9 More on Sampling

9.1 Notations

T_s denotes the sampling time in second. $\Omega_s = 2\pi/T_s$ and $\Omega_s/2$ are, respectively, the sampling frequency and Nyquist frequency in rad/sec.

Ω and ω denote, respectively, frequency in rad/s (used in continuous-time signal analysis), and frequency in rad (for discrete-time signal processing). We have

$$\omega = \Omega T_s \tag{12}$$

We use square bracket, e.g. $x[n]$, to indicate a discrete sequence; and regular parenthesis, e.g. $x_c(t)$, to indicate a continuous-time signal.

$$X_c(j\Omega) = \mathcal{F}\{x_c(t)\} \triangleq \int_{-\infty}^{\infty} x_c(t)\mathrm{e}^{-j\Omega t}\mathrm{d}t$$

denotes the Fourier transform of a continuous-time signal $x_c(t)$;

$$X(\mathrm{e}^{j\omega}) = \mathcal{F}_d\{x[n]\} \triangleq \sum_{n=-\infty}^{\infty} x[n]\,\mathrm{e}^{-j\omega n}$$

denotes the discrete-time Fourier transform of a discrete sequence $x[n] = x_c(nT_s)$.

$\mathscr{S}$ denotes the sampler that samples a continuous-time signal to discrete-time sequences.

$\mathscr{H}$ denotes a holder performing, e.g. zero order hold, to transform a discrete-time sequence to a continuous-time signal.

9.2 Impulse and impulse trains

- (Spectrum of an impulse) The Fourier transform of a shifted delta function is $\mathcal{F}(\delta(t-a)) = e^{-j\Omega a}$. In other words,[5]

$$\underline{\delta(t-a) = \mathcal{F}^{-1}\left(e^{-j\Omega a}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\Omega a} e^{j\Omega t}\mathrm{d}\Omega} \quad (13)$$

  Understanding the result: Recall that in Laplace transform, shifted impulse in time corresponds to delay in frequency domain.

- (Spectrum of sinusoidals) The Fourier transform of a single-frequency complex sinusoidal signal $x_c(t) = e^{j\Omega_0 t}$ is

$$\mathcal{F}\left(e^{j\Omega_0 t}\right) = 2\pi\delta(\Omega - \Omega_0)$$

  [by using (13)].

- Real signals have conjugate symmetric spectra. In other words, if

$$F(j\Omega) = \int_{-\infty}^{\infty} f(t) e^{-j\Omega t}\mathrm{d}t$$

  then

$$\overline{F(j\Omega)} = \int_{-\infty}^{\infty} f(t) e^{j\Omega t}\mathrm{d}t = F(-j\Omega)$$

- In particular, the Fourier transform of a real sinusoidal signal $x_c(t) = \cos(\Omega_0 t) = \left(e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right)/2$ is

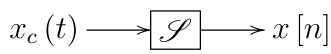
$$\mathcal{F}(\cos(\Omega_0 t)) = \pi\delta(\Omega - \Omega_0) + \pi\delta(\Omega + \Omega_0) \quad (14)$$

[5] The equality also provides an alternative definition of the delta function:

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\Omega t}\mathrm{d}\Omega$$

9.3 C/D process

- The C/D process from a continuous signal $x_c(t)$ to the discrete sampled sequence $x[n] = x_c(nT_s)$, namely



can be mathematically represented as Fig. 6, where

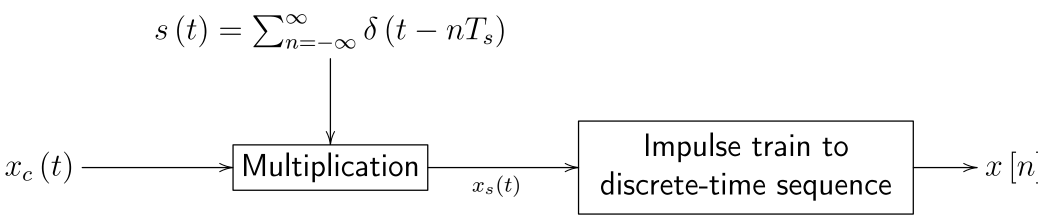


Figure 6 – Mathematical representation of the C/D process

$$x_s(t) = x_c(t)\, s(t) = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x_c(nT_s)\, \delta(t - nT_s) \tag{15}$$

