwer. This oversight is especially surprising because Peirce explicitly denies that the law of excluded middle holds for propositions concerning real numbers. This paper provides a detailed comparison of C.S. Peirce and L.E.J. Brouwer's concepts of continuity and the logic of real numbers. I will trace three major themes in their respective work, which highlight the striking similarities in their views about the *creation, composition, and logic* of the continuum.

1 Introduction

In his 1908 "The Unreliability of the Logical Principles", Luitzen Egbertus Jan Brouwer rejected the law of excluded middle (LEM) and sparked what Weyl would later call "the revolution" in the foundations of mathematics. Five years earlier, Charles Sanders Peirce had reached similar conclusions: he claimed that the concept of continuity required abandoning LEM as well. In an unpublished note, Peirce wrote, "Now if we are to accept the common idea of continuity...we must either say that a continuous line contains no points...or that the law of excluded middle does not hold of these points. The principle of excluded middle applies only to an individual...but places being mere possibilities without actual existence are not individuals."[1] Ultimately, Peirce endorsed both halves of the disjunction above. For Peirce,

[1]Peirce, C.S. 1931-58. The Collected Papers of Charles Sanders Peirce, eds. C. Hartshorne, P. Weiss (Vols. 6) and A. Burks (Vols. 7-8). (Cambridge MA: Harvard University Press). See section 168 in volume six. Because the pagination within this edi-

the continuum is not simply a collection of points, and moreover, LEM fails to hold for propositions about real numbers.

The purpose of this paper is to show that Peirce and Brouwer's common rejection of LEM is not simply a coincidence, but rather, stems from a deep underlying similarity in their respective philosophical analyses of the continuum.[2] In the Peircean spirit, I analyze three common themes in Peirce and Brouwer's work. The first theme concerns Peirce and Brouwer's views about the *creation* of the continuum. For both, the experience of two distinct moments in thought, connected by a continuous succession of ideas in consciousness, is the philosophical basis for the concept of continuity. Importantly, both argue that there is a strong connection between the continuity of thought and the continuity of time.

The second major theme concerns Peirce and Brouwer's views about the *composition* of the continuum. Both Peirce and Brouwer argue that the continuum is irreducible, in the sense that the continuum is not a set of zero-dimensional points, but rather is composed of many small intervals (or "infinitesimals" for Peirce). Each part of the continuum is, in a sense, a continuum itself. Peirce and Brouwer are, of course, not the only philosophers who have held that the continuum was irreducible in this sense. In describing the historical backdrop for his development of smooth infinitesimal analysis, John Bell has found that Aristotle, Leibniz, and Kant, for example, all held similar views about the continuum at one point or another. But even more

tion of the collected works is both unhelpful and erratic, references to the collected works of Peirce will henceforth be given in the form CW X.Y, where X designates the volume number and Y denotes the section number.

[2]To my knowledge, John Bell is the only logician/historian who has noticed the similarity between Peirce and Brouwer's philosophical views about continuity and even hinted that this similarity might be explained by their common rejection of LEM. In providing motivation for smooth infinitesimal analysis (SIA), Bell surveys a number of philosophical figures since antiquity who have held views about the composition of the continuum that differ from those views which have become dominant over the past century and a half within real analysis and set theory. Bell notes that both Peirce and Brouwer reject the law of excluded middle, and further, that they both claim the continuum is irreducible to a set of points. This paper expands on Bell's observations, identifying the common philosophical foundation for continuity in Peirce and Brouwer's work. For an enlightening survey of related philosophical views about the continuum, see John Bell. *A Primer of Infinitesimal Analysis,* (Cambridge University Press, 1998). Also, see his two unpublished lectures on infinitesimals entitled "Dissenting Voices: Divergent Conceptions of the Continuum in 19th and Early 20th Century Mathematics and Philosophy." and "An Invitation to Smooth Infinitesimal Analysis."

striking similarities in Peirce and Brouwer's views about the composition of the continuum emerge when comparing their respective attempts to reconcile their simultaneous rejections of the use of completed infinite totalities and their acceptance of Cantor's proof of the uncountability of real numbers.

Finally, given their similar analyses of the creation and composition of the continuum, it is not surprising that Peirce and Brouwer reached similar conclusions about the *logic* of real numbers.[3] As noted above, both Peirce and Brouwer rejected LEM as it pertained to propositions about real numbers. For both, this rejection stemmed from the fact that the irreducibility of the continuum implied that some real numbers were "indistinguishable" from one another.

There are a number of other striking similarities between Peirce and Brouwer's views about logic and mathematics, which unfortunately, I cannot describe in depth here. For example, both deny that logic is a legitimate foundation for mathematics. Rather, for both Peirce and Brouwer, logic depends upon or is a report of mathematical activity. One might also investigate the parallel between Peirce's voluntarism about beliefs and Brouwer's ideas about creation in mathematics.[4] Analyzing these similarities may prove illuminating in understanding Peirce and Brouwer's philosophical views on logic and mathematics in general.

