# ME 547: Linear Systems
# Review of Matrices and Linear Algebra

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ver. April 23, 2022

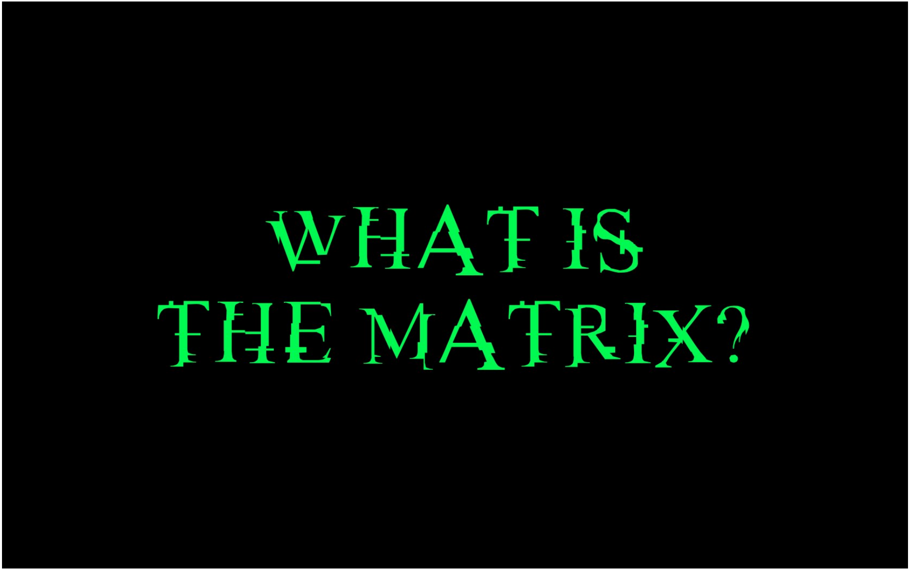



Figure: source: https://steemit.com/life/@suraj0651/what-is-the-matrix


Figure: source: the Matrix film

# Basic concepts of matrices and vectors

A linear equation set

$$\begin{aligned} 3x_1 + 4x_2 + 10x_3 &= 6 \\ x_1 + 4x_2 - 10x_3 &= 5 \\ 4x_2 + 10x_3 &= -1 \end{aligned} \tag{1}$$

can be simply written as

$$\begin{bmatrix} 3 & 4 & 10 \\ 1 & 4 & -10 \\ 0 & 4 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ -1 \end{bmatrix} \tag{2}$$

Imagine using (1) instead of (2) when we have thousands of unknowns...

# Basic concepts of matrices and vectors

Formally, we write an $m \times n$ matrix $A$ as

$$A = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \dots & \dots & a_{2n} \\ \vdots & \dots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- $m \times n$ (reads $m$ by $n$) is the dimension/size of the matrix.
- Each element $a_{jk}$ is an entry of the matrix.
- If $m = n$, $A$ is a square matrix.
- Concepts of diagonal entries, upper triangular matrices, lower triangular matrices, diagonal matrices, and the identity matrix

# Basic concepts of matrices and vectors

## Vectors: special matrices whose row or column number is one.

- A $1 \times n$ row vector: $a = [a_1, a_2, \ldots, a_n]$
- An $m \times 1$ column vector:

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

# Matrix addition and multiplication

- The **sum** of two matrices $A$ *and* $B$ *of the same size:*

$$A + B = [a_{jk} + b_{jk}]$$

- The **product** between an $m \times n$ matrix $A$ and a scalar $c$:

$$cA = [ca_{jk}]$$

- The **matrix product** $C = AB$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \boxed{a_{21}} & \boxed{a_{22}} & \boxed{a_{23}} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}_{4\times 3} \begin{bmatrix} \boxed{b_{11}} & b_{12} \\ \boxed{b_{21}} & b_{22} \\ \boxed{b_{31}} & b_{32} \end{bmatrix}_{3\times 2} = \begin{bmatrix} c_{11} & c_{12} \\ \boxed{c_{21}} & c_{22} \\ c_{31} & c_{32} \\ c_{41} & c_{42} \end{bmatrix}_{4\times 2}$$

where $c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} =$
"second row of $A$" $\times$ "first column of $B$"

