9.4 Matrix inversion lemma

Fact 44 (Matrix inversion lemma). *Assume A is nonsingular and $(A+BC)^{-1}$ exists. The following is true*

$$\boxed{(A+BC)^{-1} = A^{-1}\left(I - B\left(CA^{-1}B+I\right)^{-1}CA^{-1}\right)} \tag{30}$$

Proof. Consider

$$(A+BC)\,x = y$$

We aim at getting $x = (*)\,y$, where $(*)$ will be our $(A+BC)^{-1}$. First, let

$$Cx = d$$

We have

$$\begin{aligned} Ax + Bd &= y \\ Cx - d &= 0 \end{aligned}$$

Solving the first equation yields

$$x = A^{-1}(y - Bd)$$

Then

$$CA^{-1}(y - Bd) = d$$

gives

$$d = \left(CA^{-1}B + I\right)^{-1} CA^{-1}y$$

Hence

$$\begin{aligned} x &= A^{-1}\left(y - B\left(CA^{-1}B+I\right)^{-1}CA^{-1}y\right) \\ &= A^{-1}\left(I - B\left(CA^{-1}B+I\right)^{-1}CA^{-1}\right)y \end{aligned}$$

and (30) follows. □

Exercise 45. The matrix inversion lemma is a powerful tool useful for many applications. One application in adaptive control and system identification uses

$$\left(A + \phi\phi^T\right)^{-1} = A^{-1}\left(I - \frac{\phi\phi^T A^{-1}}{\phi^T A^{-1}\phi + 1}\right)$$

Prove the above result. Prove also the general case (called rank one update):

$$\left(A + bc^T\right) = A^{-1} - \frac{1}{1 + c^T A^{-1} b}\left(A^{-1}b\right)\left(c^T A^{-1}\right)$$