Stability of Parameter Adaptation Algorithms

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Big picture

- For
$$\hat{\theta}(k+1)=\hat{\theta}(k)+[\text{correction term}]$$
we haven't talked about whether $\hat{\theta}(k)$ will converge to the true value θ if $k\to\infty$. We haven't even talked about whether $\hat{\theta}(k)$ will stay bounded or not!
- Tools of stability evaluation: Lyapunov-based analysis, or hyperstability theory (often easier for adaptive control, especially when a Lyapunov function is nontrivial to find)

Outline

1. Big picture
2. Hyperstability theory
    - Passivity
    - Main results
    - Positive real and strictly positive real
    - Understanding the hyperstability theorem
3. Procedure of PAA stability analysis by hyperstability theory
4. Appendix
    - Strictly positive realness is equivalent to strict passivity
    - (Strict) passivity implies (asymptotic) stability

Hyperstability theory

history

Vasile M. Popov:

- born in 1928, Romania
- retired from University of Florida in 1993
- developed hyperstability theory independently from Lyapnov theory

Hyperstability theory

Consider a closed-loop system in Fig. 1

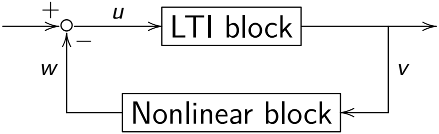

Figure 1: Block diagram for hyperstability analysis

The linear time invariant (LTI) block is realized by

continuous-time case:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$v(t) = Cx(t) + Du(t)$$

discrete-time case:

$$x(k+1) = Ax(k) + Bu(k)$$
$$v(k) = Cx(k) + Du(k)$$

Hyperstability discusses conditions for “nice” behaviors in x.

Passive systems

Definition (Passive system)

The square system $v \longrightarrow \boxed{\text{System}} \longrightarrow w$ is called passive if

$$\int_0^{t_1} w^T(t)\,v(t)\,\mathrm{d}t \geq -\gamma^2,\ \forall t_1 \geq 0 \text{ or } \sum_{k=0}^{k_1} w^T(k)\,v(k) \geq -\gamma^2,\ \forall k_1 \geq 0$$

for any input v, where $\gamma^2 < \infty$ depends on the initial conditions.

- intuition: $\int_0^{t_1} w^T(t)\,v(t)\,\mathrm{d}t$ is the work/supply delivered to the system. By conservation of energy,

  $$\text{Energy}(t_1) \leq \text{Energy}(0) + \int_0^{t_1} w^T(t)\,v(t)\,\mathrm{d}t$$

  and energy is nonnegative:
  $\text{Energy}(t_1) \geq 0 \Rightarrow \int_0^{t_1} w^T(t)\,v(t)\,\mathrm{d}t \geq -\text{Energy}(0) \geq -\gamma^2$.

Passive systems

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for any input v, where $\gamma^2 < \infty$ depends on the initial conditions.

- passive systems can be linear or nonlinear
- passive systems must be square
- example: (linear system) $w = kv$ where $k > 0$
- example: (sector nonlinearity) $w = f(v)$ where $f(v)$ lies between $\underline{\alpha}v$ and $\bar{\alpha}v$ ($\bar{\alpha} > \underline{\alpha} \geq 0$)

Strictly passive systems

If the equality is *strict* in the passivity definition, with

$$\int_0^{t_1} w^T(t)\,v(t)\,\mathrm{d}t \geq -\gamma^2 + \delta \int_0^{t_1} v^T(t)\,v(t)\,\mathrm{d}t + \varepsilon \int_0^{t_1} w^T(t)\,w(t)\,\mathrm{d}t,\ \forall t_1 \geq 0$$

or in the discrete-time case

$$\sum_{k=0}^{k_1} w^T(k)\,v(k) \geq -\gamma^2 + \delta \sum_{k=0}^{k_1} v^T(k)\,v(k) + \varepsilon \sum_{k=0}^{k_1} w^T(k)\,w(k),\ \forall k_1 \geq 0$$

where $\delta \geq 0$, $\varepsilon \geq 0$, but not both zero, the system is *strictly passive.*

Strictly passive systems

the system is

- *input strictly passive* if

$$\sum_{k=0}^{k_1} w^T(k)v(k) \geq -\gamma^2 + \delta \sum_{k=0}^{k_1} v^T(k)v(k), \ \forall k_1 \geq 0$$

  where $\delta > 0$, $\gamma^2 < \infty$.

