Controllability and Observability

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The concept of controllability and observability

Controllability:
- inputs do not act directly on the states but via state dynamics:
  \[
  \dot{x}(t) = Ax(t) + Bu(t) \quad \text{or} \quad x(k+1) = Ax(k) + Bu(k) \quad (1)
  \]
- can the inputs drive the system to any value in the state space in finite time?

Observability:
- states are not all measured directly but instead impact the output via the output equation:
  \[
  y = Cx + Du
  \]
- can we infer fully the initial state from the outputs and the inputs? (can then reveal the full state trajectory through (1))
The concept of controllability and observability

\[ \dot{x}_1 = x_2, \quad \dot{x}_3 = x_4 \]

- Assume \( x(0) = 0 \)
- Because of symmetry, we always have
  \[ x_1(t) = x_3(t), \quad x_2(t) = x_4(t), \quad \forall t \geq 0 \]
- State cannot be arbitrarily steered \( \Rightarrow \) uncontrollable
Controllability definition in discrete time

Definition
A discrete-time linear system $x(k + 1) = A(k)x(k) + B(k)u(k)$ is called controllable at $k = 0$ if $\exists$ a finite time $k_1$ such that $\forall$ initial state $x(0)$ and target state $x_1$, there exists a control sequence $\{u(k); k = 0, 1, \ldots, k_1\}$ that will transfer the system from $x(0)$ at $k = 0$ to $x_1$ at $k = k_1$. 
Controllability of LTI systems

\[ x(k + 1) = Ax(k) + Bu(k) \Rightarrow x(n) = A^nx(0) + \sum_{k=0}^{n-1} A^{n-1-k}Bu(k) \]

\[ \Rightarrow x(n) - A^nx(0) = \begin{bmatrix} B, AB, A^2B, \ldots, A^{n-1}B \end{bmatrix} P_d \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix} \]

- given any \( x(n) \) and \( x(0) \) in \( \mathbb{R}^n \), \( u_n \) can be solved if the columns of \( P_d \) span \( \mathbb{R}^n \)
- equivalently, system is controllable if \( P_d \) has rank \( n \) (full row rank)
Controllability of LTI systems Cont’d

\[ x(k + 1) = Ax(k) + Bu(k) \Rightarrow \]

\[ x(n) - A^n x(0) = \begin{bmatrix} B, AB, A^2B, \ldots, A^{n-1}B \end{bmatrix}_{P_d} \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix} \]

▶ also, no need to go beyond \( n \): adding \( A^nB, A^{n+1}B, \ldots \) does not increase the rank of \( P_d \) (Cayley Halmilton Theorem):

\[ x(k_1) - A^{k_1} x(0) = \begin{bmatrix} B & AB & \ldots & A^{n-1}B & \ldots & A^{k_1-1}B \end{bmatrix}_{\text{rank} = \text{rank}(P_d)} \begin{bmatrix} u(k_1 - 1) \\ u(k_1 - 2) \\ \vdots \\ u(0) \end{bmatrix} \]
Theorem (Cayley Hamilton Theorem)

Let \( A \in \mathbb{R}^{n \times n} \). \( A^n \) is linearly dependent with \( \{ I, A, A^2, \ldots A^{n-1} \} \)

Proof.

Consider characteristic polynomial

\[
p(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 = \det(\lambda I - A)
\]

\[
= (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_p)^{m_p}
\]

\[
\Rightarrow p(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_1A + c_0I
\]

\[
= (A - \lambda_1 I)^{m_1} \cdots (A - \lambda_p I)^{m_p}, \quad m_1 + m_2 + \cdots + m_p = n
\]

