Robotics, Vision, & Mechatronics for Manufacturing.

Chapter 2: representing rotation and translation

Sp. 2021

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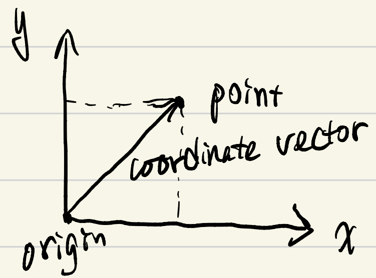
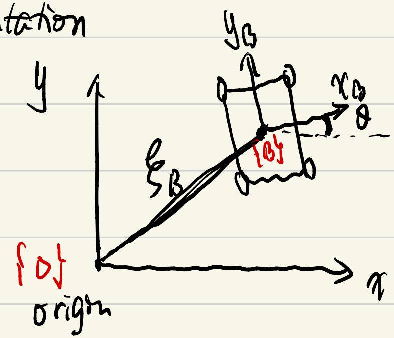
Announcements:

* Please complete survey : link on canvas, announced by Dan Wang

Recap: pose : location & orientation

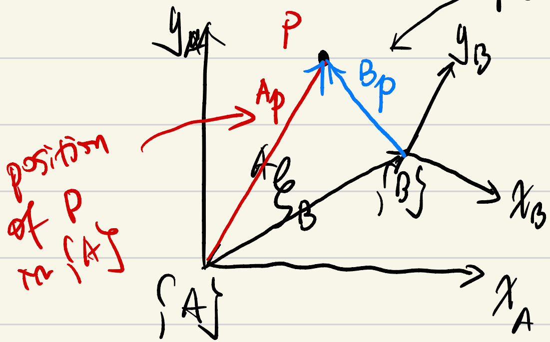




notation: ${}^A\xi_B$

- relating two frames $\{A\}$ & $\{B\}$
- describing the relative pose of $\{B\}$ w.r.t. $\{A\}$
- leading superscript : reference frame
- subscript : frame being described
- when leading script is omitted, the reference frame is $\{O\}$, the world coordinate syst.

position of point P in {B}



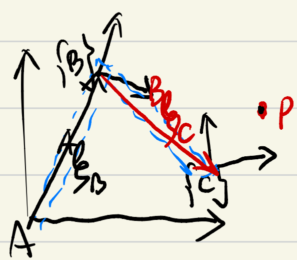
$$ {}^A p = {}^A\xi_B \cdot {}^B p $$

operator that transforms a vector from one coordinate frame to another.

important characteristic of relative poses:

$$ {}^A\xi_C = {}^A\xi_B \oplus {}^B\xi_C $$

the pose of {C} relative to {A} can be obtained by computing the relative poses from {A} to {B} & from {B} to {C}.



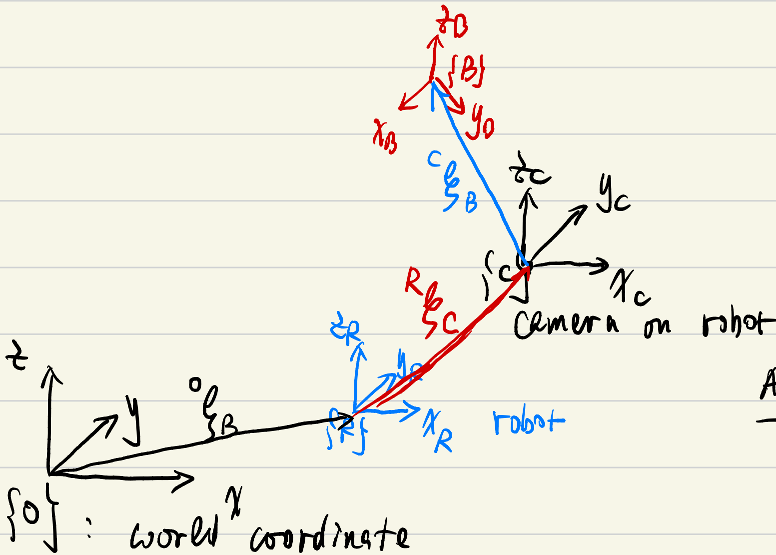
$\oplus$: the composition of relative frames.

$$ \Rightarrow \quad {}^A p = ({}^A\xi_B \oplus {}^B\xi_C)\, {}^C p $$

Usual coordinate systems: 2d & 3d

2d coordinate frame: approiate for e.g. mobile robots

3d 〃 〃 : needed by. e.g. the pose of flying airplanes/drones, underwater robots, or the end of a tool carried by a robotic arm.