9.3.1 Continuous-time signal to continuous-time impulse train

- By Fourier series expansion,[6] the continuous-time impulse train contains infinite amount of frequency components:

$$\sum_{k=-\infty}^{\infty}\delta\left(t-kT_s\right)=\frac{1}{T_s}\sum_{k=-\infty}^{\infty}e^{j\Omega_s kt}$$

  The Fourier transformation of the impulse train in Fig. 6 is thus

$$S\left(j\Omega\right)=\mathcal{F}\left(s\left(t\right)\right)=\frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta\left(\Omega-k\Omega_s\right)=\frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta\left(\Omega-k\frac{2\pi}{T_s}\right)$$

  where $\Omega_s=2\pi/T_s$ is the sampling frequency in rad/sec.

- By convolution property, the Fourier transform of $x_s\left(t\right)$ in (15) is

$$\boxed{X_s\left(j\Omega\right)=\mathcal{F}\left(x_c\left(t\right)s\left(t\right)\right)=\frac{1}{2\pi}X_c\left(j\Omega\right)*S\left(j\Omega\right)=\frac{1}{T_s}\sum_{k=-\infty}^{\infty}X_c\left(j\left(\Omega-k\Omega_s\right)\right)} \quad (16)$$

  Hence if the spectrum of $x_c\left(t\right)$ contains components beyond the Nyquist frequency $\Omega_s/2$, aliasing will occur in obtaining the spectrum of $x_s\left(t\right)$.

  - example: consider sampling $x_c\left(t\right)=\cos\left(\Omega_0 t\right)$ and $x_c^{'}\left(t\right)=\cos\left(\left(\Omega_0+\Omega_s\right)t\right)$ at sampling frequency Ω_s, with $\Omega_0<\Omega_s/2$.

---

[6] If $f(t)$ is periodic with period T_s, then

$$f(t)=\frac{1}{T_s}\sum_{k=-\infty}^{\infty}<f(x),e^{j\omega_s kt}>e^{j\omega_s kt}$$

where $\omega_s=2\pi/T_s$ and $<f(x),e^{j\omega_s kt}>$ is the inner product defined by

$$<f(x),e^{j\omega_s kt}>=\int_{-T_s/2}^{T_s/2}\overline{f(x)}e^{j\omega_s kt}\mathrm{d}t$$

* $x[n]$ will have a spectrum of periodic spikes with base pattern at Ω_0 and $-\Omega_0$ [recall (14)]
* $x^{'}[n]$ will have a spectrum of periodic spikes with base pattern at Ω_0 and $-\Omega_0$ as well [recall (16)]!

- Example: if $x_c(t) = e^{j\Omega_o t}$, then

$$x_s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) e^{j\Omega_o t} \tag{17}$$

and

$$X_s(j\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \delta(j(\Omega - k\Omega_s))$$

Hence by inverse Fourier transform

$$x_s(t) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{j(\Omega_o + \Omega_s n)t}$$

which is an equivalent form of (17).

9.3.2 Continuous-time impulse train to discrete-time sequence

- The discrete-time Fourier transform of $x[n] = x_c(nT)$ is

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Directly taking the Fourier transfrm of (15) yields

$$X_s(j\Omega) = \mathcal{F}\left(\sum_{n=-\infty}^{\infty} x_c(nT_s)\delta(t - nT_s)\right) = \sum_{n=-\infty}^{\infty} x(nT) e^{-j\Omega T_s n}$$

Hence discrete-time Fourier transform is a frequency scaled version of the continuous-time Fourier transform:[7]

$$X\left(e^{j\omega}\right) = X_s\left(j\Omega\right)\big|_{\Omega=\omega/T_s}$$

Using (16), we finally have

$$\boxed{X\left(e^{j\omega}\right) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi}{T_s}k\right)\right)\Bigg|_{\Omega=\frac{\omega}{T_s}}} \tag{18}$$

9.4 *Sampling signals beyond Nyquist frequency

Recall that:

