Outline:

- when does a dynamical system have a unique solution?
- state-space and transfer-function representations of LTI systems
- state-space canonical forms
- matrix computation: determinants

1 State-space and transfer-function descriptions of LTI systems

- why are we learning them?
- relations between the two:

Table 1: Relations between state-space (ss) and transfer-function (tf) system representations

ss	tf
$\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t) + Du(t)$	$G(s) = C(sI - A)^{-1}B + D$
$x(k+1) = Ax(k) + Bu(k),\ y(k) = Cx(k) + Du(k)$	$G(z) = C(zI - A)^{-1}B + D$

2 Math review

2.1 Computing determinants

- 2×2 matrices:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

- 3×3 matrices:

$$\begin{aligned} \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} &= a \det \begin{bmatrix} e & f \\ h & k \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & k \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix} \\ &= aek + bfg + cdh - gec - bdk - ahf \end{aligned}$$

2.2 Computing the inverse of a matrix

There are several ways to compute a matrix inverse. One approach for low-order matrices is the method of using adjugate matrix (aka adjoint matrix):

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

We explain the computation by two examples:

- 2×2 example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} (-1)^{1+1} d & (-1)^{1+2} b \\ (-1)^{2+1} c & (-1)^{2+2} a \end{bmatrix}$$

  where b in $(-1)^{1+2} b$ is obtained by:

  - noticing b is at row 1 column 2 of A;
  - looking at the element at row 2 column 1 of A;
  - constructing a submatrix of A by removing row 2 and column 1 from it, i.e., $[b]$ in this 2×2 example;
  - computing the determinant of this submatrix.

- adding $(-1)^{1+2}$ as a scalar

- 3×3 example:

$$A^{-1}=\begin{bmatrix} a & b & c\\ d & e & f\\ g & h & k\end{bmatrix}^{-1}=\frac{1}{\det A}\begin{bmatrix} \begin{vmatrix} e & f\\ h & k\end{vmatrix} & -\begin{vmatrix} b & c\\ h & k\end{vmatrix} & \begin{vmatrix} b & c\\ e & f\end{vmatrix}\\ -\begin{vmatrix} d & f\\ g & k\end{vmatrix} & \begin{vmatrix} a & c\\ g & k\end{vmatrix} & -\begin{vmatrix} a & c\\ d & f\end{vmatrix}\\ \begin{vmatrix} d & e\\ g & h\end{vmatrix} & -\begin{vmatrix} a & b\\ g & h\end{vmatrix} & \begin{vmatrix} a & b\\ d & e\end{vmatrix}\end{bmatrix}$$

where $|\cdot|$ denotes the determinant of a matrix. Similar as before, the row 1 column 2 element $-\begin{vmatrix} b & c\\ h & k\end{vmatrix}$ is obtained via

$$(-1)^{2+1}\det\left(A \text{ with } [d,e,f], \begin{bmatrix} a\\ d\\ g\end{bmatrix} \text{ removed}\right)$$

3 Canonical forms

$$G=\frac{b_2s^2+b_1s+b_0}{s^3+a_2s^2+a_1s+a_0}$$

- Controllable canonical form

$$A=\begin{bmatrix} & 1 & \\ & & 1\\ -a_0 & -a_1 & -a_2\end{bmatrix},\ B=\begin{bmatrix} 0\\ 0\\ 1\end{bmatrix},\ C=\begin{bmatrix} b_0 & b_1 & b_2\end{bmatrix},\ D=0$$

  - understanding the formula: the transfer function is given by $G(s)=C(sI-A)^{-1}B+D$, where the poles of the system come from $\det(sI-A)=0$. Suppose we don't know the order of a_0, a_1 and a_2 in the last row of A, and use $\star$ as a temporary representation in A. Looking at

$$\det(sI-A)=\det\begin{bmatrix} s & -1 & \\ & s & -1\\ \star & \star & s+\star\end{bmatrix}$$

    we see that the only way for s^2 to appear is from the term $s^2(s+\star)$ in the determinant computation. Hence the location of $-a_2$ has to be at the bottom right corner of A.
  - exercise: write down the controllable canonical form for the following systems
    * $G(s)=\frac{s^2+1}{s^3+2s+10}$
    * $G(s)=\frac{b_0s^2+b_1s+b_2}{s^3+a_0s^2+a_1s+a_2}$

- Observable canonical form

$$A=\begin{bmatrix} -a_2 & 1 & \\ -a_1 & & 1\\ -a_0 & & \end{bmatrix},\ B=\begin{bmatrix} b_2\\ b_1\\ b_0\end{bmatrix},\ C=\begin{bmatrix} 1 & 0 & 0\end{bmatrix},\ D=0$$

- Diagonal form: for systems described by

$$G(s)=\frac{k_1}{s-p_1}+\frac{k_2}{s-p_2}+\frac{k_3}{s-p_3}$$

  one state-space form is

$$A=\begin{bmatrix} p_1 & & \\ & p_2 & \\ & & p_3\end{bmatrix},\ B=\begin{bmatrix} 1\\ 1\\ 1\end{bmatrix},\ C=\begin{bmatrix} k_1 & k_2 & k_3\end{bmatrix},\ D=0$$

- Jordan form: if

$$G(s)=\frac{k_1}{s-p_1}+\frac{k_2}{(s-p_m)^2}+\frac{k_3}{s-p_m}$$

then one state-space form of the system is

$$A=\begin{bmatrix} p_1 & & \\ & p_m & 1 \\ & & p_m \end{bmatrix},\ B=\begin{bmatrix} 1\\0\\1 \end{bmatrix},\ C=\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix},\ D=0$$

  - understanding the formula: why $B=[1,0,1]^T$? What if

$$G(s)=\frac{k_1}{(s-p_1)^2}+\frac{k_1}{s-p_1}+\frac{k_2}{(s-p_m)^2}+\frac{k_3}{s-p_m}$$

- Modified Jordan form: this is for systems with complex poles:

$$G(s)=\frac{k_1}{s-p_1}+\frac{\alpha s+\beta}{(s-\sigma)^2+\omega^2}$$

we have

$$A=\begin{bmatrix} p_1 & & \\ & \sigma & \omega \\ & -\omega & \sigma \end{bmatrix},\ B=\begin{bmatrix} 1\\0\\1 \end{bmatrix},\ C=\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix},\ D=0$$