- *output strictly passive* if

$$\sum_{k=0}^{k_1} w^T(k)v(k) \geq -\gamma^2 + \varepsilon \sum_{k=0}^{k_1} w^T(k)w(k), \ \forall k_1 \geq 0$$

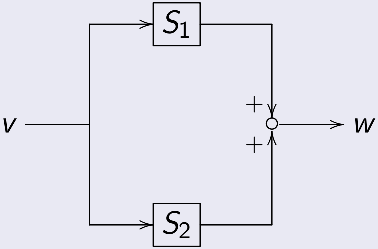
  where $\varepsilon > 0$, $\gamma^2 < \infty$.

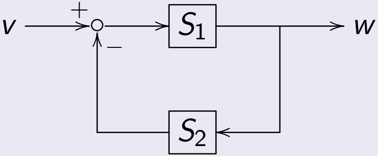
e.g. (sector nonlinearity): $w = f(v)$ where $f(v)$ lies between $\underline{\alpha}v$ and $\bar{\alpha}v$ $(\bar{\alpha} > \underline{\alpha} \geq 0)$ is input strictly passive

Passivity of combined systems

Fact (Passivity of connected systems)

If two systems S_1 and S_2 are both passive, then the following parallel and feedback combination of S_1 and S_2 are also passive





Passivity of combined systems

- Passivity of linear time invariant systems relates to the concept of positive real systems originated in circuit analysis: (strict) passivity ⇔ (strict) positive realness
- Passivity of nonlinear systems characterized by the inequality:

$$\int_0^{t_1} w^T(t)\,v(t)\,\mathrm{d}t \geq -\gamma^2,\ \forall t_1 \geq 0 \text{ or } \sum_{k=0}^{k_1} w^T(k)\,v(k) \geq -\gamma^2,\ \forall k_1 \geq 0$$

  referred to as the Popov Inequality.
- Hyperstability theory leverages the concept of passivity of two feedback connected systems.

Hyperstability theory

Definition (Hyperstability)

The feedback system in Fig. 1 is hyperstable if and only if there exist positive constants $\delta > 0$ and $\gamma > 0$ such that

$$\|x(t)\| < \delta \|x(0)\| + \gamma, \ \forall t > 0 \text{ or } \|x(k)\| < \delta \|x(0)\| + \gamma, \ \forall k > 0$$

for all passive feedback blocks, i.e., satisfying the *Popov inequality*

$$\int_0^{t_1} w^T(t)\, v(t)\, \mathrm{d}t \geq -\gamma^2, \ \forall t_1 \geq 0 \text{ or } \sum_{k=0}^{k_1} w^T(k)\, v(k) \geq -\gamma^2, \ \forall k_1 \geq 0$$

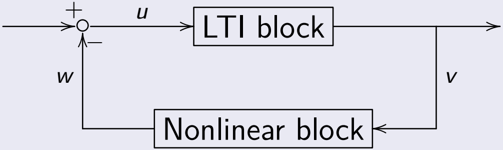
In other words, the LTI block is bounded in states for any initial conditions for any passive nonlinear blocks.

Hyperstability theory

Definition (Asymptotic hyperstability)

The feedback system below is asymptotically hyperstable if and only if it is hyperstable and for all *bounded* w satisfying the Popov inequality we have

$$\lim_{\sigma\to\infty} x(\sigma) = 0,\ \sigma = t \text{ or } k$$



Hyperstability theory

Theorem (Hyperstability)

The feedback system in Fig. 1 is hyperstable if and only if the nonlinear block satisfies ***Popov inequality*** *(i.e., it is passive) and the LTI transfer function is* ***positive real****.*

Theorem (Asymptotical hyperstability)

The feedback system in Fig. 1 is asymptotically hyperstable if and only if the nonlinear block satisfies ***Popov inequality*** *and the LTI transfer function is* ***strictly positive real****.*

intuition: a strictly passive system in feedback connection with a passive system gives an asymptotically stable states for the strictly passive system.