\( \forall \) eigenvector or generalized eigenvector \( t_i \), say, associated to \( \lambda_i \):

\[
p(A) t_i = (A - \lambda_1 I)^{m_1} \cdots (A - \lambda_p I)^{m_p} t_i =
\]

\[
(A - \lambda_1 I)^{m_1} \cdots (A - \lambda_p I)^{m_p-1} (\lambda_i t_i - \lambda_p t_i) = (\lambda_i - \lambda_1)^{m_1} \cdots (\lambda_i - \lambda_p)^{m_p} t_i = 0
\]

\[\triangleright\text{ therefore } p(A) [t_1, t_2, \ldots, t_n] = 0\]

\[\triangleright\text{ but } T = [t_1, t_2, \ldots, t_n] \text{ is invertible. Hence } p(A) = 0\]

\[
\Rightarrow A^n = -c_0 I - c_1A - \cdots - c_{n-1}A^{n-1}
\]
Arthur Cayley: 1821-1895, British mathematician
- algebraic theory of curves and surfaces, group theory, linear algebra, graph theory, invariant theory, ...
- extraordinarily prolific career: ~1,000 math papers

William Hamilton: 1805-1865, Irish mathematician
- optics and classical mechanics in physics, dynamics, algebra, quaternions, ...
- quaternions: extending complex numbers to higher spatial dimensions: 4D case

\[ i^2 = j^2 = k^2 = ijk = -1 \]

now used in computer graphics, control theory, orbital mechanics, e.g., spacecraft attitude-control systems
Theorem (Controllability Theorem)

The $n$-dimensional $r$-input LTI system with
\[ x(k+1) = Ax(k) + Bu(k), \quad A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times r} \]
is controllable if and only if either one of the following is satisfied:

1. the $n \times nr$ controllability matrix

\[
P_d = [B, AB, A^2B, \ldots, A^{n-1}B]\
\]

has rank $n$
2. the controllability gramian

\[
W_{cd} = \sum_{k=0}^{k_1} A^k BB^T (A^T)^k
\]

is nonsingular for some finite $k_1$
Proof: from controllability matrix to gramian

Recall

\[
x(n) - A^nx(0) = \begin{bmatrix} B, AB, A^2B, \ldots, A^{n-1}B \end{bmatrix} u_n
\]

(2)

\[\text{P}_d\text{ is full row rank} \Rightarrow \text{P}_d\text{P}_d^T = \sum_{k=0}^{n} A^k B B^T (A^T)^k \text{ is nonsingular}
\]

\[\text{W}_{cd} \text{ at } k_1 = n\]

\[\text{a (least-square) solution to (2) is}
\]

\[u_n = \left(\text{P}_d\text{P}_d^T\right)^{-1} \left[ x(n) - A^nx(0) \right]\]
Example

\[ A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

\[ P_d = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \lambda_2 + \lambda_2 \\ 1 & \lambda_2 & \lambda_2^2 \end{bmatrix} \Rightarrow \text{rank}(P_d) = 2 < 3 \Rightarrow \text{uncontrollable} \]

Intuition: \( \dot{x}_1 = \lambda_1 x_1 \) is not impacted by the control input at all.
Example

\[ \dot{x}_1 = x_2, \quad \dot{x}_3 = x_4 \]

\[
\begin{bmatrix}
  x_1(k + 1) \\
  x_2(k + 1) \\
  x_3(k + 1) \\
  x_4(k + 1)
\end{bmatrix} =
\begin{bmatrix}
  0.4 & 0.4 & 0 & 0 \\
  -0.9 & -0.07 & 0 & 0 \\
  0 & 0 & 0.4 & 0.4 \\
  0 & 0 & -0.9 & -0.07
\end{bmatrix}
\begin{bmatrix}
  x_1(k) \\
  x_2(k) \\
  x_3(k) \\
  x_4(k)
\end{bmatrix} +
\begin{bmatrix}
  0.3 \\
  0.4 \\
  0.3 \\
  0.4
\end{bmatrix} u(k)
\]

\[ \text{rank}(P_d) = \text{rank}
\begin{bmatrix}
  0.3 & 0.28 & -0.0072 & -0.0953 \\
  0.4 & -0.298 & -0.2311 & 0.0227 \\
  0.3 & 0.28 & -0.0072 & -0.0953 \\
  0.4 & -0.298 & -0.2311 & 0.0227
\end{bmatrix} = 2 \implies \text{uncontrollable} \]
Example