# Matrix multiplication is not commutative

$$\underset{[m \times n]}{A} \quad \underset{[n \times p]}{B} \quad = \quad \underset{[m \times p]}{C}$$

$AB$ in general does not equal to $BA$, e.g.:

$$ABC = (AB)\,C = A\,(BC) \neq BCA$$

# Matrices as combination of vectors

$$Ax = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 \end{bmatrix}$$ is the weighted sum of the columns of $A$:

$$Ax = \left[\begin{array}{c|c|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix}$$

So for $\begin{bmatrix} 3 & 4 & 10 \\ 1 & 4 & -10 \\ 0 & 4 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ -1 \end{bmatrix}$ to have a solution, $\begin{bmatrix} 6 \\ 5 \\ -1 \end{bmatrix}$ must be a linear combination of the columns of $\begin{bmatrix} 3 & 4 & 10 \\ 1 & 4 & -10 \\ 0 & 4 & 10 \end{bmatrix}$.

# Matrix transposition

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \dots & \dots & a_{2n} \\ \vdots & \dots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n} \Rightarrow A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & \dots & \dots & a_{m2} \\ \vdots & \dots & \dots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}_{n\times m}$$

Properties:

- $\left(A^T\right)^T = A$
- $(A+B)^T = A^T + B^T$
- $(cA)^T = cA^T$
- $(AB)^T = B^T A^T$

$A$ is called *symmetric* if $A = A^T$, *skew-symmetric* if $A = -A^T$.

## Example (Matrix and quadratic forms)

We can use matrices to express general quadratic functions of vectors. For instance

$$f(x) = x^T A x + 2bx + c$$

is equivalent to

$$f(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

# Vector space, linear independence

Given nonzero vectors $a_1$, $a_2$, ..., $a_m$ with the same size,

- $k_1 a_1 + k_2 a_2 + \cdots + k_m a_m$ is a linear combination of the vectors.
- $a_1$ is *linearly dependent* on $a_2$, $a_3$, ..., $a_m$ if $\exists k_2, \ldots, k_m \in \mathbb{R}$ such that $a_1 = k_2 a_2 + k_3 a_3 + \cdots + k_m a_m$.
  - The set $\{a_1, a_2, \ldots, a_m\}$ is then a linearly dependent set.
- Why is linear independence important? - getting the smallest "truly essential" set in a possibly large set of vectors.
- the concept of a vector space

# Span and basis

- The *span* of the $n$ vectors: all the possible linear combinations of these vectors.
- A *basis* of V is a set B of vectors in V, such that any $v \in \mathrm{V}$ can be uniquely expressed as a finite linear combination of vectors in B.

**Example**

In $\mathbb{R}^2$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is a linearly independent set and forms a basis.

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ v_3 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

is not a linearly independent set.

# Matrix defines linear transformations between vector spaces

## Example (A person with two ID cards)

| General Info: | Company 1 | Company 2 |
|---|---|---|
| name, | | |
| height, | $x_1 = \text{X}$ | $y_1 = \text{X}$ |
| birthday | $x_2 = 6.0$ (ft) | $y_2 = 6019901201$ |
| | $x_3 = 19901201$ | $y_3 = 182.88$ (cm) |

The two ID cards are related by

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 10^9 & 1 \\ 0 & 30.48 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{3}$$

To shift data base, just perform a matrix multiplication.