- Sampling maps the continuous-time frequency

  $$-\frac{\pi}{T_s} < \Omega < \frac{\pi}{T_s}$$

  onto the unit circle

- Sampling also maps the continuous-time frequencies $\frac{\pi}{T_s} < \Omega < 3\frac{\pi}{T_s}$, $3\frac{\pi}{T_s} < \Omega < 5\frac{\pi}{T_s}$, etc, onto the unit circle

Consider sampling signals beyond Nyquist frequency:

From (18), sampling maps frequency components beyond Nyquist frequency onto the same discrete-time frequency region $[-\pi, \pi]$. The mapping is periodic: portions of the continuous-time $j\Omega$ axis, for Ω in the ranges of $\left[-\frac{\pi}{T_s}, \frac{\pi}{T_s}\right]$, $\left[\frac{\pi}{T_s}, \frac{3\pi}{T_s}\right]$, and $\left[\frac{3\pi}{T_s}, \frac{5\pi}{T_s}\right]$, etc, map to the same unit circle in the discrete-time domain.

[7] Understanding the scaling: in time domain, consecutive samples in $x_s(t)$ are spaced by the sampling time T_s; samples in $x[n]$ are indexed by integers and hence consecutive samples are spaced by 1. The normalization by $1/T_s$ in time domain corresponds to a normalization by T_s in the frequency domain, hence $\omega = \Omega T_s$.

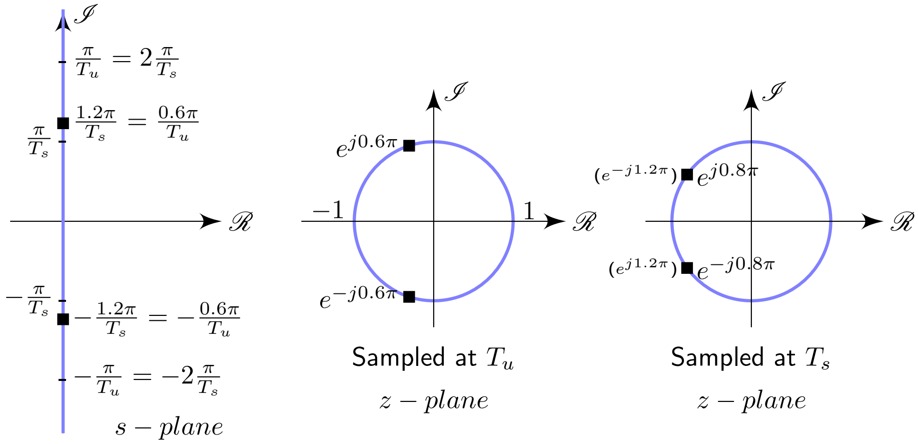


Figure 7 – Example: sampling signals beyond Nyquist frequency

For instance, consider a continuous-time sinusoidal signal $\cos(\Omega_o t)$ with frequency satisfying $\Omega_o = 1.2\pi/T_s = 1.2\pi$ rad/sec. The continuous-time Fourier transform of the signal is $\pi\delta(\Omega - 1.2\pi) + \pi\delta(\Omega + 1.2\pi)$ (The signal contains a positive and a negative frequency components at the same frequency).

If sampled at $T_u = T_s/2$, Ω_o is below the Nyquist frequency $\pi/T_u = 2\pi$ rad/sec, and mapped to $\omega_o = \Omega_o \times T_u = 0.6\pi$. The Fourier transform of the sampled signal is [by using (18)] $2\sum_{k=-\infty}^{\infty} \pi [\delta(\omega - 0.6\pi + 2\pi k) + \delta(\omega + 0.6\pi + 2\pi k)]$. If sampled at T_s, Ω_o is beyond Nyquist frequency. Aliasing occurs and the discrete-time Fourier transform of the sampled signal becomes

$$\sum_{k=-\infty}^{\infty} \pi\delta(\omega \pm 1.2\pi + 2\pi k) = \sum_{k=-\infty}^{\infty} \pi\delta(\omega \pm 0.8\pi + 2\pi k)$$

, i.e., within Nyquist frequency, the observable spectral peaks are at $0.8\pi/T_s = 0.8\pi$ rad/sec and $-0.8\pi/T_s = -0.8\pi$ rad/sec, instead of the actual frequency 1.2π rad/sec. Graphically, the result is demonstrated in Fig. 7.