\[ \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} 0.4 & 0.4 & 0 & 0 \\ -0.9 & -0.07 & 0 & 0 \\ 0 & 0 & 0.4 & 0.4 \\ 0 & 0 & -0.9 & -0.07 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} 0.3 \\ 0.4 \\ 0.3 \\ 0.4 \end{bmatrix} u(k) \]

```
import numpy as np
import control as ct
A = np.array([[0.4, 0.4, 0, 0], [-0.9, -0.07, 0, 0], [0, 0, 0.4, 0.4], [0, 0, -0.9, -0.07]])
B = np.array([[0.3], [0.4], [0.3], [0.4]])
P = ct.ctrb(A,B)
print(np.linalg.matrix_rank(P))
```
Example

\[
\frac{d}{dt} \begin{bmatrix} \nu_m \\ F_{k_1} \\ F_{k_2} \end{bmatrix} = \begin{bmatrix} -\frac{b}{m} & -\frac{1}{m} & -\frac{1}{m} \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \nu_m \\ F_{k_1} \\ F_{k_2} \end{bmatrix} + \begin{bmatrix} \frac{1}{m} \\ 0 \\ 0 \end{bmatrix} F
\]

\[
P = \begin{bmatrix} \frac{1}{m} & -\frac{b}{m^2} & \frac{b^2}{m^3} - \frac{k_1}{m^2} - \frac{k_2}{m^2} \\ 0 & \frac{k_1}{m} & -\frac{b k_1}{m^2} \\ 0 & \frac{k_2}{m} & -\frac{b k_2}{m^2} \end{bmatrix} \Rightarrow \text{rank}(P) = 2
\]

\Rightarrow \text{uncontrollable}
Analysis: controllability and controllable canonical form

\[ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

▶ controllability matrix

\[ P_d = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & -a_1 + a_2^2 \end{bmatrix} \]

has full row rank

▶ system in controllable canonical form is controllable
Recap

General LTI state-space models:

\[
\dot{x}(t) = Ax(t) + Bu(t) \quad \text{or} \quad x(k + 1) = Ax(k) + Bu(k)
\]

\[
y =Cx + Du
\]

<table>
<thead>
<tr>
<th></th>
<th>continuous time</th>
<th>discrete time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lyapunov Eq.</td>
<td>(A^T P + PA = -Q)</td>
<td>(A^T PA - P = -Q)</td>
</tr>
<tr>
<td>unique sol.</td>
<td>(\lambda_i(A) + \lambda_j(A) \neq 0) \quad \forall i, j</td>
<td>(</td>
</tr>
<tr>
<td>cond.</td>
<td></td>
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<tr>
<td>solution</td>
<td>(P = \int_0^\infty e^{A^T t} Q e^{At} dt) (if (A) is Hurwitz stable)</td>
<td>(P = \sum_{k=0}^{\infty} (A^T)^k QA^k) (if (A) is Schur stable)</td>
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</tbody>
</table>
Analysis: controllability gramian and Lyapunov Eq.

\[ W_{cd} = \sum_{k=0}^{k_1} A^k BB^T (A^T)^k \]

- If \( A \) is Schur, \( k_1 \) can be set to \( \infty \)

\[ W_{cd} = \sum_{k=0}^{\infty} A^k B B^T \underbrace{(A^T)^k}_Q \]

which can be solved via the Lyapunov Eq.

\[ AW_{cd} A^T - W_{cd} = -BB^T \]
Analysis: controllability and similarity transformation

\[
\begin{align*}
    x(k+1) &= Ax(k) + Bu(k) \\
    y(k) &=Cx(k) + Du(k)
\end{align*}
\]

\[
x = T x^*
\]

\[
\begin{align*}
    x^*(k+1) &= T^{-1}ATx^*(k) + T^{-1}Bu(k) \\
    y(k) &= CTx^*(k) + Du(k)
\end{align*}
\]

- controllability matrix

\[
P_d^* = \begin{bmatrix} \tilde{B}, \tilde{A}\tilde{B}, \ldots, \tilde{A}^{n-1}\tilde{B} \end{bmatrix}
\]

\[
= \begin{bmatrix} T^{-1}B, T^{-1}AB, \ldots, T^{-1}A^{n-1}B \end{bmatrix} = T^{-1}P_d
\]

hence \((A, B)\) controllable \(\iff\) \((T^{-1}AT, T^{-1}B)\) controllable

- The controllability property is invariant under any coordinate transformation.
* Popov-Belevitch-Hautus (PBH) controllability test