# Matrix defines linear transformations between vector spaces

## Example (Rotation matrix)

A vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is rotated by an angle of $\theta$ in the 2-dimensional vector space. Let $x_1 = r\cos\alpha$ and $x_2 = r\sin\alpha$. The rotated vector has the following representation

$$y_1 = r\cos(\theta+\alpha) = r\cos\theta\cos\alpha - r\sin\theta\sin\alpha$$
$$y_2 = r\sin(\theta+\alpha) = r\sin\theta\cos\alpha + r\cos\theta\sin\alpha$$

namely,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

# Matrix properties

- *Rank*: the maximum number of linearly independent row or column vectors

## Example (Dyad)

$A = uv^T$ is called a dyad, where $u$ and $v$ are vectors of proper dimensions. It is a rank 1 matrix, as can be seen that $A = uv^T$ is formed by linear combinations of the vector $u$, where the weights of the combinations are coefficients of $v$.

- *Range space* $\mathcal{R}(A)$: the span of all the column vectors
- *Null space* $\mathcal{N}(A)$ for $A \in \mathbb{R}^{n \times n}$: $\{x \in \mathbb{R}^n : \; Ax = 0\}$
- *Nullity*: the dimension of the null space

# Determinants

- $2 \times 2$ matrices:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

- $3 \times 3$ matrices:

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = a \det \begin{bmatrix} e & f \\ h & k \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & k \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

$$= aek + bfg + cdh - gec - bdk - ahf$$

# Determinants

**Theorem**

*The determinant of $A$ is nonzero if and only if $A$ is full rank.*

**Definition**

A linear transformation is called *singular* if the determinant of the corresponding transformation matrix is zero.

- If $A$ and $B$ are square matrices, then

$$\det(AB) = \det(BA) = \det A \det B$$
$$\det\left(A^T\right) = \det(A)$$

# Matrix and linear equations

Consider again, using now concepts in range and null spaces of matrices, the linear equations

$$Ax = y \tag{4}$$

- *Existence* of solutions requires that $y \in \mathcal{R}(A)$
- *Solutions*, if exist, are constructed from

$$x = x_o + z : \ Az = 0 \tag{5}$$

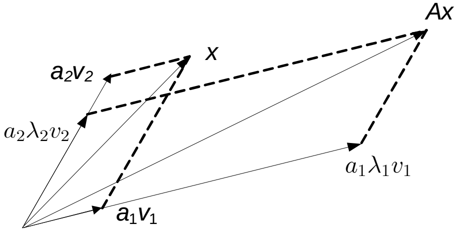
  where $x_0$ is any (fixed) solution of (4) and $z$ runs through all vectors in the null space of $A$.
- *Uniqueness* of a solution: if $\mathcal{N}(A) = \emptyset$, the solution is unique.
- If $A$ is square and singular, $Az = 0$ has infinite many solutions.

# Matrix, mappings, and eigenvectors

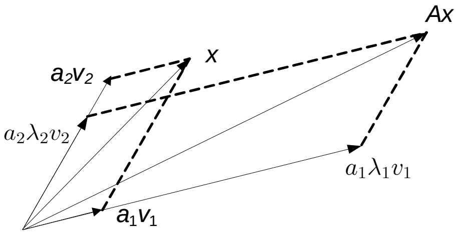
- $A$ defines a linear operator: $x \mapsto Ax$, e.g., 1D projection:

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow A_0 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

- eigenvectors: "special directions/vectors" for $A$ such that $Av_i = \lambda_i v_i$: $Av_i$ is aligned with the original vector $v_i$, scaled by the eigenvalue $\lambda_i$.
- turns out that if $\lambda_1 \neq \lambda_2$, then any $x \in \mathbb{R}^2$ can be written as $x = a_1 v_1 + a_2 v_2$
- thus $Ax = a_1 A v_1 + a_2 A v_2 = a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2$:



# Matrix, mappings, and eigenvectors




- Knowing $\lambda_i$ and $v_i$ thus can directly tell us how $Ax$ looks like. More important, we have decomposed $Ax$ into small modules that are handy for analyzing the system properties.
- *Eigenvalues* are also called *characteristic values* of a matrix. The set of all the eigenvalues of $A$ is called the *spectrum* of $A$. The largest of the *absolute* values of the eigenvalues of $A$ is called the *spectral radius* of $A$.