$$\xi_B = \xi_R \oplus {}^R\xi_C \oplus {}^C\xi_B$$

Algebraic Rules:

$$\xi \oplus 0 = \xi, \quad \xi \ominus 0 = \xi$$

$$\xi \ominus \xi = 0, \quad \ominus\xi \oplus \xi = 0$$

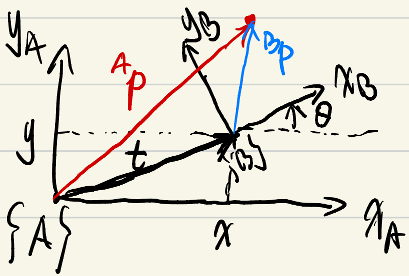
Note: $\xi_1 \oplus \xi_2 \neq \xi_2 \oplus \xi_1$ composition is not commutative.

Exception $\xi_1 \oplus \xi_2 = 0 = \xi_2 \oplus \xi_1$

$$\underline{\ominus \xi_F} \oplus \xi_F \oplus {}^F\xi_R = \underline{\ominus \xi_F} \oplus \xi_R \Rightarrow {}^F\xi_R = \ominus \xi_F \oplus \xi_R$$

What is a relative pose mathematically?

2D case :



decomposition: origin of $\{B\}$ is $\begin{cases} \text{displaced by } t = \begin{bmatrix} x \\ y \end{bmatrix} \\ \text{rotated by } \theta \text{ counter clockwisely} \end{cases}$

$\Rightarrow$ A complete representation of ${}^A\xi_B$ is therefore (x, y, θ)

Mathematically: rotation operation

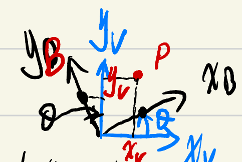


Fig. 1

Goal: identify the relationship between $\{B\}$ & $\{V\}$

Important observation: $\underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{coordinate transformation matrix}} \underbrace{\begin{bmatrix} {}^Bx \\ {}^By \end{bmatrix}}_{\text{coordinates in } \{B\}} = \underbrace{\begin{bmatrix} {}^Vx \\ {}^Vy \end{bmatrix}}_{\text{coordinates in } \{V\}}$

$${}^Vp = \underbrace{{}^Vx}_{\text{coordinates in } \{V\}} \hat{x}_V + \underbrace{{}^Vy}_{\text{coordinates in } \{V\}} \hat{y}_V$$

$$= [\hat{x}_V \ \hat{y}_V] \begin{bmatrix} {}^Vx \\ {}^Vy \end{bmatrix}$$

$\hat{x}_V$: unit vector along x_V

$\hat{y}_V$: unit vector along y_V

same vector in Fig. 1

$${}^Bp = [\hat{x}_B \ \hat{y}_B] \begin{bmatrix} {}^Bx \\ {}^By \end{bmatrix} = [1 \cdot \cos\theta \, \hat{x}_V + 1 \cdot \sin\theta \, \hat{y}_V, \ -1 \sin\theta \, \hat{x}_V + \cos\theta \, \hat{y}_V] \begin{bmatrix} {}^Bx \\ {}^By \end{bmatrix} = [\hat{x}_V \ \hat{y}_V] \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} {}^Bx \\ {}^By \end{bmatrix}$$

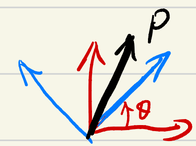
$$\underbrace{\begin{bmatrix} {}^Vx \\ {}^Vy \end{bmatrix}}_{{}^Vp} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{{}^V\xi_B \triangleq {}^VR_B} \underbrace{\begin{bmatrix} {}^Bx \\ {}^By \end{bmatrix}}_{{}^Bp}$$

describes how points are transformed from $\{B\}$ to $\{V\}$ when rotating frames

The rotation matrix:

Properties

$${}^VR_B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



- $\det({}^VR_B) = \cos^2\theta + \sin^2\theta = 1$

$${}^Vp = {}^VR_B\,{}^Bp$$

$$\Rightarrow \|{}^Vp\| = \|{}^VR_B\,{}^Bp\| = \|{}^Bp\|$$

- $r_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ & $r_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

$r_1 \perp r_2$: $r_1 \cdot r_2 = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$

columns are perpendicular to each other.