- the full rank condition of the controllability matrix

\[
P_d = [B, AB, A^2B, \ldots, A^{n-1}B]
\]

is equivalent to: the matrix \([A - \lambda I, B]\) having full row rank at every eigenvalue, \(\lambda\), of \(A\)

- to see this: if \([A - \lambda I, B]\) is not full row rank then there exists nonzero vector (a left eigenvector) such that

\[
v^T [A - \lambda I B] = 0
\]

\[
\iff v^T A = \lambda v^T
\]

\[
v^T B = 0
\]

i.e., the input vector \(B\) is orthogonal to a left eigenvector of \(A\).
Example

\[
A = \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 1 \\
0 & 0 & \lambda_2
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

\[
[A - \lambda_1 I, \quad B] =
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \lambda_2 - \lambda_1 & 1 & 0 \\
0 & 0 & \lambda_2 - \lambda_1 & 1
\end{bmatrix}
\]

not full row rank \Rightarrow uncontrollable

Intuition: \( \dot{x}_1 = \lambda_1 x_1 \) is not impacted by the control input at all.
1. Concepts

2. DT controllability
   Controllability and controllable canonical form
   Controllability and Lyapunov Eq.

3. DT observability
   Observability and observable canonical form

4. CT cases

5. The degrees of controllability and observability

6. Transforming controllable systems into controllable canonical forms

7. Transforming observable systems into observable canonical forms
Observability of LTI systems

Definition

A discrete-time linear system

\[ x(k+1) = A(k)x(k) + B(k)u(k) \]
\[ y(k) = C(k)x(k) + D(k)u(k) \]

is called observable at \( k = 0 \) if \( \exists \) a finite time \( k_1 \) such that \( \forall \) initial state \( x(0) \), the knowledge of input \( \{ u(k); k = 0, 1, \ldots, k_1 \} \) and \( \{ y(k); k = 0, 1, \ldots, k_1 \} \) suffice to determine the state \( x(0) \). Otherwise, the system is said to be unobservable at time \( k = 0 \).
Observability of LTI systems

Let us start with the unforced system

\[
x(k + 1) = Ax(k), \ A \in \mathbb{R}^n
\]

\[
y(k) = Cx(k), \ y \in \mathbb{R}^m
\]

\[x(k) = A^k x(0) \text{ and } y(k) = Cx(k)\] give

\[
\begin{bmatrix}
y(0) \\
y(1) \\
\vdots \\
y(n-1)
\end{bmatrix}
= \begin{bmatrix} C \\
CA \\
CA^2 \\
\vdots \\
CA^{n-1}
\end{bmatrix}
\begin{bmatrix} y_n \\
Q_d: nm \times n
\end{bmatrix}
\]

- If the linear matrix equation has a nonzero solution \( x(0) \), the system is observable
Observability of LTI systems

generalizing to

\[ x(k + 1) = Ax(k) + Bu(k), \quad y(k) = Cx(k) + Du(k): \]

\[ x(k) = A^k x(0) + \sum_{j=0}^{k-1} A^{k-1-j} Bu(j) \]

\[ y(k) = CA^k x(0) + C \sum_{j=0}^{k-1} A^{k-1-j} Bu(j) + Du(k) \]

\[
\begin{bmatrix}
    y(0) - y_{\text{forced}}(0) \\
    y(1) - y_{\text{forced}}(1) \\
    \vdots \\
    y(n-1) - y_{\text{forced}}(n-1)
\end{bmatrix}
= \begin{bmatrix}
    C \\
    CA \\
    \vdots \\
    CA^{n-1}
\end{bmatrix}
\begin{bmatrix}
x(0) \\
\vdots \\
\bar{y}_n: \text{available from measurements and inputs}
\end{bmatrix}
\]

\[ \bar{y}_n: \text{available from measurements and inputs} \]

\[ Q_d: nm \times n \]
Observability of LTI systems

\[
\begin{bmatrix}
    y(0) - y_{\text{forced}}(0) \\
    y(1) - y_{\text{forced}}(1) \\
    \vdots \\
    y(n-1) - y_{\text{forced}}(n-1)
\end{bmatrix}
\begin{bmatrix}
    C \\
    CA \\
    \vdots \\
    CA^{n-1}
\end{bmatrix}
\begin{bmatrix}
    x(0)
\end{bmatrix}
= \bar{y}_n
\]