Finally, before I begin, I should note that Peirce and Brouwer seemed to have no knowledge of each other's work. Brouwer might have learned of Peirce's ideas on semiotics in the 1920's through his association with Lady

[3]Strangely, almost all discussions of Peirce's analysis of the law of excluded middle fail to discuss intuitionistic logic at all. The only exceptions are John Bell's lectures and Timothy Herron's paper, which builds on Bell's work. See Timothy Herron. "C. S. Peirces Theory of Infinitesimals." *Transactions of the Charles S. Peirce Society.* Vol. 33. No.3. Summer 1997. The most extensive discussion of Peirce on the law of excluded middle can be found in Robert Lane "Peirce's Entanglement with The Principle of Excluded Middle and Contradiction." *Transactions of the Charles S. Peirce Society.* Vol. 33. No.3. Summer 1997. For Lane, the failure to discuss intuitionism is understandable: Lane argues that Peirce does not understand the law of excluded middle to be the principle $p \vee \neg p$. Rather, according to Lane, Peirce rejects the law of excluded middle because he is reacting to a common mistake of his contemporaries concerning the difference between negating an entire sentence and negating the predicate of the sentence. That is, Lane argues that Peirce endorses classical logic, despite the initial appearance. While Lane is correct in noticing Peirce's concern with the scope of negation (and quantification), I will argue that his interpretation becomes problematic when considering Peirce's discussion of LEM and its relation to continuity

[4]This parallel was suggested to me by Teddy Seidenfeld

Welby, a renowned Peirce scholar, in the International Academy of Philosophy. However, the two most likely worked independently, and the commonalities in their views might be explained, in part, as a reaction to the late 19th and early 20th century developments in set theory and the foundations of real analysis.

2 The Creation of the Continuum

Both Peirce and Brouwer were heavily influenced by Kant's philosophy of mathematics. For Brouwer, this influence becomes manifest in his discussion of time. Brouwer argues that Kant is correct in attributing to rational agents a fundamental intuition of time (though, according to Brouwer, Kant was incorrect about the intuition of space). Like Kant, Brouwer claims the intuition of time allows one to represent an infinite succession of discrete objects. This infinite succession, then, provides the philosophical basis for the natural numbers and arithmetic operations. He writes:[5]

> The first act of intuitionism separates mathematics from mathematical language, in particular from the phenomena of language which are described by theoretical logic, and recognizes that intuitionist mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time, i.e. the falling part of a life moment into two distinct things, one of which gives ay to the other, but is retained by memory. If the two-ity thus born is divested of all quality, there remains the empty form of the common substratum of all two-ities. It is this common substratum, this empty form, which is the basic intuition of mathematics.

Brouwer's terminology is somewhat opaque, but the idea is very simple. Right now, I am thinking, and I am experiencing particular sights, sounds, smells, and so on. At the same time, I can remember the sights I saw, the sounds I heard, and the thoughts I had just a moment ago. This is what Brouwer calls "the falling apart of a moment of life into two distinct things." The two distinct moments in my thought generate an abstract, mathematical

[5] L.E.J. Brouwer. "Historical Background, Principles and Methods of Intuitionism." *South African Journal of Science*, 49:139146, 1952.

concept of "two-ity," which is what remains when I 'subtract' the content of the two experiences. As I now experience this moment and my memory of the past, a new set of thoughts and experiences begin to enter my mind, and the falling apart of my life occurs again. Brouwer, therefore, argues that the abstract notion of an infinite succession of discrete points (i.e. the natural numbers) originates with our experience of having our mental life repeatedly "fall apart" into two halves: the past and the now. In "Mathematics, Science, and Language," he explains more fully how the successive falling apart of life into two stages generates the natural numbers:[6]

> Mathematical Attention as an act of the will serves the instinct for self preservation of individual man; it comes into being in two phases; time awareness and causal attention. The first phase is nothing but the fundamental intellectual phenomenon of the falling apart of a moment of life into two qualitatively different things of which one is experienced as giving away to the other and yet is retained by an act of memory. At the same time this split moment of life is separated from the Ego and moved into a world of its own, the world of perception. Temporal two-ity, born from this time awareness, or the two-membered sequence of time phenomena, can itself again be taken as one of the elements of a new two-ity, so creating temporal three-ity, and so on. In this way, by means of the self-unfolding of the fundamental phenomenon of the intellect, a time sequence of phenomena is created of arbitrary multiplicity.

Importantly, in the first passage above, Brouwer argues that the "perception of a move of time," and not time simpliciter, provides the philosophical foundations for arithmetic. Here, Brouwer's choice of words is not simply rhetorical. Unlike Kant, Brouwer distinguishes between two types of time: subjective and objective. Subjective time is our experience of time, whereas objective time is a quantity measurable by clocks. The distinction is important because the former is the philosophical basis for mathematics, whereas the latter is, in some sense, a quality of the world independent of human beings. Therefore, for Brouwer, the concepts of mathematics are *created or*

[6] Brouwer, "Mathematics, Science, and Language." Reprinted in *From Brouwer to Hilbert.* Ed. Paolo Mancosu. (Oxford University Press, 1998). pp.45

constructed by agents, and so mathematical theorems are not facts about the world.