Positive real and strictly positive real

Positive real transfer function (continuous-time case): a SISO transfer function $G(s)$ is called *positive real* (**PR**) if

- $G(s)$ is real for real values of s
- $\mathrm{Re}\{G(s)\} > 0$ for $\mathrm{Re}\{s\} > 0$

The above is intuitive but not practical to evaluate. Equivalently, $G(s)$ is PR if

1. $G(s)$ does not possess any pole in $\mathrm{Re}\{s\} > 0$ (no unstable poles)
2. any pole on the imaginary axis $j\omega_0$ does not repeat and the associated residue (i.e., the coefficient appearing in the partial fraction expansion) $\lim_{s\to j\omega_0}(s-j\omega_0)G(s)$ is non-negative
3. $\forall \omega \in \mathbb{R}$ where $s = j\omega$ is not a pole of $G(s)$, $G(j\omega)+G(-j\omega) = 2\mathrm{Re}\{G(j\omega)\} \geq 0$

The need for stability: unstable-pole case

If $G(s)$ has a simple pole p_u **on the right half plane**, partial fraction expansion gives

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \sum_{i=1}^{N_d} \sum_{m_i=1}^{\mu_i} \frac{k_{i,m_j}}{(s-p_i)^{m_i}}$$

where m_i is the multiplicity of pole p_i, and $\sum_{i=1}^{N_d} \mu_i = n$. When s is evaluated on the right half plane and close to p_u, say $s = p_u + \varepsilon e^{j\phi}$ with a small radius $\varepsilon > 0$ such that $\operatorname{Re}\{s\} > 0$, the terms

$$\frac{k_{u1}}{(s-p_u)^{m_u}} + \frac{k_{u2}}{(s-p_u)^{m_u-1}} + \cdots = \frac{k_{u1}}{(\varepsilon e^{j\phi})^{m_u}} + \frac{k_{u2}}{(\varepsilon e^{j\phi})^{m_u-1}} + \dots$$

dominate and in particular

$$G(s) \approx \frac{k_{u1}}{\varepsilon^{m_u} e^{j m_u \phi}}$$

which cannot be guaranteed to have a positive real part because $e^{j m_u \phi}$ $(\phi \in [0, 2\pi))$ can be both positive and negative.

The need for stability: marginally-stable-pole case

If $G(s)$ has a simple pole p_u **on the imaginary axis**, we have

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \dots b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0} = \sum_{i=1}^{N_d} \sum_{m_i=1}^{\mu_i} \frac{k_{i,m_j}}{(s-p_i)^{m_i}}$$

When s is evaluated on the right half plane and close to p_u, say $s = p_u + \varepsilon e^{j\phi}$ with a small radius $\varepsilon > 0$ and $\phi \in (-\pi/2, \pi/2)$ such that $\mathrm{Re}\{s\} > 0$, the terms

$$\frac{k_{u1}}{(s-p_u)^{m_u}} + \frac{k_{u2}}{(s-p_u)^{m_u-1}} + \dots = \frac{k_{u1}}{(\varepsilon e^{j\phi})^{m_u}} + \frac{k_{u2}}{(\varepsilon e^{j\phi})^{m_u-1}} + \dots$$

dominate and in particular

$$G(s) \approx \frac{k_{u1}}{\varepsilon^{m_u} e^{j m_u \phi}}$$

which cannot be guaranteed to have a positive real part unless $m_u = 1$ and $k_{u1} \geq 0$.

Positive real and strictly positive real

Strictly positive real transfer function (continuous-time case): a SISO transfer function $G(s)$ is *strictly positive real* (**SPR**) if

1. $G(s)$ does not possess any pole in $\mathrm{Re}\{s\} \geq 0$ (i.e., all poles are on the open left half plane)
2. $\forall \omega \in \mathbb{R}$, $G(j\omega) + G(-j\omega) = 2\,\mathrm{Re}\{G(j\omega)\} > 0$

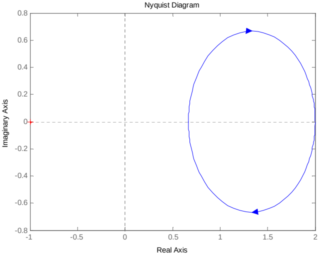


Figure: example Nyquist plot of a SPR transfer function

Positive real and strictly positive real

discrete-time case

A SISO discrete-time transfer function $G(z)$ is positive real (**PR**) if:

1. it does not possess any pole outside of the unit circle
2. any pole on the unit circle does not repeat and the associated redsidue is non-negative
3. $\forall|\omega| \leq \pi$ where $z = e^{j\omega}$ is not a pole of $G(z)$, $G(e^{-j\omega}) + G(e^{j\omega}) = 2\,\mathrm{Re}\left\{G(e^{j\omega})\right\} \geq 0$

