1 Controllability and Observability

- Controllability
  - Key idea in the definition: steer the system to any point in the state space within a finite time
  - Cayley Halmilton theorem: A^n is linearly dependent on $\{I,\ A,\ A^2, \cdots A^{n-1}\}$
    Proof: Let $p(\lambda) \triangleq \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$. We have $p(\lambda) = (\lambda - \lambda_1)^{m_1} \ldots (\lambda - \lambda_p)^{m_p}$. Thus

$$\begin{aligned} p(A) &= A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I \\ &= (A - \lambda_1 I)^{m_1} \ldots (A - \lambda_p I)^{m_p} \end{aligned}$$

    Take any eigenvector or generalized eigenvector t_i, we have $p(A)\,t_i = 0$. Therefore $p(A)[t_1, t_2, \ldots, t_n] = 0$. But $T = [t_1, t_2, \ldots, t_n]$ is invertible, hence $p(A) = 0$, namely, $A^n = -\left(c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I\right)$, which is a linear combination of $\{I,\ A,\ A^2, \cdots A^{n-1}\}$.
  - Essential equations:

$$x(n) = A^n x(0) + \sum_{j=0}^{n-1} A^{n-1-j} B u(j) = A^n x(0) + \left[\begin{array}{cccc} A^{n-1}B & A^{n-2}B & \cdots & B \end{array}\right] \left[\begin{array}{c} u(0) \\ u(1) \\ \vdots \\ u(n-1) \end{array}\right]$$

    and Cayley Halmilton theorem
  - Theorem: The following statements are equivalent:
    1. $x(k+1) = Ax(k) + Bu(k)$ is controllable.
    2. The columns of the controllability matrix $P = [B\, AB\, A^2B \cdots A^{n-1}B]$ span $\mathbb{R}^n$. (equivalence between 1 and 2: use $x(n) = A^n x(0) + \left[\begin{array}{cccc} A^{n-1}B & A^{n-2}B & \cdots & B \end{array}\right] \left[\begin{array}{c} u(0) \\ u(1) \\ \vdots \\ u(n-1) \end{array}\right]$)
    3. The controllability matrix $P = [B\, AB\, A^2B \cdots A^{n-1}B]$ is rank n (full row rank). (equivalence between 2 and 3: rank $n \Leftrightarrow$ spans $\mathbb{R}^n$)
    4. The controllability grammian $W_c(k_1) = \sum_{k=0}^{k_1} A^k BB^T (A^T)^k$ is positive definite, for some finite integer k_1. ($3 \Leftrightarrow 4$: consider $W_c(n) = PP^T$)
    5. (Popov-Belevitch-Hautus (PBH) test) The matrix $[A - \lambda I\ B]$ has full row rank at every eigenvalue, λ, of A. (intuition: if 5 does not hold, then $\exists v$ such that $v^T[A - \lambda I\ \ B] = 0 \Rightarrow v^T A = \lambda v^T,\ v^T B = 0$, yielding $v^T[B\, AB\, A^2B \cdots A^{n-1}B] = 0$, which violates the full row rank condition for P)
  - Example 1: identify controllability of (A, B):

$$A = \left[\begin{array}{ccc} \lambda_1 & & \\ & \lambda_2 & 1 \\ & & \lambda_2 \end{array}\right],\ B = \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$$

  - The controllability property is invariant under any coordinate transformation:

$$(A, B) \text{ is controllable.} \Longleftrightarrow (\bar{A}, \bar{B}) = (TAT^{-1}, TB) \text{ is controllable.}$$

    where T is any nonsingular matrix. (Proof: $\bar{P} = TP$ where T is invertible)
  - If A is asymptotically stable, k_1 can be set to ∞ in the controllability grammian, and we may obtain the grammian by solving the Lyapunov equation

$$AW_cA^T - W_c = -BB^T$$

    The solution is positive definite if and only if the system is controllable.

- Observability
  - Key idea in the definition: knowing the outputs and the inputs can determine the initial state (Question: why initial state here?)
  - Essential equations: assuming $u(k) = 0$, then $y(0) = Cx(0)$, $y(1) = Cx(1) = CAx(0)$, $y(2) = Cx(2) = CA^2x(0)$,... (Question: what if $u \neq 0$?)
  - Theorem: the following statements are equivalent:
    1. $x(k+1) = Ax(k)$, $y = Cx(k)$ is observable.
    2. The observability grammian $W_o(k_1) = \sum_{k=0}^{k_1}(A^T)^k C^T CA^k$ is positive definite, for some finite integer k_1.
    3. The observability matrix $Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$ is rank n (full column rank).
    4. The matrix $\begin{bmatrix} A-\lambda I \\ C \end{bmatrix}$ has full column rank at every eigenvalue, λ, of A.
  - The observability property is invariant under any coordinate transformation:
  $$(A, C) \text{ is observable.} \Longleftrightarrow (\bar{A}, \bar{C}) = (TAT^{-1}, CT^{-1}) \text{ is observable.}$$
  where T is any nonsingular matrix.
  - If A is asymptotically stable, k_1 can be set to ∞ in the observability grammian, and we may look for the grammian by solving the Lyapunov equation
  $$A^T W_o A - W_o = -C^T C$$
  The solution is positive definite if and only if the system is observable.

- Example 2: analyze the controllability, observability, stabilizability, and detectability of the following systems
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  $$A = \begin{bmatrix} 1 & & \\ & -2 & \\ & & 3/10 \end{bmatrix}, \; B = \begin{bmatrix} 23 \\ 0 \\ 3/20 \end{bmatrix}$$
  $$C = \begin{bmatrix} 0 & 132 & 0 \end{bmatrix}$$
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  $$A = \begin{bmatrix} -2 & 1 \\ & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
  $$C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

- the grammians for the time-invariant cases ($t_0 = 0$)

	Controllability Grammian	Observability Grammian
continuous time	$\int_0^t e^{A\tau}BB^T\left(e^{A\tau}\right)^T d\tau$	$\int_0^t \left(e^{A\tau}\right)^T C^T C e^{A\tau} d\tau$
Lyapunov equation (when $t \to \infty$)	$AW_c + W_cA^T + BB^T = 0$	$A^TW_o + W_oA + C^TC = 0$
discrete time	$\sum_{k=0}^{k_1} A^kBB\left(A^T\right)^k$ for some k_1	$\sum_{k=0}^{k_1}(A^T)^kC^TCA^k$ for some k_1
Lyapunov equation (when $k_1 \to \infty$)	$AW_cA^T - W_c + BB^T = 0$	$A^TW_oA - W_o + C^TC = 0$