Outline:

- solution of LTI systems (and some intuitions)

1 Fundamental Theorem of Differential Equations

Knowing the existance of a solution is the first step towards getting the answer. The following theorem addresses the question of whether a dynamical system has a *unique* solution or not.

Theorem 1. *Consider* $\dot{x} = f(x,t)$, $x(t_0) = x_0$, *with:*

- $f(x,t)$ piecewise continuous in t
- $f(x,t)$ Lipschitz continuous in x

then there exists a *unique* function of time $\phi(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$ which is continuous almost everywhere and satisfies

- $\phi(t_0) = x_0$
- $\dot{\phi}(t) = f(\phi(t), t)$, $\forall t \in \mathbb{R}_+ \backslash D$, where D is the set of discontinuity points for f as a function of t.

Note:

- piecewise continuous: continuous except at finite points of discontinuity.
  - exercise: are these functions piecewise continuous?–$f(t) = |t|$ and

$$f(x,t) = \begin{cases} A_1 x, & t \le t_1 \\ A_2 x, & t > t_1 \end{cases}$$

- Lipschitz continuous: if $f(x,t)$ satisfies the following cone-shape constraint:

$$||f(x,t) - f(y,t)|| \le k(t)\, ||x - y||$$

  where $k(t)$ is piecewise continuous.
  - exercise: is $f(x) = Ax + B$ Lipschitz continuous?

2 Solution of LTI systems

Consider a state equation

$$\dot{x}(t) = Ax(t) + Bu(t); \quad x(t_0) = x_0$$

Note that $f(x,t) = Ax + Bu$ satisfies the conditions in Fundamental Theorem for Differential Equations. A unique solution thus exists. The solution is given by

$$x(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau)\, d\tau \tag{1}$$

For discrete-time systems, we have

$$x(k+1) = Ax(k) + Bu(k) \Rightarrow x(k) = A^{k-k_0} x(k_o) + \sum_{j=k_0}^{k-1} A^{k-1-j} B u(j) \tag{2}$$

Understanding (2):

- why $k-1$ but not k in the summation $\sum_{j=k_0}^{k-1}$?: observe in $x(k) = Ax(k-1) + Bu(k-1)$, that only the inputs at or before the $k-1$ time instance are required to obtain $x(k)$.

- another form of (2):

$$x(k) = A^{k-k_0}x(k_o) + \sum_{j=k_0}^{k-1} A^{k-1-j}Bu(j) = A^{k-k_0}x(k_o) + \begin{bmatrix} A^{k-k_0-1}B & A^{k-k_0-2}B & \cdots & B \end{bmatrix} \begin{bmatrix} u(k_0) \\ u(k_0+1) \\ \vdots \\ u(k-1) \end{bmatrix}$$

From here we see that the system is indeed linear, and $x(k)$ is an affine function of $u(i)$, $k_0 \le i \le k-1$.

Expressing $x(t)$ and $x(k)$ as (1) and (2) is usually not enough to reveal detailed properties of the states. Specifically for (1), we usually want to get a more detailed form of e^{At}. Here are some special cases:

A	e^{At}
$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$	$\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$
$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$	$\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$
$\begin{bmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix}$	$\begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2!}t^2e^{\lambda t} \\ & e^{\lambda t} & te^{\lambda t} \\ & & e^{\lambda t} \end{bmatrix}$
$\begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}$	$\begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} & \frac{t^3}{3!}e^{\lambda t} \\ & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ & & e^{\lambda t} & te^{\lambda t} \\ & & & e^{\lambda t} \end{bmatrix}$

Understanding the results: why does the term $te^{\lambda t}$ occur in e^{At} for $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$?

- We could do the usual Taylor expansion of e^{At}. But we could also gain intuition from the Laplace perspective. Notice that $\det(sI - A) = (s-\lambda)^2$ and $\frac{1}{(s-\lambda)^2}$ corresponds to $te^{\lambda t}$ in time domain. This gives a motivation of using Laplace or inverse Laplace transforms. Consider the free response of the system $\dot{x} = Ax$, $x(0) = x_0$. Performing the Laplace transform yields

$$\begin{aligned} sX(s) - x(0) &= AX(s) \\ \Rightarrow X(s) &= (sI-A)^{-1}x(0) \\ &= \frac{1}{(s-\lambda)^2}\begin{bmatrix} s-\lambda & 1 \\ & s-\lambda \end{bmatrix} x(0) \\ &= \begin{bmatrix} \frac{1}{s-\lambda} & \frac{1}{(s-\lambda)^2} \\ & \frac{1}{s-\lambda} \end{bmatrix} x(0) \end{aligned}$$

Applying inverse Laplace transform, we have

$$x(t) = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} x(0)$$

Comparing the above with

$$x(t) = e^{At}x(0)$$

we get

$$e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}.$$