We've seen that, like Kant, Brouwer argues that the natural numbers are representable in the intuition of time. Unlike Kant, however, Brouwer argues that continuity is also representable in intuition, as there is a "between-ness" that connects adjacent elements in the infinite series.

> We shall go further into the basic intuition of mathematics (and of every intellectual activity) as the substratum, divested of all quality, of any perception of change, a unity of continuity and discreteness, a possibility of thinking together several entities, connected by a between, which is never exhausted by the insertion of new entities.

Brouwer's notion of "falling apart," therefore, provides the philosophical basis for the concept of continuity as well. Here, we see that Brouwer claims that the adjacent moments in the succession of our thoughts are connected by an irreducible "betweeen-ness." In the next section, I'll attempt to clarify what Brouwer meant by "between-ness," but one remark is appropriate now. When an agent experiences his or her life "falling apart" into two distinct moments, say A and B, he or she cannot pinpoint a third distinct moment in between A and B, in which the falling apart had occurred. The concept of continuity that is derived from intuition, therefore, is not composed of a set of discrete points.

Though many historians of logic have found Brouwer's views to be rather idiosyncratic and unique, Peirce embraced strikingly similar beliefs about the continuum and its relation to continuity of thought. To understand Peirce's view, however, it is necessary to make a few distinctions. First, Peirce distinguishes between the "common sense" notion of continuity and the "mathematical" notion of continuity.[7] Despite its name, the "common sense" notion of continuity *is* amenable to rigorous, mathematical development. What Peirce means to emphasize is that mathematicians had chosen to define the set of real numbers and continuity in a particular way, mainly, by relying on constructions involving limits. Peirce calls the use of limits as the foundational concept of analysis "the doctrine of limits." The "doctrine of limits," however, fails to capture the "common sense" notion of continuity according to Peirce, and so many of his writings are dedicated to rigorously

[7] CW 6.168

developing an alternative definition of the continuum and continuity in general. Peirce writes:

> In the calculus and the theory of functions, it is assumed that between any two rational points (or points at distances along the line expressed by rational fractions) there are rational points and further that for every convergent series of such fractions...there is just one limiting point; and such a collection of points is called continuous. *But this does not seem to be the common sense idea of continuity.* It is only a collection of independent points. Breaking grains of sand more and more will only make the sand more broken. It will not weld the grains into unbroken continuity.

How does Peirce define the "common sense" notion of continuity? He begins by saying, "On the whole, therefore, I think we must say that continuity is the relation of the parts of an unbroken space or time."[8] Peirce's notion of the continuum, therefore, requires an analysis of space or time. We shall see time is the more central concept of these two. Like Brouwer, Peirce's analysis of time in turn requires one to analyze thought. In an 1892 article entitled "The Law of Mind," Peirce struggles to analyze how a "past idea can be present."[9] That is, Peirce is concerned with a number of related questions about our mental lives and personal identity, such as, "How is memory possible?" and "How can a succession of thoughts and sensations be considered part of one, unified consciousness?"

Peirce's solution to the problem involves two clams: (1) an individual's past ideas are connected to the present ones through a series of infinitesimal intervals of thought, and (2) past ideas are capable of *affecting* present ones. The conjunction of these two claims is the the thesis to Peirce's article, and he calls the thesis "the law of mind." Although what Peirce means by one idea "affecting" another is unclear, what is crucial is that the affectability relation on ideas gives rise to an ordering on instants in time. Peirce claims, "One of the most marked features about the law of mind is that it makes time to have a definite direction of flow from past to future."[10] In particular, Peirce argues the affectability relation between ideas is (to use modern terms) a dense ordering, and so the ordering induced on moments in time is also a

[8] Ibid; Peirce discusses in several passages, however, that he believes that time is the "spectacles" through which we must understand all other continuums.

[9] CW 6.107

[10] CW 6.127

dense. Similarly, Peirce also claims that the continuity of intensity of feeling provides a way for defining the continuity of time.[11] In sum, Peirce argues that an analysis of continuity requires an analysis of time, which in turn requires an analysis of continuity of thought and intensity of feelings.

However, for Peirce, the relationship between time and consciousness is not so straightforward as I have represented. Peirce also claims that thoughts do not occur at discrete instants, but rather, they are spread out over very small intervals of time:[12]

> But yet consciousness must essentially cover an interval of time; for if it did not, we could get no knowledge of time, and no merely no veracious cognition of it, but no conception whatsoever. We are, therefore, forced to say that we are immediately conscious through an infinitesimal interval of time.

The problem is that we seem to have encountered a vicious circle. As noted above, Peirce argues the concept of time requires an analysis of consciousness. But here, Peirce seems to be claiming that to understand consciousness, we must have an understanding of the continuity of time. The only way I can see to avoid this circle is to read Peirce as postulating a distinction, like Brouwer, between two types of time. The type of time that provides the philosophical basis for continuity is experienced: it is the continuity that we experience as connecting our ideas or the continuity we feel in the ranges of our feelings. The second type of time, over which conscious thought is spread, is something that is experimentally measured. This would explain why Peirce claims that "consciousness must essentially cover an interval of time" in the middle of a discussion of experimental psychology of his day.

