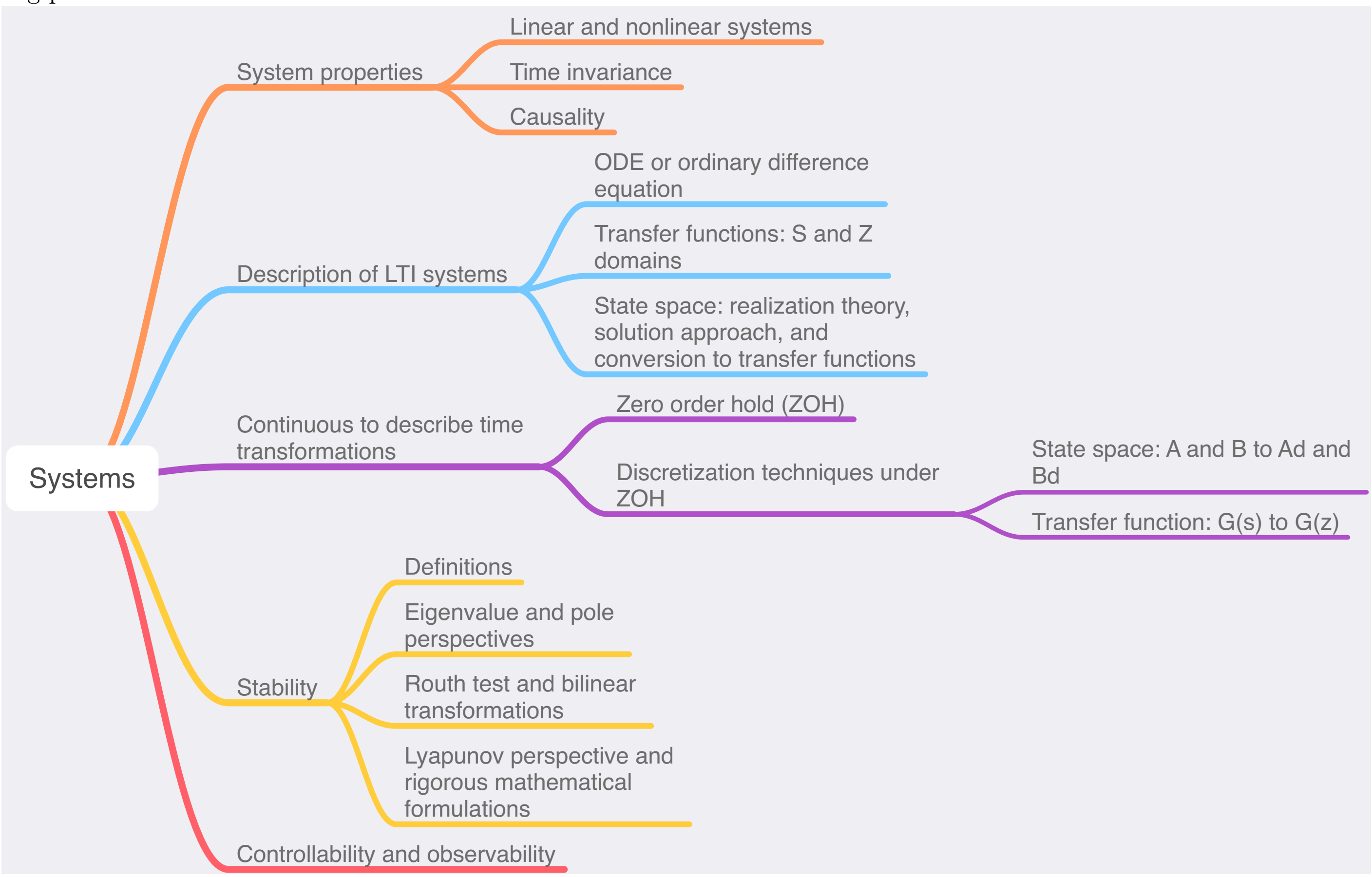
Midterm Review

Big picture:



(controllability and observability will come later after the midterm)

Laplace and Z transforms: Practice explaining the following concepts on a piece of white paper:

- definitions: $F(s) = \int_0^\infty f(t) e^{-st} dt$, $F(z) = \sum_{k=0}^\infty f(k) z^{-k}$
- properties: linearity, convolution, time delay, differentiation/integration ($\mathcal{L}$), step advance/delay ($\mathcal{Z}$), initial value theorem, final value theorem
- inverse Laplace and inverse Z transforms, partial fraction expansion
- table of Laplace and Z transform pairs
- differential/difference equations, transfer functions, and their relationships
- examples: $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+a)^2}\right\}$, $f(k) = \mathcal{Z}^{-1}\left\{\frac{Tz^{-1}}{(a-z^{-1})^2}\right\}$ (solution: te^{-at} and $k\frac{T}{a}\left(\frac{1}{a}\right)^k$)

State-space representation of LTI systems

state space	transfer function
$\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t) + Du(t)$	$G(s) = C(sI - A)^{-1}B + D$
$x(k+1) = Ax(k) + Bu(k),\ y(k) = Cx(k) + Du(k)$	$G(z) = C(zI - A)^{-1}B + D$

Practice explaining the following concepts on a piece of white paper:

- relationship to transfer functions:

- canonical forms: controllable canonical form, observable canonical form, diagonal form, Jordan form, modified Jordan form
- example: write down all the possible canonical forms for $G(s) = \frac{2s+1}{s^2+2s+1}$