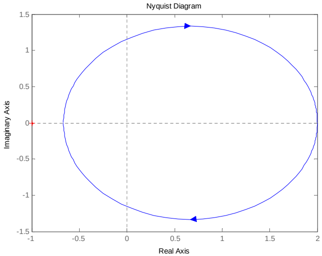
$G(z)$ is strictly positive real (**SPR**) if:

1. $G(z)$ does not possess any pole outside of or on the unit circle on z-plane
2. $\forall|\omega| < \pi,\ G(e^{-j\omega}) + G(e^{j\omega}) = 2\,\mathrm{Re}\left\{G(e^{j\omega})\right\} > 0$

Examples of PR and SPR transfer functions

- $G(z) = c$ is SPR if $c > 0$
- $G(z) = \frac{1}{z-a}$, $|a| < 1$ is asymptotically stable but not PR:

$$2\,\mathrm{Re}\left\{G\left(e^{j\omega}\right)\right\} = \frac{1}{e^{j\omega}-a} + \frac{1}{e^{-j\omega}-a}$$
$$= 2\frac{\cos\omega - a}{1 + a^2 - 2a\cos\omega}$$



- $G(z) = \frac{z}{z-a}$, $|a| < 1$ is asymptotically stable and SPR
- as homework, show that $G(z) = 1/\left(1 - a_1 z^{-1} - a_2 z^{-1} - \cdots - a_n z^{-n}\right)$, $\sum_{i=1}^{n} |a_i| < 1$ is SPR

Strictly positive real implies strict passivity

It turns out [see Appendix]:
Lemma: the LTI system $G(s) = C(sI - A)^{-1} B + D$ (in minimal realization)

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

is

- passive if $G(s)$ is positive real
- strictly passive if $G(s)$ is strictly positive real

Analogous results hold for discrete-time systems.

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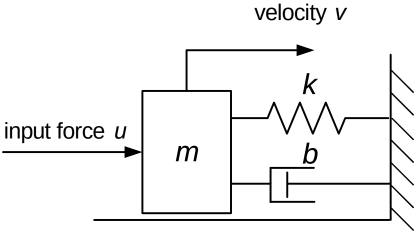
Understanding the hyperstability theorem

Example: consider a mass-spring-damper system

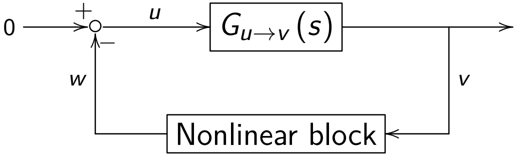


$$m\ddot{x}+b\dot{x}+kx=u\Rightarrow$$

$$G_{u\rightarrow x}(s)=\frac{1}{ms^2+bs+k}$$

$$G_{u\rightarrow v}(s)=\frac{s}{ms^2+bs+k}$$

with a general nonlinear feedback control law



- $\int_0^{t_1}u(t)v(t)\mathrm{d}t$ is the total energy supplied to the system

Understanding the hyperstability theorem

- if the nonlinear block satisfies the Popov inequality

$$\int_0^{t_1} w(t)v(t)\mathrm{d}t \ge -\gamma_0^2,\ \forall t_1 \ge 0$$

  then from $u(t) = -w(t)$, the energy supplied to the system is bounded by

$$\int_0^{t_1} u(t)v(t)\mathrm{d}t \le \gamma_0^2$$

- energy conservation (assuming $v(0) = v_0$ and $x(0) = x_0$):

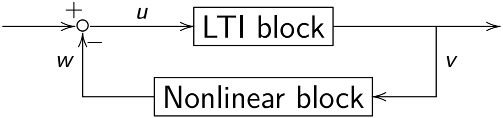
$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 - \frac{1}{2}mv_0^2 - \frac{1}{2}kx_0^2 = \int_0^{t_1} u(t)v(t)\mathrm{d}t \le \gamma_0^2$$

- define state vector $x = [x_1, x_2]^T$, $x_1 = \sqrt{\frac{k}{2}}x$, $x_2 = \sqrt{\frac{m}{2}}v$, then

$$||x(t)||_2^2 \le ||x(0)||_2^2 + \gamma_0^2 \le (||x(0)||_2 + \gamma_0)^2$$

  which is a special case in the hyperstability definition

Understanding the hyperstability theorem



intuition from the example:

