Outline:

- Recap: controllability and observability
- Linear state feedback control

1 Recap: controllability and observability

- introductory examples: discuss controllability, observability conditions for the following system realizations[1]

$$\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & B_2 \\ \hline 0 & C_2 & D \end{array}\right], \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ & A_{22} & 0 \\ \hline C_1 & C_2 & D \end{array}\right], \left[\begin{array}{cc|c} A_{11} & 0 & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & 0 & D \end{array}\right], \left[\begin{array}{cc|c} A_{11} & 0 & 0 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$$

- exercise: find the uncontrollable and unobservable states for the following system

$$A = \left[\begin{array}{c|c|c|c} A_{11} & 0 & A_{13} & 0 \\ \hline A_{21} & A_{22} & A_{23} & A_{24} \\ \hline 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{array}\right], \quad B = \left[\begin{array}{c} B_1 \\ \hline B_2 \\ \hline 0 \\ 0 \end{array}\right]$$

$$C = \left[\begin{array}{cccc} C_1 & 0 & C_3 & 0 \end{array}\right]$$

- why do we study the decomposition forms?
  - if controllability and observability fail, these decomposed forms provide detectability and stabilizability for us. These are essential for a good engineering judgment of the problem we are solving. For example, if the system is not stabilizable, then either it is not properly designed or some desired control goals are simply not achievable (turn back the problem to the hardware designer or give up).

- intuitions and extensions of the decomposition: *the column space (a.k.a. range) of* $P = \left[B, AB, A^2B, \dots, A^{n-1}B\right]$ *is* $\mathcal{A}$*-invariant.*
  - Definition: consider a vector space $(\mathbb{V}, \mathbb{F})$ and a linear mapping $A : V \to V$, A subspace M is said to be $\mathcal{A}$-invariant if $\forall x \in M$, $\mathcal{A}(x) \in M$. Example: the null space of A, $\mathcal{N}(A)$, is $\mathcal{A}$-invariant.
  - Application in controllability analysis:
    - We first prove (outline only) the range of P *is* $\mathcal{A}$*-invariant:* $\forall x \in R(P)$, there exists y such that $x = Py$, yielding

$$Ax = APy = \left[AB, A^2B, \dots, A^nB\right] y \in R(P)$$

      where we used Caylay Halmilton theorem in the last $\in$ sign.
    - Now let M_1 be composed of the independent column vectors from P. We will next slightly abuse the notation M_1 to denote both a matrix and the subspace spanned by its column vectors.
    - Complete the basis for $\mathbb{R}^n$ with $\mathbb{R}^n = M_1 \oplus M_2$ where $\dim M_1 = k$, $\dim M_2 = n - k$. Here $\mathbb{R}^n = M_1 \oplus M_2$ means the combination of the two subspaces form $\mathbb{R}^n$, and in addition, the two subspaces are perpendicular to each other (i.e., $\forall m_i \in M_1$ and $m_j \in M_2$, the inner product $m_i^T m_j = 0$).
    - Now let $M = [M_1, M_2]$. Since the column space of P is $\mathcal{A}$*-invariant*, all columns of AM_1 are in the subspace spanned by M_1. Thus there exists

$$AM_1 = [M_1, M_2] \left[\begin{array}{c} \tilde{A}_{11} \\ 0 \end{array}\right]$$

      where the lower block in the right most matrix above has to be zero in order to let the columns of AM_1 stay in M_1. Hence we have

$$AM = M \left[\begin{array}{cc} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{array}\right]$$

[1] We use the compact form $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ to denote a system with state-space matrices $\{A, B, C, D\}$.

yielding

$$M^{-1}AM = M^{-1}M \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}$$

which is the similarity transform to decompose the system intro controllable and uncontrollable modes.

- exercise: work out the case for observability

2 Notes about linear state/output feedback control

- State variable feedback control theorem: the pair $\{A, B\}$ is controllable if and only if the roots of the closed loop characteristic equation (closed-loop eigenvalues) can be arbitrarily assigned[2] in the complex plane.
  Proof: "$\Rightarrow$": as exercise; "$\Leftarrow$": if not controllable, then there exists a Kalman canonical realization such that

$$\frac{d}{dt}\begin{bmatrix} \tilde{x}_c \\ \tilde{x}_{uc} \end{bmatrix} = \begin{bmatrix} \tilde{A}_c & \tilde{A}_{12} \\ 0 & \tilde{A}_{uc} \end{bmatrix}\begin{bmatrix} \tilde{x}_c \\ \tilde{x}_{uc} \end{bmatrix} + \begin{bmatrix} \tilde{B}_c \\ 0 \end{bmatrix}$$

  and the eigenvalues of $\tilde{A}_{uc}$ cannot be changed.

- Controllability is preserved under (linear or nonlinear) state and output feedback control.

[2]Complex roots must be accompanied by their complex conjugates - symmetry about real axis.