Given this discussion, the parallel between Peirce's and Brouwer's ideas is clear. Both argue that the continuum of real numbers is defined in terms of continuity of subjective or experienced time. Continuity of time, in turn, is defined in terms of two distinct moments in consciousness, connected by a flow of thought. For both, the flow of thought has a definite ordering. For Peirce, the flow of ideas is ordered by affectability relation between the two ideas, and for Brouwer, the ordering is induced by the "falling apart" of time into a past and a now. Finally, because the concept of the continuum is a

[11] CW 6.132
[12] CW 6.110

product of our subjective experience of time, both Peirce and Brouwer argue that the continuum is “created” by human beings in one way or another.[13] In the next section, I’ll show that their similar ideas about the origin of the concept of the continuum give rise to strikingly similar views about what the continuum *contains*.

3 The Composition of The Continuum

Peirce and Brouwer both argue the continuum is irreducible, in the sense that the continuum does not consist of points, but rather, is composed of many tiny intervals. Brouwer, therefore, argues that neither continuity nor discreteness can be defined in terms of the other. He writes:

> continuity and discreteness occur as inseparable complements, both having equal rights and being equally clear, it is impossible to avoid one of them as a primitive entity, trying to construe it from the other one, the latter being put forward as self-sufficient.

How does Brouwer reach this conclusion? For Brouwer, the inability to define continuity and discreteness in terms of one another is a consequence of the fact that our thoughts are not merely discrete, disjoint points. Rather, our thoughts and experiences occur one after another in such a way that they overlap, and are spread out over intervals, not instants, of time. Mark Van Atten has provided both a clear summary and diagrammatic representation of Brouwer’s view.[14] He writes:

> ...we see that particular stages are not cut off from one another as thought there were isolated, atomic points. Rather, there will always be connections between earlier and later phases by way of retentions and protentions, along with secondary memory. There is always an overlap of phases. The stream of consciousness is a continuous fabric.

[13]Peirce explicitly uses the word “creation” in describing the continuum in 6.211 and ?

[14]See Mark Van Atten, Dirk Van Dalen, and Richard Tieszen. “The Phenomenology and Mathematics of the Intuitive Continuum.” *Philosophia Mathematica.* Vol. 10. No. 2. 2002. pg. 203-206.

Here, Van Atten uses the word, "retention" to mean, roughly, "awareness or memory of a moment ago." Van Atten uses this terminology to emphasize that the awareness or memory of a moment becomes fainter as the moment sinks further and further into the past, and moreover, that retentions overlap so that they may be present in continuous degrees as time passes. Similarly, "pretention" is the expectation of how a thought or idea will be completed. Terminology aside, there are two important features of Brouwer's position. First, one cannot isolate a thought as a occurring at a discrete instant in (objective) time, but rather, thoughts are spread out over small intervals of time. Second, past thoughts are connected to current ones through overlapping, small intervals of conscious thought.

Both of these features of Brouwerian thought are likewise present in the writings of Peirce. In an attempt to analyze how past ideas can affect present ones, Peirce argues that thought must occur over an infinitesimal interval of time. What is an infinitesimal for Peirce? Though Peirce worked in the late 19$^{\text{th}}$ and early 20$^{\text{th}}$ century, there is good reason to suspect that he anticipated many of the features of a more modern theory of infinitesimal analysis.[15] In particular, Peirce understood that any field that contained infinitesimals would need to be non-Archimedean, and thus, for Peirce, there was no contradiction in asserting that between any two points, there existed an arbitrarily large finite number of intervals (of non-zero length). Peirce, therefore, concluded that past ideas could affect current ones through a sequence of overlapping, infinitesimally small intervals of thought.

> How can a past idea be present? Not vicariously. Then, only by direct perception. In other words, to be present, it must be *ipso facto* present. That is, it cannot be wholly past; it can only be going, infinitesimally past, less past than any assignable past date. We are thus brought to the conclusion that the present is connected with the past by a series of real infinitesimal steps.

Therefore, because continuity of thought provides the foundations for the common-sense concept of the continuum, and because thought is spread out over infinitesimal intervals of (objective) time, the continuum does not consist of points but rather is composed of infinitesimally small intervals. Peirce traces this idea back to Kant:[16]

[15] Herron. "Peirce's Theory of Infinitesimals."
[16] CW. 6.168

> Kant's definition, that a continuum is that of which every part has itself parts of the same kind, seems to be correct...In accordance with this, it seems necessary to say that a continuum, where it is continuous, contains no definite parts; that its parts a created in the act of defining them and the precise definition of them breaks continuity.

It now becomes clear why both Peirce and Brouwer argue that the continuum is irreducible to a set of discrete points. In the first section, we saw that both argue that the philosophical foundations for continuity require an analysis of subjective time, which in turn requires an analysis of conscious thought. For both, thought is not discrete, but rather, is spread out over some interval of objective time. Therefore, the continuum must likewise consist of intervals, rather than discrete points.