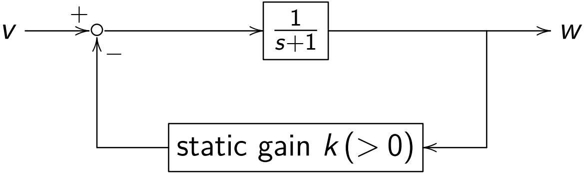
The passive nonlinear block (satisfying Popov inequality) assures bounded supply to the LTI system. Based on energy conservation, the energy of the LTI system is bounded. If the energy function is positive definite with respect to the states, then the states will be bounded.

more intuition:

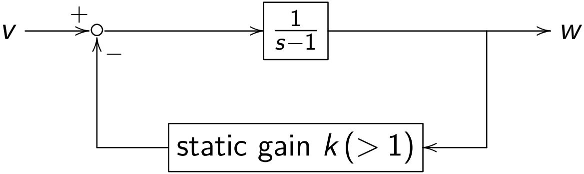
If the LTI system is strictly passive, it consumes energy. The bounded supply will eventually be all consumed, hence the convergence to zero for the states.

A remark about hyperstability

An example of a system that is asymptotically hyperstable and stable:



Stable systems may however be not hyperstable: for instance



is stable but not hyperstable ($\frac{1}{s-1}$ is unstable and hence not SPR)

Outline

PAA stability analysis by hperstability theory

- step 1: translate the adaptation algorithm to a feedback combination of a LTI block and a nonlinear block, as shown in Fig. 1
- step 2: verify that the feedback block is passive (satisfies the Popov inequality)
- step 3: check that the LTI block is strictly positive real
- step 4: show that the output of the feedback block is bounded. Then from the definition of asymptotic hyperstability, we conclude that the state x converges to zero

Example: hyperstability of RLS with constant adaptation gain

Recall PAA with recursive least squares:

- *a priori* parameter update

$$\hat{\theta}(k+1)=\hat{\theta}(k)+\frac{F(k)\phi(k)}{1+\phi^{T}(k)F(k)\phi(k)}\varepsilon^{o}(k+1)$$

- *a posteriori* parameter update

$$\hat{\theta}(k+1)=\hat{\theta}(k)+F(k)\phi(k)\varepsilon(k+1)$$

We use the *a posteriori* form to prove that the RLS with $F(k)=F\succ 0$ is **always asymptotically hyperstable.**

Example cont'd

step 1: transformation to a feedback structure

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F\phi(k)\varepsilon(k+1)$$

parameter estimation error (vector) $\tilde{\theta}(k) = \hat{\theta}(k) - \theta$:

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + F\phi(k)\varepsilon(k+1)$$

a posteriori prediction error $\varepsilon(k+1) = y(k+1) - \phi^T(k)\hat{\theta}(k+1)$:

$$\begin{aligned}\varepsilon(k+1) &= \phi^T(k)\theta - \phi^T(k)\hat{\theta}(k+1)\\ &= -\phi^T(k)\tilde{\theta}(k+1) = -\tilde{\theta}^T(k+1)\phi(k)\end{aligned}$$

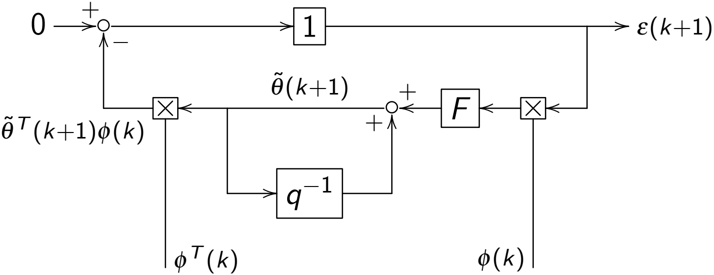
Example cont'd

step 1: transformation to a feedback structure

PAA equations:

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + F\phi(k)\varepsilon(k+1)$$
$$\varepsilon(k+1) = -\tilde{\theta}^T(k+1)\phi(k)$$

equivalent block diagram:



where at the product junction at the lower left corner, we used the fact that $\tilde{\theta}^T(k+1)\phi(k) = \phi^T(k)\tilde{\theta}(k+1)$

