1 Controllability and Observability

- **Controllability**
  - Key idea in the definition: steer the system to any point in the state space within a finite time
  - Cayley Halmilton theorem: \( A^n \) is linearly dependent on \( \{I, A, A^2, \ldots, A^{n-1}\} \)
  - Proof: Let \( p(\lambda) \triangleq \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 \). We have \( p(\lambda) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_p)^{m_p} \).
  - Thus
    \[
    p(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_1A + c_0I = (A - \lambda_1I)^{m_1} \cdots (A - \lambda_pI)^{m_p}
    \]
  - Take any eigenvector or generalized eigenvector \( t_i \), we have \( p(A)t_i = 0 \). Therefore \( p(A)[t_1, t_2, \ldots, t_n] = 0 \). But \( T = [t_1, t_2, \ldots, t_n] \) is invertible, hence \( p(A) = 0 \), namely, \( A^n = -(c_{n-1}A^{n-1} + \cdots + c_1A + c_0I) \), which is a linear combination of \( \{I, A, A^2, \ldots, A^{n-1}\} \).
  - Essential equations:
    \[
    x(n) = A^n x(0) + \sum_{j=0}^{n-1} A^{n-1-j}Bu(j) = A^n x(0) + \begin{bmatrix} A^{n-1}B & A^{n-2}B & \cdots & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(n-1) \end{bmatrix}
    \]
  - and Cayley Halmilton theorem
  - Theorem: The following statements are equivalent:
    1. \( x(k+1) = Ax(k) + Bu(k) \) is controllable.
    2. The columns of the controllability matrix \( P = [B A B A^2 B \cdots A^{n-1} B] \) span \( \mathbb{R}^n \). (equivalence between 1 and 2: use \( x(n) = A^n x(0) + \begin{bmatrix} A^{n-1}B & A^{n-2}B & \cdots & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(n-1) \end{bmatrix} \))
    3. The controllability matrix \( P = [B A B A^2 B \cdots A^{n-1} B] \) is rank \( n \) (full row rank). (equivalence between 2 and 3: rank \( n \) \( \iff \) spans \( \mathbb{R}^n \))
    4. The controllability grammian \( W_c(k_1) = \sum_{k=0}^{k_1} A^k B B^T (A^T)^k \) is positive definite, for some finite integer \( k_1 \).
    (3 \( \iff \) 4: consider \( W_c(n) = PPT^T \))
    5. (Popov-Belevitch-Hautus (PBH) test) The matrix \( [A - \lambda I B] \) has full row rank at every eigenvalue, \( \lambda \), of \( A \). (intuition: if 5 does not hold, then \( \exists v \) such that \( v^T [A - \lambda I B] = 0 \Rightarrow v^T A = \lambda v^T, v^T B = 0 \), yielding \( v^T [B A B A^2 B \cdots A^{n-1} B] = 0 \), which violates the full row rank condition for \( P \))
  - Example 1: identify controllability of \((A, B)\):
    \[
    A = \begin{bmatrix} \lambda_1 & 1 \\ \lambda_2 & \lambda_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
    \]
  - The controllability property is invariant under any coordinate transformation:
    \[
    (A, B) \text{ is controllable. } \iff (\bar{A}, \bar{B}) = (TAT^{-1}, TB) \text{ is controllable.}
    \]
    where \( T \) is any nonsingular matrix. (Proof: \( \bar{P} = TP \) where \( T \) is invertible)
  - If \( A \) is asymptotically stable, \( k_1 \) can be set to \( \infty \) in the controllability grammian, and we may obtain the grammian by solving the Lyapunov equation
    \[
    AW_c A^T - W_c = -BB^T
    \]
    The solution is positive definite if and only if the system is controllable.
• Observability

  – Key idea in the definition: knowing the outputs and the inputs can determine the initial state (Question: why initial state here?)
  
  – Essential equations: assuming \( u(k) = 0 \), then \( y(0) = Cx(0), \ y(1) = Cx(1) = CAx(0), \ y(2) = Cx(2) = CA^2x(0) \)...

  – Theorem: the following statements are equivalent:

  1. \( x(k+1) = Ax(k), \ y = Cx(k) \) is observable.

  2. The observability grammian \( \mathcal{W}_o(k_1) = \sum_{k=0}^{k_1} (A^T)^k C T C A^k \) is positive definite, for some finite integer \( k_1 \).

  3. The observability matrix \( Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \) is rank \( n \) (full column rank).

  4. The matrix \( \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} \) has full column rank at every eigenvalue, \( \lambda \), of \( A \).

  – The observability property is invariant under any coordinate transformation:

  \((A, C)\) is observable. \( \iff (\bar{A}, \bar{C}) = (T A T^{-1}, C T^{-1}) \) is observable.

  where \( T \) is any nonsingular matrix.

  – If \( A \) is asymptotically stable, \( k_1 \) can be set to \( \infty \) in the observability grammian, and we may look for the grammian by solving the Lyapunov equation

  \[
  A^T \mathcal{W}_o A - \mathcal{W}_o = -C^T C
  \]

  The solution is positive definite if and only if the system is observable.

• Example 2: analyze the controllability, observability, stabilizability, and detectability of the following systems

  \[
  A = \begin{bmatrix} 1 & -2 & 3/10 \\ \end{bmatrix}, \quad B = \begin{bmatrix} 23 \\ 0 \\ 3/20 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 132 & 0 \end{bmatrix}
  \]

  \[
  A = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}
  \]

• the grammians for the time-invariant cases (\( t_0 = 0 \))

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<thead>
<tr>
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<th>Controllability Grammian</th>
<th>Observability Grammian</th>
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<tbody>
<tr>
<td>continuous time</td>
<td>( \int_0^t e^{A\tau} BB^T (e^{A\tau})^T d\tau )</td>
<td>( \int_0^t (e^{A\tau})^T C T C e^{A\tau} d\tau )</td>
</tr>
<tr>
<td>Lyapunov equation (when ( t \to \infty ))</td>
<td>( AW_c + W_c A^T + BB^T = 0 )</td>
<td>( A^T W_o + W_o A + C^T C = 0 )</td>
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<td>discrete time</td>
<td>( \sum_{k=0}^k A^k BB (A^T)^k ) for some ( k_1 )</td>
<td>( \sum_{k=0}^k (A^T)^k C T C A^k ) for some ( k_1 )</td>
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