- \( x(0) \) can be solved if \( Q_d \) has rank \( n \) (full column rank):
  - if \( Q_d \) is square, \( x(0) = Q_d^{-1}\bar{y}_n \)
  - if \( Q_d \) is a tall matrix, pick \( n \) linearly independent rows from \( Q_d \)
Observability of LTI systems Cont’d

\[
\begin{bmatrix}
  y(0) - y_{\text{forced}}(0) \\
  y(1) - y_{\text{forced}}(1) \\
  \vdots \\
  y(n-1) - y_{\text{forced}}(n-1)
\end{bmatrix}
\begin{bmatrix}
  \bar{y}_n
\end{bmatrix}
= \begin{bmatrix}
  C \\
  CA \\
  \vdots \\
  CA^{n-1}
\end{bmatrix}
\begin{bmatrix}
  x(0)
\end{bmatrix}
\]

- also, no need to go beyond \( n \) in \( Q_d \): adding \( CA^n, CA^{n+1}, \ldots \) does not increase the column rank of \( Q_d \) (Cayley Halmilton Theorem)
Theorem (Observability Theorem)

System \( x(k+1) = Ax(k) + Bu(k), \ y(k) = Cx(k) + Du(k), \)
\( A \in \mathbb{R}^{n \times n}, \ C \in \mathbb{R}^{m \times n} \) is observable if and only if either one of the following is satisfied:

1. the observability matrix \( Q_d = \begin{bmatrix} C & CA & \cdots & CA^{n-1} \end{bmatrix} \) has full column rank

2. the observability gramian \( W_{od} = \sum_{k=0}^{k_1} \left( A^T \right)^k C^T C A^k \) is nonsingular for some finite \( k_1 \)

3. * PBF test: The matrix \( \begin{bmatrix} A - \lambda I & C \end{bmatrix} \) has full column rank at every eigenvalue, \( \lambda \), of \( A \).
Proof: from observability matrix to gramian

\[ Q_d = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \]

\[ W_{od} = \sum_{k=0}^{k_1} (A^T)^k C^T CA^k \]

\[ Q_d \text{ is full column rank} \Rightarrow Q_d^T Q_d = \sum_{k=0}^{n} (A^T)^k C^T CA^k \text{ is nonsingular} \]

\[ W_{od} \text{ at } k_1=n \]
Observability check

- Analogous to the case in controllability, the observability property is invariant under any coordinate transformation:

\[(A, C) \text{ is observable } \iff (T^{-1}AT, CT) \text{ is observable}\]

- If $A$ is Schur, $k_1$ can be set to $\infty$ in the observability gramian

\[W_{od} = \sum_{k=0}^{\infty} (A^T)^k C^T CA^k\]

and we can compute by solving the Lyapunov equation

\[A^T W_{od} A - W_{od} = -C^T C\]

The solution is nonsingular if and only if the system is observable. In fact, $W_{od} \succeq 0$ by definition $\Rightarrow$ “nonsingular” can be replaced with “positive definite”.

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Observability and observable canonical form

\[ A = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \]

- observability matrix

\[ Q_d = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -a_2 & 1 & 0 \\ a_2 - a_1 & -a_2 & 1 \end{bmatrix} \]

has full column rank

- system in observable canonical form is observable
* PBH test for observability

The matrix $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has full column rank at every eigenvalue, $\lambda$, of $A$.

