Inverted Pendulum Modeling

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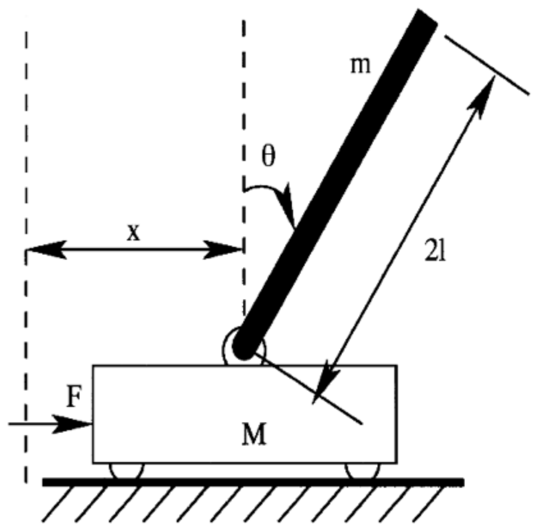
A cart with an inverted pendulum is shown below:



The mass of the cart is m_c, the mass and moment of inertia (about mass center) of the pendulum are m_p, I_p respectively. The pendulum is attached to the cart at a frictionless pivot point. The pendulum's center of mass is l units from the pivot. The coordinate x measures the position of the cart (a point-mass) relative to an inertial frame. F is an applied external horizontal force to the cart.

1. Find the equations of motion. The following hints may be useful.

   - Hint 1: Analyze the force balance in the x-direction for the system. Note that the x-position of the center of gravity of the pendulum is a function of x and θ.

   - Hint 2: Analyze the torque balance of the pendulum with respect to the pivot point (mounted on a moving base). In this case, the moment of inertia about the pivot point is $I_p + m_p l^2$.

2. Assume that x and θ are small. Find the linearized equations of motion about $x = 0,\ \theta = 0$.

3. Let $m_c = m_p = l = 1$ and $I_p = 1/3$. Approximate g with $g = 10$. Find the transfer functions

$$H_1(s) = \frac{X(s)}{F(s)}, \quad H_2(s) = \frac{\Theta(s)}{F(s)}$$

4. Let the force F be chosen to be a linear feedback,

$$F(t) = -c_1\theta(t) - c_2\dot{\theta}(t) - c_3 x(t) - c_4\dot{x}(t),$$

   - find the transfer function of this controller.
   - find the closed-loop characteristic equation.

5. Find necessary (but not sufficient) conditions on $c_1,\ c_2,\ c_3,\ c_4$ to guarantee the stability of the linearized system.

Solution:

1. Nonlinear equation of motion: x_p is related to x by

$$x_p = x + l\sin\theta.$$

   We have

$$F = m_c\ddot{x} + m_p\ddot{x}_p = (m_c + m_p)\ddot{x} + m_p l(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)$$
$$\Rightarrow \ddot{x} = -\frac{m_p l}{m_c + m_p}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + \frac{F}{m_c + m_p}$$

   For the torque balance of the pendulum, note that this is in a moving frame with acceleration $\ddot{x}$. There is a force component in the -x direction that equals $m_p\ddot{x}$. Hence

$$(I_p + m_p l^2)\ddot{\theta} = m_p g l\sin\theta - m_p\ddot{x} l\cos\theta.$$

   or

$$\ddot{\theta} = \frac{m_p l}{I_p + m_p l^2}(g\sin\theta - \ddot{x}\cos\theta)$$

   (When the rod is uniform, the moment of inertia satisfies $I_p = m_p(2l)^2/12$.)

2. Linearization: When θ is very small,

$$\cos\theta \approx 1$$

$$\sin\theta \approx 0$$

   and

$$\dot{\theta}^2 \approx 0$$

yielding

$$\ddot{x} = \frac{1}{m_c + m_p}(-m_p l \ddot{\theta} + F) \tag{1}$$

$$\ddot{\theta} = \frac{m_p l}{I_p + m_p l^2}(g\theta - \ddot{x}) \tag{2}$$

3. Transfer functions: Applying Laplace transform and assuming zero initial conditions give

$$\frac{X(s)}{F(s)} = \frac{60 - 8s^2}{60s^2 - 5s^4}$$

and

$$\frac{\Theta(s)}{F(s)} = \frac{6s^2}{60s^2 - 5s^4} = \frac{6}{60 - 5s^2}$$

4. TBD
5. TBD

Controllability of the linearized system

Let the state vector be $\eta = [x, \dot{x}, \theta, \dot{\theta}]^T$. Substituting (1) into (2) yields

$$\ddot{\theta} = \frac{m_p l}{I_p + m_p l^2}(g\theta - \ddot{x}) = \frac{m_p l}{I_p + m_p l^2}\left(g\theta - \frac{1}{m_c + m_p}(-m_p l\ddot{\theta} + F)\right)$$

$$\Rightarrow \ddot{\theta} = \frac{m_p l}{I_p + \frac{m_c m_p l^2}{m_c + m_p}}\left(g\theta - \frac{1}{m_c + m_p}F\right)$$

Substituting the above back into (1), we then have

$$\ddot{x} = \frac{-m_p^2 l^2 g}{I_p\left(m_c + m_p\right) + m_c m_p l^2}\theta + \frac{m_p l^2 + I_p}{I_p\left(m_c + m_p\right) + m_c m_p l^2}F$$

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-m_p^2 l^2 g}{I_p(m_c+m_p)+m_c m_p l^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{m_p l g (m_c+m_p)}{I_p(m_c+m_p)+m_c m_p l^2} & 0 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{m_p l^2 + I_p}{I_p(m_c+m_p)+m_c m_p l^2} \\ 0 \\ \frac{-m_p l}{m_c I_p + m_p I_p + m_c m_p l^2} \end{bmatrix} F$$

The controllability matrix

$$P = \begin{bmatrix} 0 & \frac{m_p l^2 + I_p}{I_p(m_c+m_p)+m_c m_p l^2} & 0 & \frac{m_p^2 l^2 g m_p l}{(I_p(m_c+m_p)+m_c m_p l^2)^2} \\ \frac{m_p l^2 + I_p}{I_p(m_c+m_p)+m_c m_p l^2} & 0 & \frac{m_p^2 l^2 g m_p l}{(I_p(m_c+m_p)+m_c m_p l^2)^2} & 0 \\ 0 & \frac{-m_p l}{I_p(m_c+m_p)+m_c m_p l^2} & 0 & \frac{-m_p^2 l^2 g (m_c+m_p)}{(I_p(m_c+m_p)+m_c m_p l^2)^2} \\ \frac{-m_p l}{I_p(m_c+m_p)+m_c m_p l^2} & 0 & \frac{-m_p^2 l^2 g (m_c+m_p)}{(I_p(m_c+m_p)+m_c m_p l^2)^2} & 0 \end{bmatrix}$$

has full row rank.

Extensions

- double pendulum on a cart: https://www.youtube.com/watch?v=B6vr1x6KDaY
- triple pendulum on a cart: https://www.youtube.com/watch?v=FFW52FuUODQ
- Example in Matlab:
  - single pendulum: http://ctms.engin.umich.edu/CTMS/index.php?example=InvertedPendulum&sectio http://ctms.engin.umich.edu/CTMS/index.php?example=InvertedPendulum&section=SimulinkCont
  - `double pendulum: sm_cart_double_pendulum`