Example cont'd

step 2: Popov inequality

for the feedback nonlinear block, need to prove

$$\sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1) \geq -\gamma_0^2,\ \forall k_1 \geq 0$$

$\tilde{\theta}(k+1) - \tilde{\theta}(k) = F\phi(k)\varepsilon(k+1)$ gives

$$\begin{aligned}&\sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1)\\ =&\sum_{k=0}^{k_1}\left(\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k+1) - \tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k)\right)\end{aligned}$$

Example cont'd

step 2: Popov inequality

"adding and substracting terms" gives

$$\sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1)$$

$$= \sum_{k=0}^{k_1} \left(\tilde{\theta}^T(k+1) F^{-1} \tilde{\theta}(k+1) - \tilde{\theta}^T(k+1) F^{-1} \tilde{\theta}(k) \right)$$

$$= \sum_{k=0}^{k_1} \Bigg(\tilde{\theta}^T(k+1) F^{-1} \tilde{\theta}(k+1) \pm \frac{1}{2} \tilde{\theta}^T(k) F^{-1} \tilde{\theta}(k)$$

$$- \tilde{\theta}^T(k+1) F^{-1} \tilde{\theta}(k) \Bigg)$$

Example cont'd

step 2: Popov inequality

Combining terms yields

$$\sum_{k=0}^{k_1}\left(\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k+1)\pm\frac{1}{2}\tilde{\theta}^T(k)F^{-1}\tilde{\theta}(k)-\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k)\right)$$

$$=\underline{\sum_{k=0}^{k_1}\frac{1}{2}\left(\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k+1)-\tilde{\theta}^T(k)F^{-1}\tilde{\theta}(k)\right)}$$

$$+\sum_{k=0}^{k_1}\frac{1}{2}\underbrace{\left(\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k+1)-2\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k)+\tilde{\theta}^T(k)F^{-1}\tilde{\theta}(k)\right)}_{[\bigstar]}$$

- $[\bigstar]$ is equivalent to

$$\left(F^{-1/2}\tilde{\theta}(k+1)-F^{-1/2}\tilde{\theta}(k)\right)^T\left(F^{-1/2}\tilde{\theta}(k+1)-F^{-1/2}\tilde{\theta}(k)\right)\geq 0$$

Example cont'd

step 2: Popov inequality

- the underlined term is also lower bounded:

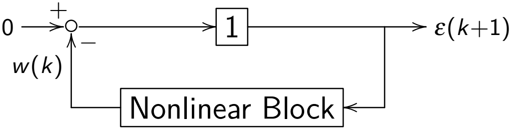
$$\begin{aligned}\sum_{k=0}^{k_1}\frac{1}{2}\left(\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k+1)-\tilde{\theta}^T(k)F^{-1}\tilde{\theta}(k)\right)\\ =\frac{1}{2}\tilde{\theta}^T(k_1+1)F^{-1}\tilde{\theta}(k_1+1)-\frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0)\\ \geq-\frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0)\end{aligned}$$

hence

$$\boxed{\sum_{k=0}^{k_1}\tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1)\geq-\frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0)}$$

Example cont'd

step 3: SPR condition

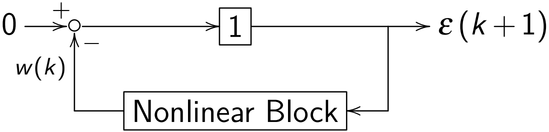


the identity block in the LTI path is always SPR

- from steps 1-3, we conclude the adaptation system is asymptotically hyperstable
- this means $\varepsilon(k+1)$ will be bounded, and if $w(k)$ is further shown to be bounded, $\varepsilon(k+1)$ converge to zero

Example cont'd

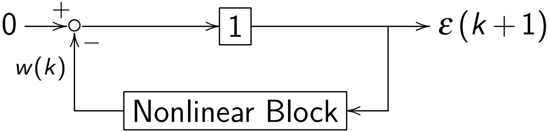
step 4: boundedness of the signal



- $\varepsilon(k+1) = -w(k)$, so $w(k)$ is bounded if $\varepsilon(k+1)$ is bounded
- thus hyperstability theorem gives that $\varepsilon(k+1)$ converges to zero

Example cont'd

intuition



For this simple case, we can intuitively see why $\varepsilon(k+1) \to 0$: passivity of the nonlinear block (Popov inequality) gives $\sum_{k=0}^{k_1} \varepsilon(k+1) w(k) \geq -\gamma_0^2$; as $w(k) = -\varepsilon(k+1)$, so

$$\sum_{k=0}^{k_1} \varepsilon^2(k+1) \leq \gamma_0^2$$

Let $k_1 \to \infty$. $\varepsilon(k+1)$ must converge to 0 to ensure the boundedness.