There are two other central issues concerning the composition of the continuum. First, although we know both Peirce and Brouwer argue the concept of the continuum is rooted in some intuition of time, how are elements of the continuum mathematically constructed? That is, do Peirce and Brouwer both use sequences, bounded sets, or some other type of mathematical construction to define real numbers in terms of rational numbers. Second, what is the cardinality of the continuum, and how does the concept of cardinality fit into their respective philosophies?

Although I have thus far stressed the similarities between Peirce and Brouwer's views, I must confess that their views on these matters diverge. Peirce identifies real numbers with a bounded, monotone increasing sequences of rational numbers,[17] whereas a least upperbound of such a sequence may not exist within intuitionistic mathematics. Peirce defines a real number in this way, instead of by using Cauchy sequences, because he argues that the concept of the continuum does not require metrical notions. In contrast, Brouwer constructs real numbers by use of choice sequences. A choice sequence is a sequence of nested intervals of rational numbers such that each interval is at most half the size of the previous one. Importantly, a choice sequence is not a finished infinite object, but rather, one whose properties are determined more and more precisely as time flows forward. Brouwer

[17]See 6.122. In this section, Peirce argues that density plus the least upperbound property are sufficient to distinguish the real numbers from the rationals

explains:[18]

> In intuitionist mathematics a mathematical entity is not necessarily predeterminate, and may, in its state of free growth, at some time acquire a property which it did not possess before.

For Brouwer, then, mathematical truth is not time independent, in the sense that a proposition p is not true until its truth is experienced by the creating subject. Although Peirce did not define real numbers in terms of choice sequence, we shall see, in the next section, that Brouwer's concept of a choice sequence does have a precise analog in Peirce's theory of signs (mainly, in Peirce's concept of "generality").

With regards to the cardinality of the continuum, Peirce and Brouwer share some views but diverge on others. Both accept Cantor's proof that the real numbers are not countable. Brouwer, therefore, refers to the real numbers as "denumerably unfinished."[19] A set S is denumerably unfinished if whenever one constructs a countable subset $S' \subset S$ one can then construct an element $x \in S \backslash S'$. Likewise, Peirce argues that there is no limit to the series of "abnumeral multitudes" amongst "multitudes of distinct individuals."[20] That is, Peirce accepts Cantor's proof that the power set *of a set of distinct individuals* has more members than the original set. He therefore accepts that one can construct increasingly large infinite sets in one way or another.

Brouwer maintains, however, that no cardinal numbers beyond the continuum exist. This is a consequence of his view that a mathematical object exists if and only if it can be constructed by one of the two acts of intuitionism. For Brouwer, the first act permits one to construct the *potentially* infinite set of natural numbers (and more generally, countable ordinals) and the *actually* infinite continuum. No other such infinite sets, however, can be constructed according to the two acts of intuitionism.

Does Brouwer's rejection of larger cardinals constitute a disagreement with Peirce? Yes and no. Although Peirce accepts Cantor's proof, he denies

[18] L.E.J. Brouwer. The Effect of Intuitionism on Classical Algebra of Logic. *Proceedings of the Royal Irish Academy,* 57:113, 116, 1955.

[19] Brouwer. L.E.J. Brouwer. *Over de Grondslagen der Wiskunde.* Ph.D. Thesis, Universiteit van Amsterdam, 1907. Quoted from the English translation in L.E.J. Brouwer. *Collected works I. Philosophy and Foundations of Mathematics.* Ed. A. Heyting. North-Holland, Amsterdam, 1975. pp.13

[20] CW. 6.185

that the continuum is a set of *distinct individuals*, as the continuum is irreducible to a set of points. The power set operation, therefore, is inapplicable to the continuum. This explains why Peirce claims the continuum is more multitudinous than any collection of individuals.[21] In a sense, then, Peirce and Brouwer agree that there are no infinite sets larger than the continuum.

Additionally, both Peirce and Brouwer pay close attention to the distinction between potentially infinite and actually infinite sets. Peirce, for instance, claims that although one can finish counting from 1 to any fixed finite whole number (given a long enough life time), the set of all whole numbers cannot be completely counted, as finishing would require that there be a greatest whole number. He therefore refers to the whole numbers as a "potential collection."[22] In general, Peirce argues that all infinite sets are potential collections. Likewise, for Brouwer, the "falling apart" of life, which generates the mathematical concept of the natural numbers, produces a collection that is continually expanding but is never a finished infinite totality. Similarly, we've seen that a choice sequence is not a completed object, but rather is continually acquiring new properties.

Therefore, although Peirce and Brouwer diverge on their views about the existence of larger cardinals and how the real numbers ought to mathematically constructed from the rationals, they both fundamentally agree that the continuum is irreducible to a set of discrete points and that this irreducibility somehow prevents Cantor's proof from generating a set larger than the continuum. In the next section, I discuss how Peirce and Brouwer's views on the irreducibility of the continuum implies that there are real numbers that are indistinguishable in such a way that prevents LEM from holding.

4 The Logic of the Continuum

Given Peirce and Brouwer's common views about the creation and composition of the continuum, it is not surprising that both argue that LEM is not, in general, valid for propositions concerning real numbers. I will begin by briefly summarizing Brouwer's view. Because Brouwer's views on LEM are well-known, however, the majority of this section is dedicated to explaining how Peirce's concept of continuity requires the abandonment of LEM and how is view is similar to that of Brouwer.