Solution of LTI systems: Consider

$$\dot{x}(t) = Ax(t) + Bu(t); \quad x(t_0) = x_o$$
$$x(k+1) = Ax(k) + Bu(k); \quad x(k_0) = x_o$$

we have

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$
$$x(k) = A^{k-k_0}x(k_o) + \sum_{j=k_0}^{k-1} A^{k-1-j}Bu(j)$$
$$= A^{k-k_0}x(k_o) + \begin{bmatrix} A^{k-k_0-1}B & A^{k-k_0-2}B & \cdots & B \end{bmatrix} \begin{bmatrix} u(k_0) \\ u(k_0+1) \\ \vdots \\ u(k-1) \end{bmatrix}$$

- free response: the case for $u(t) = 0$ or $u(k) = 0$
- forced response: the case for $x_0 = 0$
- computation of e^{At} and A^k:
  - eigenvalue-eigenvector approach: $A = T\Lambda T^{-1}$, $e^{At} = Te^{\Lambda t}T^{-1}$,$A^k = T\Lambda^k T^{-1}$
  - inverse Laplace/Z transform approach: $e^{At} = \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}$, $A^k = \mathcal{Z}^{-1}\left\{(zI-A)^{-1}z\right\}$
  - direct approach: select $x_0 = [1, 0, \ldots, 0]^T$, $x_0 = [0, 1, \ldots, 0]^T$,... $x_0 = [0, 0, \ldots, 1]^T$ as the initial condition and solve the differential/difference equations
- eigenvectors and generalized eigenvectors: if the eigenvectors are distinct, then A can be diagonalized; if not, generalized eigenvectors may need to be introduced. Practice doing a second-order example where $(A - \lambda I)t_1 = 0$ and $A\begin{bmatrix} t_1 & t_2 \end{bmatrix} = \begin{bmatrix} t_1 & t_2 \end{bmatrix}\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \Rightarrow (A - \lambda I)t_2 = t_1$.[1]

A	e^{At}
$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$	$\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$
$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$	$\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$
$\begin{bmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix}$	$\begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2!}t^2e^{\lambda t} \\ & e^{\lambda t} & te^{\lambda t} \\ & & e^{\lambda t} \end{bmatrix}$
$\begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}$	$\begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} & \frac{t^3}{3!}e^{\lambda t} \\ & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ & & e^{\lambda t} & te^{\lambda t} \\ & & & e^{\lambda t} \end{bmatrix}$

[1]In the notes, we used $(\lambda I - A)t_2 = -t_1$.

J	J^k
$\begin{bmatrix}\lambda_1 & \\ & \lambda_2\end{bmatrix}$	$\begin{bmatrix}\lambda_1^k & \\ & \lambda_2^k\end{bmatrix}$
$\begin{bmatrix}\lambda & 1\\ & \lambda\end{bmatrix}$	$\begin{bmatrix}\lambda^k & k\lambda^{k-1}\\ & \lambda^k\end{bmatrix}$
$\begin{bmatrix}\lambda & 1 & \\ & \lambda & 1\\ & & \lambda\end{bmatrix}$	$\begin{bmatrix}\lambda^k & k\lambda^{k-1} & \frac{1}{2!}k\left(k-1\right)\lambda^{k-2}\\ & \lambda^k & k\lambda^{k-1}\\ & & \lambda^k\end{bmatrix}$
$\begin{bmatrix}\lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1\\ & & & \lambda\end{bmatrix}$	$\begin{bmatrix}\lambda^k & k\lambda^{k-1} & \frac{1}{2!}k\left(k-1\right)\lambda^{k-2} & \frac{1}{3!}k\left(k-1\right)\left(k-2\right)\lambda^{k-3}\\ & \lambda^k & k\lambda^{k-1} & \frac{1}{2!}k\left(k-1\right)\lambda^{k-2}\\ & & \lambda^k & k\lambda^{k-1}\\ & & & \lambda^k\end{bmatrix}$

- Exercise: write down J^k and e^{Jt} for $J = \begin{bmatrix}-0.5 & & \\ & -0.5 & 1\\ & & -0.5\end{bmatrix}$ and $J = \begin{bmatrix}-10 & 1 & & & \\ & -10 & & & \\ & & -2 & & \\ & & & -1/2 & \sqrt{3}/2\\ & & & -\sqrt{3}/2 & -1/2\end{bmatrix}$

System discretization: from continuous- to discrete-time When we have a ZOH:

- State-space:

$$A_d = e^{A\Delta t},\ B_d = \int_0^{\Delta t} e^{A\tau}Bd\tau$$

  (For other holders, the formulas will be different.)

- Transfer function:

$$G\left(z\right) = \left(1 - z^{-1}\right)\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G\left(s\right)}{s}\right\}\right\}$$

Stability

- Definition
  - Stability in the sense of Lyapunov
  - Asymptotic stability
  - BIBO stability etc
- Stability analysis of linear time invariant systems ($\dot{x} = Ax$ or $x(k+1) = Ax(k)$):
  Is the equilibrium, $x = 0$, asymptotically stable?
  - Based on the eigenvalues of A:

$$\text{Asymptotic stability} \Longleftrightarrow \begin{cases}\text{CT:} & Re\{\lambda_i\} < 0\\ \text{DT:} & |\lambda_i| < 1\end{cases}$$

  - Routh's criterion:
    - No need to solve the characteristic equation
    - Routh's array
    - Discrete time case: bilinear transform ($z = \frac{1+s}{1-s}$)
  - Lyapunov equations: intuition and formulas