Diophantine Equation

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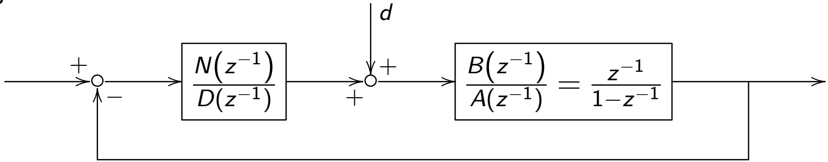
Introductory example

Consider controlling a marginally stable system $z^{-1}/\left(1-z^{-1}\right)$ using negative feedback:



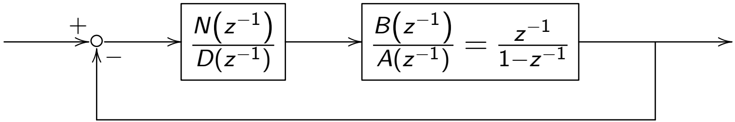
From undergraduate controls, we know we can do pole placement, e.g., to assign a stable closed-loop pole at 0.5, via designing the closed-loop characteristic polynomial:

$$D\left(z^{-1}\right)\left(1-z^{-1}\right)+N\left(z^{-1}\right)z^{-1}=1-0.5z^{-1}$$

Letting $D\left(z^{-1}\right)=1$ and $N\left(z^{-1}\right)=n_0$, for example, yields

$$1\times\left(1-z^{-1}\right)+n_0z^{-1}=1-0.5z^{-1}$$
$$\Rightarrow -1+n_0=-0.5$$
$$\Rightarrow n_0=0.5$$

Introductory example



Observations and analysis:

- Negative feedback can stabilize marginally stable and unstable systems.
- If the target closed-loop pole is at other places such as 0.6, 0.8, and 0.2, no problem at all from the pole placement design.
- Given a general plant $B\left(z^{-1}\right)/A\left(z^{-1}\right)$, do we have guarantees to place the closed-loop poles anywhere we want? How do we select the orders of the controller? Is the solution always unique? These questions can be systematically answered by leveraging the Diophantine Equation.

Coprimeness of transfer functions

Theorem

Consider $G(z) = \frac{\beta(z)}{\alpha(z)} = \frac{\beta_1 z^{n-1} + \beta_2 z^{n-2} + \cdots + \beta_n}{z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n}$. $\alpha(z)$ *and* $\beta(z)$ *are coprime (no common roots) iff* S *(Sylvester matrix) is nonsingular:*

$$S = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & \dots & 0 \\ \alpha_1 & 1 & \ddots & \vdots & \beta_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \alpha_1 & 1 & & \ddots & \ddots & 0 \\ \alpha_{n-1} & & & \alpha_1 & \ddots & & \ddots & \beta_1 \\ \alpha_n & \ddots & & \vdots & \beta_n & \ddots & & \beta_2 \\ 0 & \alpha_n & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha_{n-1} & & \ddots & \beta_n & \beta_{n-1} \\ 0 & \dots & 0 & \alpha_n & \dots & \dots & 0 & \beta_n \end{bmatrix}_{2n \times 2n}$$

Coprimeness of transfer functions

Example:

$$G(z) = \frac{z^{n-1} + \alpha_1 z^{n-2} + \cdots + \alpha_{n-1}}{z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + 0}$$

i.e.$\beta_1 = 1,\ \beta_i = \alpha_{i-1},\ \forall i \geq 2, \alpha_n = 0.$ $\alpha(z)$ and $\beta(z)$ are not coprime, and S is singular:

$$S = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & \dots & 0 \\ \alpha_1 & 1 & \ddots & \vdots & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \alpha_1 & \ddots & \ddots & \vdots \\ \vdots & & \alpha_1 & 1 & & \ddots & \ddots & 0 \\ \alpha_{n-1} & & & \alpha_1 & \ddots & & \ddots & 1 \\ 0 & \alpha_{n-1} & & \vdots & \alpha_{n-1} & \ddots & & \alpha_1 \\ 0 & 0 & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha_{n-1} & & \ddots & \alpha_{n-1} & \alpha_{n-2} \\ 0 & \dots & 0 & 0 & \dots & \dots & 0 & \alpha_{n-1} \end{bmatrix}_{2n \times 2n}$$

Solution concepts of Diophantine Equation

Theorem (Diophantine equation)