[21] CW. 6.185, 7.209

[22] CW 6.186

Recall, for an Brouwer, a proof requires a construction of some type. For example, an existential claim of the form $\exists x \in \mathbb{N}(P(x))$ requires one to explicitly provide a natural number for which P holds; it is not sufficient to simply prove the impossibility of $\neg P(x)$ for all natural numbers x. This stringent requirement of providing explicit constructions renders many classical proofs invalid. For instance, in the typical proof that there are irrational numbers a and b such that a^b is rational, one does not explicitly construct the numbers a and b. Rather, one reasons to the effect that if $\sqrt{2}^{\sqrt{2}}$ is rational, then we can let $a = b = \sqrt{2}$, and otherwise, we can let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. For Brouwer, however, such a proof only tells us how we could construct a and b given additional assumptions. Brouwer's philosophical view, then, motivates the so-called BHK interpretation for logical connectives. In particular, in order to prove $p \vee q$, one must explicitly construct a proof of p or a proof of q. Hence, the formula $\varphi \vee \neg\varphi$ is not a tautology, but rather, requires a proof of φ or $\neg\varphi$ in order to be accepted.

In several passages, Brouwer discusses the relationship between LEM and Hilbert's claim that all mathematical problems are solvable. Brouwer argues that the two claims are equivalent, as we can be ensured that a mathematical theorem is true or false only when we have a proof of it or its negation. As will become clear, Brouwer's view on the solvability of mathematical problems is part of his general view that neither mathematical truths nor mathematical objects are static: new objects and truths can come into existence through the mental efforts and constructions of mathematicians. As Hilbert's faith in the solvability of all mathematical problems requires viewing mathematical truth as a static set of theorems gradually uncovered (but not created) by mathematicians, his view must be rejected.

On first glance, Peirce's rejection of LEM seems to be motivated by different factors than is Brouwer's. In his discussions of continuity, Peirce first rejects LEM by pointing to a difference between individuals and sets of individuals:[23]

> Now if we are to accept the common sense idea of continuity, we may say either that a continuous line contains no points or we must say that the principle of excluded middle does not hold of these points. The principle of excluded middle only applies to an individual (for it is not true that "Any man is wise" nor that "Any

[23] CW 6.168

> man is not wise"). But places, being mere possibilities without actual existence, are not individuals. Hence, a point or indivisible place really does not exist unless there actually be something there to mark it, which, if there is, interrupts the continuity.

In his analysis of Peirce's view on the law of the excluded middle, Robert Lane claims that Peirce's argument in the above passage is not a rejection of the LEM, as is understood by modern logicians. Rather, according to Lane, Peirce is claiming that the logical negation of $\forall x \varphi(x)$ is not $\forall x \neg\varphi(x)$. This is the obvious way of reading the above passage, especially given Peirce's example that neither "Any man is wise" nor "Any man is not wise" is true. For Peirce, the sentence "Any man is wise" is an example of a *general* sentence, and so, roughly, Lane understands Peirce's definition of a general sentence as the natural language counterpart to a universally quantified sentence of first order logic. Later, I will argue that this is not the best way of understanding Peirce's definition of generality. According to Lane, however, Peirce is commenting on (what he perceives to be) a mistake of his contemporaries concerning the scope of negation and quantification; he is not rejecting classical logic in favor of some form of intuitionism.

While I agree with Lane that Peirce is concerned with how negation interacts with quantification, I think Lane's analysis cannot be the full story of Peirce's concern with LEM as it does not explain why Peirce's discussion of LEM often (a) employs modal language and (b) occurs within the context of a broader discussion of continuity.[24] In the above passage, for instance, notice Peirce talks about "places" as "*mere possibilities* without *actual* existence," and he further argues that a place "does not exist *unless there actually be something there to mark it*," thus employing explicit counterfactual language. Peirce's use of the modal terminology comes out more clearly in other passages:[25]

> Of course, there is a possible, or potential, point-place wherever a point might be placed; but that which only *may be* is necessarily thereby *indefinite*, and as such...it is not subject to the principle

[24]Lane recognizes (a) as a deficiency of his analysis in the last footnote of his paper, but he never mentions (b). After studying Lane's work, I believe that his failure to mention (b) is a result of his failure to analyze the relationship between continuity, possibility, and generality, which will be a central point in my comparison of Peirce and Brouwer

[25]CW 6.182

> of contradiction just as the negation of a may-be, which is of course a *must-be* (I mean that if "S *may be* P" is untrue, then "S *must be* non-P" is true)...is not subject to the principle of the excluded middle.