- if not full rank then there exists a nonzero eigenvector $v$:

$$
\begin{align*}
Av &= \lambda v \\
Cv &= 0 \\
\Rightarrow CAv &= \lambda Cv = 0 \\
&\vdots \\
CA^{n-1}v &= 0
\end{align*}

$$

$$
\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} v = 0 \Rightarrow \text{unobservable}
$$

- the reverse direction is analogous

- **interpretation**: some non-zero initial condition $x_0 = v$ will generate zero output, which is not distinguishable from the origin.
1. Concepts

2. DT controllability
   Controllability and controllable canonical form
   Controllability and Lyapunov Eq.

3. DT observability
   Observability and observable canonical form

4. CT cases

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7. Transforming observable systems into observable canonical forms
Theorem (Controllability of continuous-time systems)

The $n$-dimensional $r$-input LTI system with $\dot{x} = Ax + Bu$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times r}$ is controllable if and only if either one of the following is satisfied

1. the $n \times nr$ controllability matrix

$$P = \left[ B, AB, A^2 B, \ldots, A^{n-1} B \right]$$

has rank $n$

2. the controllability gramian

$$W_{cc} = \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau$$

is nonsingular for any $t > 0$
Theorem (Observability of continuous-time systems)

System \( \dot{x} = Ax + Bu, \ y = Cx + Du, \ A \in \mathbb{R}^{n \times n}, \ C \in \mathbb{R}^{m \times n} \) is observable if and only if either one of the following is satisfied

1. the \((mn) \times n\) observability matrix

\[
Q = \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix}
\]

has rank \(n\) (full column rank)

2. the observability gramian

\[
W_{oc} = \int_{0}^{t} e^{A^T \tau} C^T Ce^{A \tau} \, d\tau
\]

is nonsingular for any \(t > 0\)
Summary: computing the gramians

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<tr>
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<th>Controllability Gramian</th>
<th>Observability Gramian</th>
</tr>
</thead>
<tbody>
<tr>
<td>continuous time</td>
<td>$\int_0^t e^{A\tau} BB^T (e^{A\tau})^T d\tau$</td>
<td>$\int_0^t (e^{A\tau})^T C^T Ce^{A\tau} d\tau$</td>
</tr>
<tr>
<td>Lyapunov eq.</td>
<td>$AW_c + W_c A^T = -BB^T$</td>
<td>$A^T W_o + W_o A = -C^T C$</td>
</tr>
<tr>
<td>if $t \to \infty$ &amp; $A$ is Hurwitz stable</td>
<td></td>
<td></td>
</tr>
<tr>
<td>discrete time</td>
<td>$\sum_{k=0}^{k_1} A^k BB^T (A^T)^k$</td>
<td>$\sum_{k=0}^{k_1}(A^T)^k C^T CA^k$</td>
</tr>
<tr>
<td>Lyapunov eq.</td>
<td>$AW_{cd} A^T - W_{cd} = -BB^T$</td>
<td>$A^T W_{od} A - W_{od} = -C^T C$</td>
</tr>
<tr>
<td>if $k_1 \to \infty$ &amp; $A$ is Schur stable</td>
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</tr>
</tbody>
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- duality: $(A, B)$ is controllable if and only if $(\bar{A}, \bar{C}) = (A^T, B^T)$ is observable
Exercise

\[ A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \]

\[ C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \]

▶ exercise: show that the system is not observable.

▶ in fact, by similarity transform \( x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \) \( x \), we get

\[ \bar{A} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

\[ \bar{C} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \]

where the third state is not observable
1. Concepts

2. DT controllability
   - Controllability and controllable canonical form
   - Controllability and Lyapunov Eq.

3. DT observability
   - Observability and observable canonical form

4. CT cases

5. The degrees of controllability and observability

6. Transforming controllable systems into controllable canonical forms

7. Transforming observable systems into observable canonical forms
The degree of controllability

consider two systems

\[ S_1 : x(k + 1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \]

\[ S_2 : x(k + 1) = \begin{bmatrix} 0 & 0.01 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \]

both systems are controllable:

\[ P_{d_1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad P_{d_2} = \begin{bmatrix} 0 & 0.01 \\ 1 & 1 \end{bmatrix} \]

however, \( P_{d_2} \) is nearly singular \( \Rightarrow \) \( S_2 \) not “easy” to control

e.g., to move from \( x(0) = [0, 0]^T \) to \( x(1) = [1, 1]^T \) in two steps:

\[ S_1 : \{ u(0), u(1) \} = \{ 1, 1 \} \quad S_2 : \{ u(0), u(1) \} = \{ 100, -99 \} \]

