Outline:

- Bilinear transform
- Notes about Lyapunov theory for LTI systems

1 Bilinear transform

- The formula:
$$z = \frac{1+s}{1-s}$$
- Interpretation and further extensions: the bilinear transform is a special case of "conformal mapping". It is derived as follows:
$$z = e^{sT} = \frac{e^{sT/2}}{e^{-sT/2}} \approx \frac{1+s\frac{T}{2}}{1-s\frac{T}{2}} \tag{1}$$
where the last approximation sign is from the Taylor expansion $e^x = 1 + x + \frac{1}{2}x^2 + \dots$.
- Some special mapped values: $s = 0$ (zero frequency in continuous-time domain) $\Rightarrow z = 1$, $s = j\infty$ (highest frequency in continuous-time domain) $\Rightarrow z = -1$. Notice that $z = e^{j0} = 1$ (zero frequency in discrete-time domain) and $z = e^{j\pi} = -1$ (π is the highest frequency (Nyquist frequency) in discrete-time domain).
- Some additional observation: if we "walk along" the imaginary axis then $1+sT/2$ and $1-sT/2$ are complex conjugates of each other, and thus z has a unity magnitude.
- The inverse mapping: from (1) we have
$$s = \frac{2}{T}\frac{z-1}{z+1}$$
- Extension: say we want to perform another mapping such that the left-hand side of the s plane is mapped to a unit circle of radius r. Then we need
$$s = re^{sT} \approx r\frac{1+s\frac{T}{2}}{1-s\frac{T}{2}}$$

2 Quick notes of LTI Lyapunov theory

	continuous time $\dot{x} = Ax$	discrete time $x(k+1) = Ax(k)$
key equations	$\dot{V} = x^T\left(A^TP + PA\right)x$	$V(k+1) - V(k) = x^T(k)\left(A^TPA - P\right)x(k)$
Lyapunov equation	$A^TP + PA = -Q$	$A^TPA - P = -Q$
solution (when A is stable)	$P = \int_0^\infty e^{A^Tt}Qe^{At}dt$	$P = \sum_{k=0}^\infty \left(A^T\right)^k QA^k$

Interpretations:

- if A is stable[1] then there exists a quadratic Lyapunov function $V(x) = x^TPx$ and Q such that P, Q satisfy the Lyapunov equation and are positive definite
- if A is stable and Q is positive definite, then P is also positive definite
- if we find one set of positive definite P and Q that satisfy the Lyapunov equation, then A is stable
- if we find one set of V such that $V(x)$ and $\dot{V}(x)$ are both positive definite, then A is unstable

Solvability conditions:

- the continuous-time Lyapunov equation $A^TP + PA = -Q$ has a unique solution iff $\lambda_i(A) + \bar{\lambda}_j(A) \neq 0$ for all $i, j = 1, \dots, n$
- the discrete-time Lyapunov equation $A^TPA - P = -Q$ has a unique solution iff $\lambda_i(A)\lambda_j(A) \neq 1$ for all $i, j = 1, \dots, n$
- if A is stable then the Lyapunov equation has a unique solution

[1] Stable here means Hurwitz (all eigenvalues in the left half plane) in continuous-time case and Schur (all eigenvalues inside the unit circle) in discrete-time case.