One remark

Recall

$$\varepsilon(k+1) = \frac{\varepsilon^{o}(k+1)}{1+\phi^{T}(k)F\phi(k)}$$

- $\varepsilon(k+1) \to 0$ does not necessarily mean $\varepsilon^{o}(k+1) \to 0$
- need to show $\phi(k)$ is bounded: for instance, the plant needs to be input-output stable for $y(k)$ to be bounded
- see details in Landau et al, "Adaptive Control", 2nd Ed, Springer

There are different PAAs with different stability and convergence requirements

Summary

1. Big picture
2. Hyperstability theory
    - Passivity
    - Main results
    - Positive real and strictly positive real
    - Understanding the hyperstability theorem
3. Procedure of PAA stability analysis by hyperstability theory
4. Appendix
    - Strictly positive realness is equivalent to strict passivity
    - (Strict) passivity implies (asymptotic) stability

Further reading

- the "energy function" mentioned in the passivity slides is more formally called the storage function
- the storage function, denoted as $V(x)$, needs to be at least positive semidefinite: $V(x) \geq 0$ and $V(0) = 0$
- a (linear or nonlinear) passive system with a positive definite storage function is stable
- a (linear or nonlinear) strictly passive system is asymptotically stable
- another way to describe the hyperstability theorem (in the language of passive systems) is that: a strictly passive, time-invariant dynamical system with a passive (possibly time-varying) memoryless function is asymptotically stable
- reference: H. Khalil, "Nonlinear Systems", Prentice Hall

*Kalman Yakubovich Popov Lemma

Kalman Yakubovich Popov (KYP) Lemma connects frequency-domain SPR conditions and time-domain system matrices:

Lemma (Continuous-time KYP Lemma)

Consider $G(s) = C(sI - A)^{-1}B + D$ *where* (A, B) *is controllable and* (A, C) *is observable.* $G(s)$ *is* ***strictly positive real*** *if and only if there exist matrices* L, W, *a positive definite* $P = P^T \succ 0$, *and a positive constant* ε *such that*

$$
\begin{aligned}
PA + A^T P &= -L^T L - \varepsilon P \\
PB &= C^T - L^T W \\
W^T W &= D + D^T
\end{aligned}
$$

Proof: see H. Khalil, "Nonlinear Systems", Prentice Hall

*Kalman Yakubovich Popov Lemma

Lemma (Discrete-time KYP Lemma)

Consider $G(z) = C(zI - A)^{-1}B + D$ *where* (A, B) *is controllable and* (A, C) *is observable.* $G(z)$ *is* ***strictly positive real*** *if and only if there exist matrices* L, W, *a positive definite* $P = P^T \succ 0$, *and a positive constant* ε *such that*

$$A^T PA - P = -L^T L - \varepsilon P$$
$$B^T PA - C = -K^T L$$
$$D + D^T - B^T PB = K^T K$$

*Positive Real Lemma

Relaxing the ε condition gives the Positive Real Lemma:

Lemma (Continuous-time Positive Real Lemma)

Consider $G(s) = C(sI - A)^{-1}B + D$ *where* (A, B) *is controllable and* (A, C) *is observable.* $G(s)$ *is* ***positive real*** *if and only if there exist matrices* L, W, *and a positive definite* $P = P^T \succ 0$ *such that*

$$\begin{aligned} PA + A^T P &= -L^T L \\ PB &= C^T - L^T W \\ W^T W &= D + D^T \end{aligned}$$

*Positive Real Lemma

Lemma (Discrete-time Positive Real Lemma)

Consider $G(z) = C(zI - A)^{-1}B + D$ *where* (A, B) *is controllable and* (A, C) *is observable.* $G(z)$ *is* ***positive real*** *if and only if there exist matrices* $P = P^T \succ 0$, L, *and* W *such that*

$$\begin{aligned} A^T PA - P &= -L^T L \\ B^T PA - C &= -K^T L \\ D + D^T - B^T PB &= K^T K \end{aligned}$$

*Strictly positive realness implies strict passivity

From KYP lemma, the following result can be shown:

Lemma

The LTI system $G(s) = C(sI - A)^{-1}B + D$ *(in minimal realization)*

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is

- *passive if* $G(s)$ *is positive real*
- *strictly passive if* $G(s)$ *is strictly positive real*

Analogous results hold for discrete-time systems.