${}^VR_B$ is orthonormal $\left\{ \left\| \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \right\|_2 = \sqrt{\cos^2\theta + \sin^2\theta} = 1 \right.$ $\Rightarrow$ each column has unit length

- inverse rotation

$$ {}^{V}R_B(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} $$

$$ \Rightarrow {}^{V}R_B(-\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \left({}^{V}R_B(\theta)\right)^T $$

inverse rotation is transpose of original rotation matrix

$$ {}^{V}p = {}^{V}R_B \; {}^{B}p $$

$$ \Rightarrow \left({}^{V}R_B\right)^{-1} {}^{V}p = {}^{B}p \Leftrightarrow \left({}^{V}R_B\right)^T {}^{V}p = {}^{B}p $$

$$ \Updownarrow $$

$$ {}^{B}R_V \, {}^{V}p = {}^{B}p $$

$$ \boxed{\left({}^{V}R_B\right)^T = {}^{B}R_V = \left({}^{V}R_B\right)^{-1}} $$

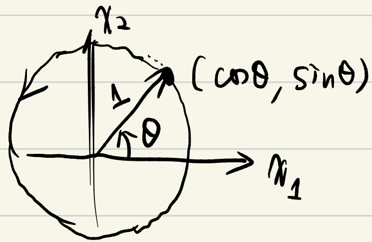
MATLAB:

```
>> R = rot2(30, 'deg')
or R = rot2(0.5)
syms theta
R = rot2(theta)
```

(after installing the Toolbox)

System representations of the rotation process & rotation matrix

$$\dot{x} = Ax \longleftrightarrow R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



$$\begin{cases} \dot{x}_1 = \frac{d}{dt}\cos\theta = -\sin\theta = -x_2 \\ \dot{x}_2 = \frac{d}{dt}\sin\theta = \cos\theta = x_1 \end{cases}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} t} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

Recall w/ linear systems

$$e^{\begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix} t} = e^{\sigma t}\begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix}$$

$$\Rightarrow e^{\begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

at t when angle is θ

$$\Rightarrow x(\theta) = e^{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\theta} x(0)$$ denoted as $x(\theta) = e^{[\theta]_x} x(0)$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} x(0)$$

$R(\theta) = e^{[\theta]_x}$

Relating two position vectors

Key: rotation closely related to the skew-symmetric $\begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} \triangleq [\theta]_x$

In MATLAB: >> $S = [0 \ \ -0.2; \ \ 0.2 \ \ 0]$

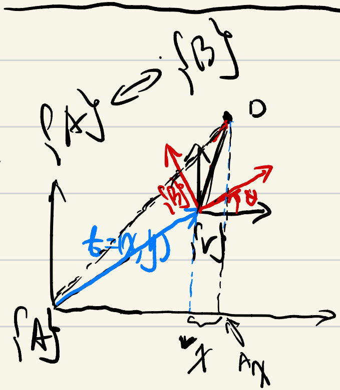
$R = \text{expm}(S)$

$$R = e^{[\theta]_\times}$$

$$[\theta]_\times = \text{logm}(R)$$

Now: add translation (homogeneous transformation)

Goal: $\{A\} \Leftrightarrow \{B\}$



For point P:

translation

$$\begin{bmatrix} {}^Ax \\ {}^Ay \end{bmatrix} = \begin{bmatrix} {}^Vx \\ {}^Vy \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= {}^VR_B \begin{bmatrix} {}^Bx \\ {}^By \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$$

because $\{V\}$ & $\{A\}$ are parallel

$$= {}^AR_B \begin{bmatrix} {}^Bx \\ {}^By \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} {}^Ax \\ {}^Ay \\ 1 \end{bmatrix}}_{\text{coordinates in homogeneous form: } {}^A\tilde{p}} = \underbrace{\left[\begin{array}{c|c} {}^AR_B & \begin{matrix} x \\ y \end{matrix} \\ \hline 0_{1\times 2} & 1 \end{array}\right]}_{\text{homogeneous transformation matrix. } {}^AT_B} \underbrace{\begin{bmatrix} {}^Bx \\ {}^By \\ 1 \end{bmatrix}}_{{}^B\tilde{p}}$$

Homogeneous transformation matrix properties:

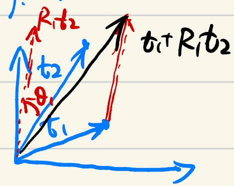
$$P = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}$$