# Computation of eigenvalues and eigenvectors

For $A \in \mathbb{R}^{n\times n}$, an eigenvalue $\lambda$ of $A$ is one for which

$$Ax = \lambda x \tag{6}$$

has a nonzero solution $x \neq 0$.

- (6) is equivalent to $(A - \lambda I)\, x = 0$
- As $x \neq 0$, the matrix $A - \lambda I$ must be singular. So

$$\det\,(A - \lambda I) = 0 \tag{7}$$

- If an $n \times n$ matrix $A$ has $n$ eigenvalues $\lambda_1, \ldots, \lambda_n$, it must be that $\det\,(A - \lambda I) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$.

**Example**

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

# Computation of eigenvalue and eigenvectors

## Theorem (Eigenvalue and determinant)

*Let $A \in \mathbb{R}^{n\times n}$. Then*

$$\det A = \prod_{i=1}^{n} \lambda_i$$

## Proof.

Letting $\lambda = 0$ in the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\ldots$$

gives

$$\det(A) = p(0) = \prod_{i=1}^{n} \lambda_i$$

□

# Matrix inversion

The inverse $A^{-1}$ of a square nonsingular matrix $A$ satisfies

$$AA^{-1} = A^{-1}A = I$$

If $A^{-1}$ exists,

- $\det A \neq 0$
- $A$ is nonsingular
- $Ax = b$ has one unique solution

# Matrix inversion

- the method of adjugate / adjoint matrix:

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)^T$$

- $2 \times 2$ example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} (-1)^{1+1} d & (-1)^{1+2} b \\ (-1)^{2+1} c & (-1)^{2+2} a \end{bmatrix}$$

# Matrix inversion

- $3 \times 3$ example:

$$A^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}^{-1} = \frac{1}{\det A} \begin{bmatrix} \begin{vmatrix} e & f \\ h & k \end{vmatrix} & -\begin{vmatrix} b & c \\ h & k \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ -\begin{vmatrix} d & f \\ g & k \end{vmatrix} & \begin{vmatrix} a & c \\ g & k \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

  - $|\cdot|$ denotes the determinant.
  - the row 1 column 2 element $-\begin{vmatrix} b & c \\ h & k \end{vmatrix}$ is obtained via

$$(-1)^{2+1} \det \left( A \text{ with } [d, e, f], \begin{bmatrix} a \\ d \\ g \end{bmatrix} \text{ removed} \right)$$

# Matrix inversion

## Example

Find the inverse matrices of

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}, \; B = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \; C = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## Theorem

*Inverse of products of matrices can be obtained from inverses of each factor:*

$$(AB)^{-1} = B^{-1}A^{-1}$$

*and more generally*

$$(AB \dots YZ)^{-1} = Z^{-1}Y^{-1} \dots B^{-1}A^{-1} \tag{8}$$

# Eigenbases and diagonalization

## Theorem (Diagonalization of a Matrix)

*Let an $n \times n$ matrix $A$ have a basis of eigenvectors $\{x_1, x_2, \ldots, x_n\}$, associated to its n distinct eigenvectors $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then*

$$A = XDX^{-1} = [x_1, x_2, \ldots, x_n] \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{bmatrix} [x_1, x_2, \ldots, x_n]^{-1} \tag{9}$$

*Also,* $A^m = XD^mX^{-1}, \quad (m = 2, 3, \ldots)$.

## Example

$A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$. Compute $A^{3000}$.

## Theorem (Spectral Mapping Theorem)

*Take any $A \in \mathbb{C}^{n \times n}$ and a polynomial function $f(\cdot)$ (more generally, analytic functions). Then*

$$\text{eig}\,(f\,(A)) = f\,(\text{eig}\,(A))$$

## Example (Compute the eigenvalues)

$$A = \begin{bmatrix} 99.8 & 2000 \\ -2000 & 99.8 \end{bmatrix}$$

Solution:

$$A = 99.8I + 2000 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \lambda(A) = 99.8 + 2000\lambda \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$
$$= 99.8 \pm 2000i$$