*Strictly positive realness implies strict passivity

Proof: Consider a storage function $V = \frac{1}{2}x^T P x$:

$$V(x(T)) - V(x(0)) = \int_0^T \dot{V}\mathrm{d}t = \int_0^T \left[\frac{1}{2}x^T\left(A^T P + PA\right)x + u^T B^T P x\right]\mathrm{d}t$$

Let u and y be the input and the output of $G(s)$. KYP lemma gives

$$V(x(T)) - V(x(0)) = \int_0^T \left[-\frac{1}{2}x^T\left(L^T L + \varepsilon P\right)x + u^T B^T P x\right]\mathrm{d}t$$

$$\begin{aligned}\int_0^T u^T y\mathrm{d}t = \int_0^T u^T(Cx + Du)\mathrm{d}t &= \int_0^T \left[u^T\left(B^T P + W^T L\right)x + u^T D u\right]\mathrm{d}t \\ &= \int_0^T \left[u^T\left(B^T P + W^T L\right)x + \frac{1}{2}u^T\left(D + D^T\right)u\right]\mathrm{d}t \\ &= \int_0^T \left[u^T\left(B^T P + W^T L\right)x + \frac{1}{2}u^T W^T W u\right]\mathrm{d}t\end{aligned}$$

*Strictly positive realness implies strict passivity

hence

$$
\begin{aligned}
&\int_0^T u^T y \mathrm{d}t - V(x(T)) + V(x(0)) \\
=&\int_0^T \left[u^T \left(B^T P + W^T L \right) x + \frac{1}{2} u^T W^T W u + \frac{1}{2} x^T \left(L^T L + \varepsilon P \right) x - u^T B^T P x \right] \mathrm{d}t \\
=&\frac{1}{2} \int_0^T (Lx + Wu)^T (Lx + Wu) \mathrm{d}t + \frac{1}{2} \varepsilon x^T P x \geq \frac{1}{2} \varepsilon x^T P x > 0
\end{aligned}
$$

*Positive realness is equivalent to passivity for LTI systems

- Through a Parseval-Theorem based analysis, it can be shown that for LTI systems:
  - the property of passivity is equivalent to the property of positive realness
  - strict passivity is equivalent to strict positive realness
- Reference that summarized and extended the literature:
  - Nicholas Kottenstette, Michael J. McCourt, Meng Xia, Vijay Gupta, and Panos J. Antsaklis, Relationships Among Passivity, Positive Realness, and Dissipativity with an Application to Passivity Based Pairing, Technical Report of the ISIS Group at the University of Notre Dame, January 2014
  - Short version: Kottenstette, N. & Antsaklis, P. J. Relationships between positive real, passive dissipative, & positive systems. in American Control Conference 409–416 (2010). doi:10.1109/acc.2010.5530779.

(Strict) passivity implies (asymptotic) stability

- From the KYP Lemma, for the A matrix of a strictly passive LTI system, we can find $P \succ 0$ such that

  $$PA + A^T P = -L^T L - \varepsilon P$$

  The right hand side of the equation is negative definite. Hence the origin is asymptotically stable.
- Analogous conclusions exist for discrete-time strictly passive LTI systems.
- It turns out for nonlinear systems,
  - passive systems with a positive definite storage function implies stability of the origin
  - strictly passive systems imply asymptotic stability of the origin

References

- Hyperstability
    - V. M. Popov, Hyperstability of control systems, Berlin: Springer Varlag, 1973.
    - K. S. Narendra and L. S. Valvani, "A comparison of Lyapunov and hyperstability approaches to adaptive control of continuous systems", IEEE Trans. AC, Vol. 25, pp. 243-247, 1980.
- Positive realness and passivity:
- H. Khalil, "Nonlinear Systems", Prentice Hall
- CI Byrnes and W Lin, "Losslessness, feedback equivalence, and the global stabilization of discrete-time nonlinear systems". IEEE Transactions on Automatic Control 1994; 39(1):83–98.
- C. A. Desoer and M. Vidyasagar, "Feedback Systems: Input-Output Properties", Academic Press, 1975