— composable:

$$P_1 \oplus P_2 = P_1 P_2$$

$$P_1 P_2 = \begin{bmatrix} R_1 & t_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 & t_2 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \overbrace{R_1 R_2}^{\text{still a rotation}} & \overbrace{t_1 + R_1 t_2}^{\text{still a translation}} \\ 0 & 1 \end{bmatrix}}_{\text{still a homogeneous transformation matrix}}$$

e.g. $R_1 : \theta_1 = 30°$



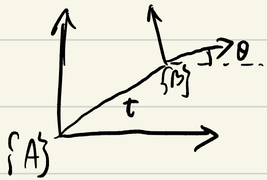
Notice:

$$P_2 P_1 = \begin{bmatrix} R_2 R_1 & t_2 + R_2 t_1 \\ 0 & 1 \end{bmatrix}$$

$$\neq P_1 P_2$$

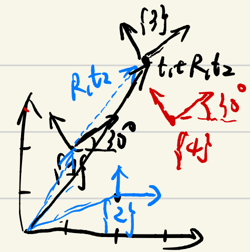
— Inverse

$$T^{-1} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^{-1}t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}$$



$$\ominus T \oplus T = 0 \Rightarrow \ominus T = T^{-1} = \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}$$

MATLAB: >> $T_1 = \underbrace{\text{transl2}(1, 2)}_{t = \begin{bmatrix} 1 \\ 2 \end{bmatrix}} \cdot \underbrace{\text{trot2}(30, \text{'deg'})}_{R:\ \theta = 30^\circ}$



>> $T_2 = \text{transl2}(2, 1)$

$$T_3 = T_1 T_2 = \begin{bmatrix} R_1R_2 & t_1 + R_1 t_2 \\ 0 & 1 \end{bmatrix}$$

$$T_4 = T_2 T_1 = \begin{bmatrix} R_2R_1 & t_2 + R_2 t_1 \\ 0 & 1 \end{bmatrix}$$

$$t_2 + R_2 t_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + I \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$R_2R_1 : \theta = 0^\circ + 30^\circ$$

coordinate

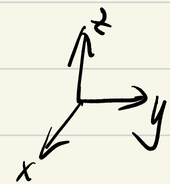
$$p = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad {}^0p = {}^0\xi_1\, {}^1p \Rightarrow {}^1p = \left({}^0\xi_1\right)^{-1} {}^0p$$

$$ {}^1\tilde{p} = \left({}^0T_1\right)^{-1} {}^0\tilde{p}$$

$$\begin{bmatrix} {}^1p \\ 1 \end{bmatrix} = T_1^{-1} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

Now : Pose in 3D

Recall: right-hand rule & right-hand coordinate frame



unit vector in the three coordinate axis:

$\hat{x}$: $\|\hat{x}\| = 1$ along x direction $\hat{y}$: $\|\hat{y}\| = 1$ along y direction $\hat{z}$: $\|\hat{z}\| = 1$ along z direction

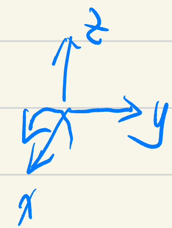
$$\hat{x} \times \hat{y} = \hat{z} \qquad \hat{y} \times \hat{z} = \hat{x} \qquad \hat{z} \times \hat{x} = \hat{y}$$

Given any p in 3D : exist $p = x\hat{x} + y\hat{y} + z\hat{z}$ $p = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

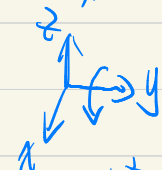
Rotation in 3D is more involved; different ways were developed in history.

Euler's Rotation thm: Any two independent orthonormal coordinate frames can be related by a sequence (≤ 3) of rotations about coordinate axes & no two successive rotations may be about the same axis.

orthonormal rotation matrix:
$$\begin{bmatrix} {}^{A}x \\ {}^{A}y \\ {}^{A}z \end{bmatrix} = \begin{bmatrix} & {}^{A}R_{B} & \end{bmatrix} \begin{bmatrix} {}^{B}x \\ {}^{B}y \\ {}^{B}z \end{bmatrix}$$

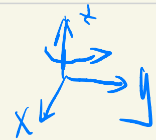


rotation of θ about x axis:
$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



〃 〃 〃 〃 y 〃 :
$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

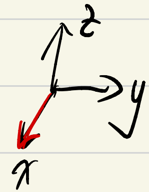
〃 〃 〃 〃 z 〃 :
$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



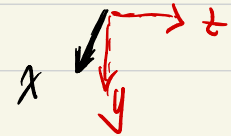
In MATLAB:

```
>> Rx = rotx(pi)
trplot(Rx)
tranimate(Rx)
```

e.g. $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$



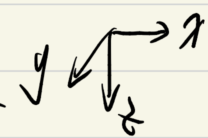
$\Rightarrow$



$-\sin\theta = 1 \Rightarrow \theta = -90^\circ$

3-angle representation: Recall Euler's rotation Thm asserts that any rotation can be achieved via successive rotation about 3 axes (can't do e.g. xxy or xzz)

- solution is not unique



- two general classes
  - Eulerian: e.g. xyx, xzx, yzy, yxy, zxz, zyz (contains repetitions)
  - Cardenian: uses all 3 axes: e.g. xyz, xzy, yxz, yzx, zxy, zyx