Here, one could understand Peirce, at face value, as arguing that one ought to be careful about the relationship between modal operators and negation. For example, if we interpret $\Box$ as necessity and $\Diamond$ as possibility, then the above passage can be taken as arguing that $\Diamond p \wedge \Diamond \neg p$ is not a logical contradiction, and the sentence $\Box p \vee \Box \neg p$ is not a logical truth. Again, however, I think this reading is too simplistic, as Peirce's discussion of LEM and modality often occurs within a broader discussion of continuity. For example, consider the following passage, quoted in its entirety, that deals with continuity of two-dimensional surfaces:[26]

> Suppose a piece of glass to be laid on a sheet of paper so as to cover half of it. Then, every part of the paper is covered or not covered; for "not" means merely outside of, or other than. But is the line under the edge of the glass covered or not? It is no more on one side of the edge than it is on the other. Therefore, it is either on both sides, or on neither side. It is not on neither side; for if it were, it would not be on either side, therefore, not on the covered side, therefore, not covered, therefore, on the uncovered side. It is not partly on one side and partly on the other, because it has no width. Hence, it is wholly on both sides, or both covered and not covered. The solution of this is, that we have supposed a part too narrow to be partly uncovered and partly covered; that is to say, a part which has no parts in a continuous surface, which by definition has no such parts. The reasoning, therefore, simply serves to reduce this supposition to absurdity.

Peirce continues:

> It may be said that there really is such a thing as a line. If a shadow falls on a surface, there really is a division between the light and the darkness. That is true. But it does not follow that because we attach a definite meaning to the part of a surface

[26]CW 6.336

> being covered, therefore we know what me mean when say a line is covered. We may define a covered line as one which separates two surfaces both of which are uncovered, or as one which separates two surfaces both of which are covered. In the former case, the line under the edge is uncovered; in the latter case, it is covered.

Here, Peirce claims that the solution to the problem of whether the boundary line is covered or not is solved by jettisoning the assumption that a two-dimensional surface contains one-dimensional lines as parts. That is, just as he claims that the continuum is irreducible to points, Peirce analogously argues that a two dimensional surface is irreducible to a collection of lines. Similar passages discussing the relationship between boundary lines on surfaces, continuity, and LEM appear elsewhere in Peirce's work.[27] In contrast to the passages discussed by Lane, it's not clear that this passage can be understood as analyzing the relationship between the scope of quantification, modal operators, and negation. On the other hand, Peirce does not seems to be rejecting LEM in this passage either. Is there any underlying theme in Peirce's discussion of LEM? A hint appears in Peirce's lecture,"The Logic of Continuity." Peirce asserts, "Continuity, as generality, is inherent in potentiality, which is essentially general."[28] Cryptic as this assertion is, it reveals Peirce's discussion of the the three above issues (generality, continuity, and possibility) are grounded in some common philosophical doctrine, which will help to shed the light on Peirce's rejection of LEM.

I begin with Peirce's discussion of generality. Peirce most clearly defines generality in a footnote in the paper "Issues of Pragmaticism."[29] There, Peirce distinguishes between two ways in which a sign could be "indeterminate." The second is generality, which "turns over to the interpreter the right to complete the determination [of a sign] as he pleases." He then calls a sentence general if either the subject or predicate of the sentence is a general sign.

Peirce provides several examples of general signs, but the most important example is the typical drawing or use of the word "triangle" in Euclidean geometry. In particular, his definition of a general sign is an explicit attempt

[27] CW 6.126, 6.203. Peirce also uses a metaphor involving a river, in which some property holds of one part of a river and not another, but one is unable to say where the property holds and where it does not in particular cross-sections 6.326

[28] CW 6.204

[29] CW 5.449

to answer Berkeley's objection to the existence of a concept of an abstract triangle, i.e one that is neither equilateral, isosceles, nor scalene.[30] Consider the general proposition "The sum of the interior angles of a triangle is 180 degrees." Suppose that the word "triangle" picks out a set of objects; the set contains some big triangles, some small ones, some scalene, some equilateral, and so on. One can then describe further and further constraints (say, by requiring the triangle be equilateral and have six inch sides) until the sign"triangle" designates a unique object. When Peirce discusses the "right of the interpreter," he is referring to this process of placing further constraints on the reference of a sign like "triangle." He calls this process *determination*. A general sentence is true, then, precisely when no matter what further constraints are placed on what the sign can signify, the sentence is true of the reference of the completely determined object.

How does the concept of generality relate to that of possibility? In the process of determination, the interpreter has many possible choices for the reference of the sign, and further, he may choose any number of *ways* of specifying the constraints that ultimately completely determine the reference. That is, generality provides the interpreter of the sign with the possibility of picking the reference of the sign and the possibility of picking *how* to determine reference of the sign. Now we can see why Peirce claims that LEM may not apply to general propositions. Suppose a general sentence is uttered, say, "The Milky Way is rather small." Suppose further that two different interpreters begin the process of determination for the sign "The Milky Way." The two interpreters, then, might determine different objects. For instance, one might determine a galaxy and the other a nearby candy bar, and so the two different interpreters may disagree about the truth value of the sentence. In this way, the original sentence, though making an assertion, is neither true nor false.