\( \Rightarrow \) more energy for \( S_2 \)!
The degree of controllability: multi-input case

consider two systems

\[
S_1 : \ x(k + 1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u(k)
\]

\[
S_2 : \ x(k + 1) = \begin{bmatrix} 0 & 0.01 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u(k)
\]

- both systems are controllable:

\[
P_{d_1} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad P_{d_2} = \begin{bmatrix} 0 & 0.01 & 0.01 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}
\]

- degree of controllability reflected in the controllability Gramian:

\[
W_{cd_1} = P_{d_1} P_{d_1}^T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad W_{cd_2} = \begin{bmatrix} 2 \times 0.01^2 & 0.02 \\ 0.02 & 3 \end{bmatrix}
\]

\(W_{cd_2}\) is almost singular (eigenvalues at 0.0001 and 3.0001)
The degree of controllability: multi-input case

- for general stable and controllable systems $\Sigma = (A, B, C, D)$, $W_{cd}$ is computed from the Lyapunov Equation
  \[ AW_{cd}A^T - W_{cd} = -BB^T \]
- if $W_{cd}$ have eigenvalues close to zero, then the system is more difficult to control in the sense that it requires more energy in the input to steer the states in the state space
The degree of observability

consider two systems

\[ S_1 : x(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(k) \quad y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \]

\[ S_2 : x(k+1) = \begin{bmatrix} 1 & 0.01 \\ 0 & 0 \end{bmatrix} x(k) \quad y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \]

▶ both systems are observable:

\[ Q_{d1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_{d2} = \begin{bmatrix} 1 & 0 \\ 1 & 0.01 \end{bmatrix} \]

▶ however, \( Q_{d2} \) is nearly singular, hinting that \( S_2 \) is not “easy” to observe

▶ e.g., to infer \( x(0) = [2, 1]^T \), the two measurements \( y(0) = 2 \) and \( y(1) = CAx(0) = 2.001 \) are nearly identical in \( S_2 \)!
The degree of observability: multi-output case

▶ for general stable and controllable systems $\Sigma = (A, B, C, D)$, the observability matrix $Q_d$ is not square

▶ the degree of observability is reflected in the eigenvalues of the observability Gramian $W_{od}$

▶ for stable systems, $W_{od}$ is computed from the Lyapunov Equation $A^T W_{od} A - W_{od} = -C^T C$

▶ if $W_{od}$ have eigenvalues close to zero, then the system is more difficult to observe
Balanced state-space realizations

we know now

- the controllability and observability Gramians represent the degrees of controllability and observability
- easily controllable systems may not be easily observable
- easily observable systems may not be easily controllable

⇒ there exists realizations that balance the two degrees of controllability and observability
Balanced state-space realizations

Consider a stable system $\Sigma = (A, B, C, D)$ in a minimal$^1$ realization.

- Minimal realization $\Rightarrow$ $\Sigma$ is controllable and observable.
- Stable $\Rightarrow$ can compute the Gramians from Lyapunov Equations.
- If $W_{cd}$ and $W_{od}$ are equal and diagonal, then $\Sigma$ is called a balanced realization.
- I.e., there exists a diagonal matrix $M = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ such that

$$M = AMA^T + BB^T$$

$$M = A^T MA + C^T C$$

\[\text{i.e., dim } A \text{ is the minimal order of the system}\]
1. Concepts

2. DT controllability
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5. The degrees of controllability and observability

6. Transforming controllable systems into controllable canonical forms

7. Transforming observable systems into observable canonical forms
Transforming single-input controllable system into ccf

Let \( x = M \tilde{x} \), where \( M = \begin{bmatrix} m_1 & m_2 & \ldots & m_n \end{bmatrix} \), then

\[
\dot{\tilde{x}} = M^{-1} \dot{x} = M^{-1} (Ax + Bu) = M^{-1} A M \tilde{x} + \underbrace{M^{-1} B}_\hat{B} u
\]