Given

$$\eta\left(z^{-1}\right)=\eta_0+\eta_1 z^{-1}+\eta_2 z^{-2}+\cdots+\eta_q z^{-q}$$
$$\alpha\left(z^{-1}\right)=1+\alpha_1 z^{-1}+\cdots+\alpha_n z^{-n}$$
$$\beta\left(z^{-1}\right)=\beta_1 z^{-1}+\beta_2 z^{-2}+\cdots+\beta_n z^{-n}$$

The Diophantine equation

$$\alpha\left(z^{-1}\right)\sigma\left(z^{-1}\right)+\beta\left(z^{-1}\right)\gamma\left(z^{-1}\right)=\eta\left(z^{-1}\right)$$

can be solved uniquely for $\sigma\left(z^{-1}\right)$ *and* $\gamma\left(z^{-1}\right)$

$$\sigma\left(z^{-1}\right)=\sigma_0+\sigma_1 z^{-1}+\cdots+\sigma_{n-1} z^{-(n-1)}$$
$$\gamma\left(z^{-1}\right)=\gamma_0+\gamma_1 z^{-1}+\cdots+\gamma_{n-1} z^{-(n-1)}$$

if the numerators of $\alpha\left(z^{-1}\right)$ *and* $\beta\left(z^{-1}\right)$ *are coprime and* $\deg\left(\eta\left(z^{-1}\right)\right)=q\leq 2n-1$

Solution concepts of Diophantine Equation

Proof of Diophantine equation Theorem (key ideas):

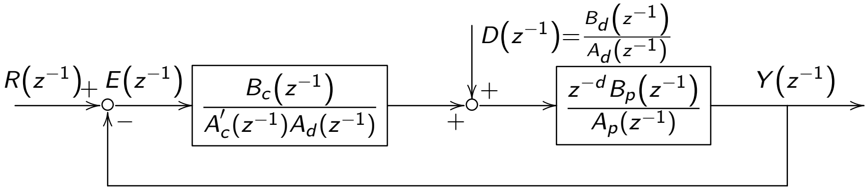
$$\alpha\left(z^{-1}\right)\underbrace{\sigma\left(z^{-1}\right)}_{\text{unknown}}+\beta\left(z^{-1}\right)\underbrace{\gamma\left(z^{-1}\right)}_{\text{unknown}}=\eta\left(z^{-1}\right)$$

Matching the coefficients of z^{-i} gives

$$S\begin{bmatrix}\sigma_0\\ \sigma_1\\ \vdots\\ \sigma_{n-1}\\ \gamma_0\\ \vdots\\ \gamma_{n-1}\end{bmatrix}=\begin{bmatrix}\eta_0\\ \eta_1\\ \vdots\\ \eta_{n-1}\\ \eta_n\\ \vdots\\ \eta_{2n-1}\end{bmatrix}$$

The coprime condition assures S is invertible. $\deg\eta\left(z^{-1}\right)\leq 2n-1$ assures the proper dimension on the right hand side of the equality.

Application: Pole placement



Pole placement assigns the closed-loop characteristic equation:

$$z^{-d}B_p\left(z^{-1}\right)B_c\left(z^{-1}\right)+A_p\left(z^{-1}\right)A_c^{'}\left(z^{-1}\right)A_d\left(z^{-1}\right) \\ =\underbrace{1+\eta_1 z^{-1}+\eta_2 z^{-2}+\cdots+\eta_q z^{-q}}_{\eta\left(z^{-1}\right)}$$

which is in the structure of a *Diophantine equation*.
Design procedure: specify the desired closed-loop dynamics $\eta\left(z^{-1}\right)$; match coefficients of z^{-i} on both sides to get $B_c\left(z^{-1}\right)$ and $A_c^{'}\left(z^{-1}\right)$.

Application: Pole placement

$$z^{-d}B_p\left(z^{-1}\right)B_c\left(z^{-1}\right)+A_p\left(z^{-1}\right)A_c^{'}\left(z^{-1}\right)A_d\left(z^{-1}\right)$$
$$=\underbrace{1+\eta_1 z^{-1}+\eta_2 z^{-2}+\cdots+\eta_q z^{-q}}_{\eta(z^{-1})}$$

Questions:

- what are the constraints for choosing $\eta\left(z^{-1}\right)$?
  — depends on desired performance. Refer to undergraduate course on linear systems for concepts related to rise time, peak time, dampling ratio, etc.
- how to guarantee a unique solution of the controller?
  — addressed by solution concepts of Diophantine Eq.