Eulerian way : e.g. zyz sequence (common in mechanical & aerospace).

$$R = R_z(\phi) R_y(\theta) R_z(\psi)$$

defines the so-called Euler angles $\Gamma = (\phi, \theta, \psi)$

MATLAB: >> $Gamma = [\phi, \theta, \psi] \overset{e.g.}{=} [0.1, 0.2, 0.3]$

>> $R = rotz(\phi) \cdot roty(\theta) \cdot rotz(\psi)$

forward $\Leftrightarrow$ $R = eul2r(0.1, 0.2, 0.3)$

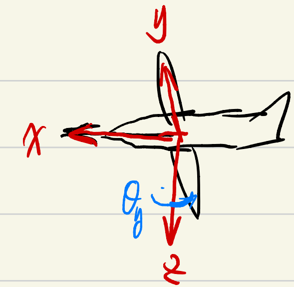
inverse >> $Gamma = tr2eul(R) \Rightarrow Gamma = [0.1, 0.2, 0.3]$

special case (singularity) : $R_y(\theta) = I \Rightarrow R = R_z(\phi) R_z(\psi) = R_z(\phi + \psi)$

$= R_z(\phi - 0.3) R_z(\psi + 0.3)$

physically : $R_y(\theta) = I \Leftrightarrow \theta = 0$

Cardenian Way: commonly xyz or zyx



usual sequence ①

yaw — direction of travel
(bank) (rotation θ_y along z axis)

pitch — elevation of the front (rotation θ_p along y)
(attitude)

roll — rotation θ_r along x axis
(heading)

zyx sequence $\Rightarrow R = R_z(\theta_y)\, R_y(\theta_p)\, R_x(\theta_r)$

MATLAB: >> R = rpy2r(0.1, 0.2, 0.3) to know more, do "help rpy2r" in MATLAB.

singularity analysis: $\theta_p = \pm\frac{\pi}{2} \Rightarrow R_y(\theta_p) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

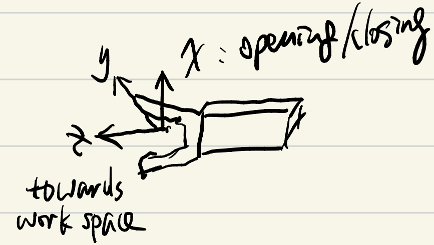
aircrafts become rockets!

Fortunately, this is outside of the range of feasible attitudes for most vehicles.

MATLAB >> tripleangle

Cardenian Way : commonly xyz or zyx

usual sequence θ : xyz for e.g. robotic grippers



Singularity occurs with using a minimal representation.

Singularity can be avoided by introducing a fourth DOF. $\Rightarrow$ quaternion

(won't talk too much for now)

Add now translation : 3D pose

$$\begin{bmatrix} {}^A x \\ {}^A y \\ {}^A z \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} {}^A R_B & t \\ 0_{1\times 3} & 1 \end{bmatrix}}_{{}^A T_B} \begin{bmatrix} {}^B x \\ {}^B y \\ {}^B z \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} {}^A x \\ {}^A y \\ {}^A z \end{bmatrix} = {}^A R_B \begin{bmatrix} {}^B x \\ {}^B y \\ {}^B z \end{bmatrix} + t$$

${}^A T_B$: 4x4 homogeneous transformation matrix

$$T_1 \cdot T_2 = \begin{bmatrix} R_1 R_2 & t_1 + R_1 t_2 \\ 0 & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}$$

MATLAB: >> T = transl(1, 0, 0) · trotx(pi/2) · transl(0, 1, 0)

Notation: $SO2 = \left\{ R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right\}$

special orthonormal matrices in 2D

$SE2 = \left\{ T = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix} \right\}$

special Euclidian group in 2D.

$SO3 = \left\{ R \mid R \in \mathbb{R}^{3\times 3} \quad R^TR = RR^T = I \right\}$

$SE3 = \left\{ T \mid T = \begin{bmatrix} R & t \\ 0_{1\times 3} & 1 \end{bmatrix}, \; R \in \mathbb{R}^{3\times 3} \quad RR^T = R^TR = I \right\}$

special Euclidian group of rigid-body motion

Wrapping up:

${}^{A}\xi_B$: pose of coordinate frame $\{B\}$ relative to frame $\{A\}$

$$\xi_B = \xi_A \oplus {}^{A}\xi_B$$

maps a coordinate vector from $\{B\}$ to $\{A\}$:

$$^{A}p = {}^{A}\xi_B \, {}^{B}p$$