So far, however, my reading of Peirce does not seem to differ in any substantial way from that of Lane or those who discuss game theoretic semantics for Peirce's general and vague sentences. However, as I read Peirce, Lane and others make two assumptions that I feel are unwarranted. First, in sentences with multiple general signs, it's not clear that all general signs ought to be treated as universally quantified variables. Again, consider the sentence, "The Milky Way is rather small." The sign "small" is also general, so does the sentence simply behave like a first order sentence with two

[30] CW 5.372 Footnote 2

universal quantifiers in front, one ranging over possible determinations of "The Milky Way" and the other over possible determinations of "small?" I doubt so. Astronomers have developed specific criteria for what constitutes a small, medium-sized, and large galaxy. While these criteria may not be completely precise and while astronomers may disagree about the size of particular galaxies, the unanimous verdict would be that the Milky Way constitutes a small galaxy. In short, determining the sign "Milky Way" to be a galaxy seems to fix (at least to some degree) how one ought to determine the sign "small."

On the other hand, if one determines the sign "Milky Way" to be a candy bar, there are any number of ways to determine the sign "small." The speaker of the sentence might have been comparing two candy bars, in which case the Milky Way may or may not have been small on the scale of candy bars. The speaker may have been considering which types of food he could fit in his pocket for traveling, in which case the Milky Way may have been compared to any number of other foods which were much larger. That is, attributions of size for everyday objects may be interpreted in any number of different ways, whereas attributions of size for terrestrial and microscopic objects seem less open to interpretation. Therefore, depending on the way one determines "The Milky Way," the choice of the quantifier type for "small" may differ.

The options I have considered, however, are not exhaustive of the ways one could understand the sentence, "The Milky Way is small." Why should one determine the sign "Milky Way" first, rather than beginning with the sign "small?" Ought one to determine the signs "Milky Way" and "sign" independently of one another, or does determining one fix the other? Must we interpret a general sign as an existential or a universal quantifier, or are there cases in which it is better represented by a quantifier of the form "most," "few," "some," or "many?" In sum, Lane and others' readings of Peirce assume too much about (a) the order of quantifiers, (b) the type of quantifiers (e.g. existential versus universal, branching versus non-branching, and so on).

The second, and more important way, in which I think others have failed to correctly interpret Peirce is that they have failed to notice that the process of determination *may never terminate with a unique object.* This is the final missing link in understanding the relationship between generality, possibility, and continuity. In his essay "The Logic of Continuity," Peirce argues that the members of the continuum are not distinguishable by either monadic properties or asymmetric dyadic relations.[31] This implies that there is a part of the continuum x and an infinite set of properties $P_1, P_2, \ldots, P_n, \ldots$ such that one could say in succession whether x satisfies each of the P_i's but fails to distinguish x from some other distinct part of the continuum. Furthermore, even if an infinite set of properties does uniquely distinguish x, the human inability to carry out an infinite process implies that there are general signs for which one may not determine a unique reference. It is precisely these type of general signs that arise in discussions of continuity, and why LEM fails for propositions concerning real numbers.

Peirce's concept of a general sign that requires infinite steps to uniquely determine, then, is analogous to Brouwer's discussion of choice sequences. Just as the creating subject can choose the nested intervals of a choice sequence, Peirce allows the interpreter of a sign to progressively determine more information about the real number in question. Peirce and Brouwer's common rejection of LEM, therefore, stems from a common philosophical doctrine. For both, determining whether a real number possesses a particular property may require the ability to survey an infinite amount of data. Because human beings (for Peirce) and the creating subject (for Brouwer) cannot carry out such an infinite process, one cannot always determine whether a given real number r possesses a particular property P. For Peirce, this implies the sentence $P(r)$ is irresolvably general, and hence, as an instance of LEM, $P(r) \vee \neg P(r)$ fails to hold. For Brouwer, our current inability to determine whether $P(r)$ holds or $\neg P(r)$ implies that we cannot assert with certainty $P(r) \vee \neg P(r)$, as we have a proof of neither disjunct. Finally, for both Peirce and Brouwer, the failure of LEM goes hand in hand with the inability to distinguish between real numbers in the continuum. The fact that members of the continuum are not necessarily distinguishable implies that the continuum is not composed of atomic points, but rather, contains only smaller continuums as parts.

We have seen that Peirce and Brouwer's views on the creation, composition, and logic of the continuum are strikingly similar. Future research might uncover other interesting connections between their work, especially with regard to their common claim that logic should not be viewed as a foundations for mathematics. However, I would like to conclude now by highlighting what I think is gained from a comparison of historical figures like Peirce and Brouwer. In addition to gaining a deeper understanding of their respective

[31]CW. 6.188

works, I think this comparison has shown us that, far from being a minor undercurrent in the history of logic, Brouwer's intuitionism brought to the forefront a growing dissatisfaction amongst many mathematicians, logicians, and technically-inclined philosophers with the (still) dominant mathematical understanding of the continuum that emerged in the mid 19$^{\text{th}}$ century. The simultaneous and independent emergence of such surprisingly similar philosophies of the continuum ought to cause historians and philosophers of mathematics to carefully investigate Brouwer's work, not as the isolated writings of a maverick in the history of logic, but rather, as a great mind reacting to perceived difficulties in the foundations of mathematics that had emerged in the previous century and were quickly becoming entrenched.