Appendix: Proof of Sylvester's Theorem

Theorem

Consider $G(z) = \frac{\beta(z)}{\alpha(z)} = \frac{\beta_1 z^{n-1} + \beta_2 z^{n-2} + \dots + \beta_n}{z^n + \alpha_1 z^{n-1} + \dots + \alpha_n}$. $\alpha(z)$ *and* $\beta(z)$ *are coprime (no common roots) iff* S *(Sylvester matrix) is nonsingular.*

Proof.

$G(z) = \frac{\beta(z)}{\alpha(z)} = \frac{\beta_1 z^{n-1} + \beta_2 z^{n-2} + \dots + \beta_n}{z^n + \alpha_1 z^{n-1} + \dots + \alpha_n} = \frac{\beta_1 z^{-1} + \beta_2 z^{-2} + \dots + \beta_n z^{-n}}{1 + \alpha_1 z^{-1} + \dots + \alpha_n z^{-n}}$

$\det S \neq 0 \Rightarrow \alpha(z)$ and $\beta(z)$ are coprime: If not, then there exist $\zeta \neq 0$ such that $z^{-n}\alpha(z) = (\zeta - z^{-1})\sigma(z^{-1}) = (\zeta - z^{-1})(\sigma_0 + \sigma_1 z^{-1} + \dots + \sigma_{n-1} z^{-(n-1)})$ and $z^{-n}\beta(z) = (\zeta - z^{-1})\gamma(z^{-1}) = (\zeta - z^{-1})(\gamma_0 + \gamma_1 z^{-1} + \gamma_2 z^{-2} + \dots + \gamma_{n-1} z^{-(n-1)})$.

Thus $\frac{z^{-n}\alpha(z)}{\sigma(z^{-1})} = \frac{z^{-n}\beta(z)}{\gamma(z^{-1})}$ and the Diophantine equation

$$\left(z^{-n}\alpha(z)\right)\gamma(z^{-1}) - \left(z^{-n}\beta(z)\right)\sigma(z^{-1}) = 0 \tag{1}$$

$$\Leftrightarrow S \begin{bmatrix} \gamma_0 & \gamma_1 & \dots & \gamma_{n-1} & -\sigma_0 & \dots & -\sigma_{n-1} \end{bmatrix}^T = 0 \tag{2}$$

has non-zero solutions for σ_i's and γ_i's, contradicting with $\det S \neq 0$. □

Appendix: Proof of Sylvester's Theorem

Theorem

Consider $G(z) = \frac{\beta(z)}{\alpha(z)} = \frac{\beta_1 z^{n-1} + \beta_2 z^{n-2} + \cdots + \beta_n}{z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n}$. $\alpha(z)$ *and* $\beta(z)$ *are coprime (no common roots) iff* S *(Sylvester matrix) is nonsingular.*

Proof.

$\alpha(z)$ and $\beta(z)$ are coprime $\Rightarrow \det S \neq 0$: If $\det S = 0$, then (2) and hence (1) has nonzero solutions $\gamma(z^{-1})$ and $\sigma(z^{-1})$. Letting $\zeta(z^{-1}) \triangleq z^{-n}\alpha(z)/\sigma(z^{-1})$ yields a common factor of $\alpha(z)$ and $\beta(z)$–a contradiction with $\alpha(z)$ and $\beta(z)$ being coprime.

□

Sylvester's Theorem: General Case

Theorem

Two polynomials $\alpha(z) = \alpha_0 z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n$ *and* $\beta(z) = \beta_0 z^n + \beta_1 z^{n-1} + \cdots + \beta_n$ *are coprime (no common roots) iff the following* S *(Sylvester matrix) is nonsingular*

$$S = \begin{bmatrix} \alpha_0 & 0 & \dots & 0 & \beta_0 & 0 & \dots & 0 \\ \alpha_1 & \alpha_0 & \ddots & \vdots & \beta_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \alpha_1 & \alpha_0 & & \ddots & \ddots & \beta_0 \\ \alpha_{n-1} & & & \alpha_1 & \ddots & & \ddots & \beta_1 \\ \alpha_n & \ddots & & \vdots & \beta_n & \ddots & & \beta_2 \\ 0 & \alpha_n & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha_{n-1} & & \ddots & \beta_n & \beta_{n-1} \\ 0 & \dots & 0 & \alpha_n & \dots & \dots & 0 & \beta_n \end{bmatrix}_{2n\times 2n}$$