If system is controllable, we show how to transform the state equation into the controllable canonical form.

\begin{itemize}
\item goal 1: \( \hat{B} \) be in controllable canonical form \( \Leftrightarrow \)
\end{itemize}

\[
M^{-1} B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \Rightarrow B = [m_1, m_2, \ldots, m_n] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = m_n
\]
Transforming SI controllable system into ccf

Let $x = M\tilde{x}$, where $M = [m_1, m_2, \ldots, m_n]$, then

$$\dot{\tilde{x}} = M^{-1}\dot{x} = M^{-1}(Ax + Bu) = M^{-1}AM\tilde{x} + M^{-1}Bu$$

Goal 2: $\tilde{A}$ be in controllable canonical form$\iff$

$$A[m_1, m_2, \ldots, m_n] = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
\vdots & 0 & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & 1 & 0 \\
0 & \ldots & 0 & 0 & 1 \\
-a_0 & -a_1 & \ldots & \ldots & -a_{n-1}
\end{bmatrix}$$
Transforming SI controllable system into ccf

Let \( x = M\tilde{x} \), where \( M = [m_1, m_2, \ldots, m_n] \), then

\[
\dot{\tilde{x}} = M^{-1}\dot{x} = M^{-1}(Ax + Bu) = M^{-1}AM\tilde{x} + M^{-1}Bu
\]

- solving goals 1 and 2 yields

\[
\begin{align*}
m_n &= B \\
m_{n-1} &= Am_n + a_{n-1}m_n \\
m_{n-2} &= Am_{n-1} + a_{n-2}m_n \\
m_{i-1} &= Am_i + a_{i-1}m_n, \quad i = n, \ldots, 2 \\
&\vdots
\end{align*}
\]

- when implementing, obtain \( a_0, a_1, \ldots, a_{n-1} \) first by calculating \( \det (sl - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 \)
Transforming single-output (SO) observable system into ocf

Let $x = R^{-1}\tilde{x}$, where $R = [r_1^T, r_2^T, \ldots, r_n^T]^T$ ($r_i$ is a row vector).

$$\dot{\tilde{x}} = R\dot{x} = R(Ax + Bu) = \underbrace{RAR^{-1}\tilde{x}}_{\tilde{A}} + RBu$$

$$y = Cx = \underbrace{CR^{-1}\tilde{x}}_{\tilde{c}}$$

If system is observable, we show how to transform the state equation into the observable canonical form.

- goal 1: $\tilde{C}$ be in observable canonical form $\iff$

$$CR^{-1} = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix} \Rightarrow C = r_1$$
Transforming SO observable system into ocf

Let \( x = R^{-1} \tilde{x} \), where \( R = [r_1^T, r_2^T, \ldots, r_n^T]^T \) (\( r_i \) is a row vector).

\[
\dot{\tilde{x}} = R \dot{x} = R (Ax + Bu) = RAR^{-1} \tilde{x} + RBu
\]

\[
y = Cx = CR^{-1} \tilde{x}
\]

\( \Rightarrow \) goal 2: \( \tilde{A} \) be in observable canonical form

\[
A = \begin{bmatrix}
  r_1 \\
r_2 \\
  \vdots \\
r_n
\end{bmatrix} \quad \begin{bmatrix}
  -a_{n-1} & 1 & 0 & \ldots & 0 \\
  0 & \ddots & \ddots & \vdots \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
  0 & \cdots & 0 \\
  -a_1 & \cdots & \cdots & 1 \\
  -a_0 & 0 & \ldots & 0 & 0
\end{bmatrix} \begin{bmatrix}
  r_1 \\
r_2 \\
  \vdots \\
r_n
\end{bmatrix}
\]
Transforming SO observable system into ocf

Let \( x = R^{-1}\tilde{x} \), where \( R = [r_1^T, r_2^T, \ldots, r_n^T]^T \) (\( r_i \) is a row vector).

\[
\dot{x} = R\dot{x} = R(Ax + Bu) = \underbrace{RAR^{-1}}_{\tilde{A}}\tilde{x} + RBu
\]

\[
y = Cx = \underbrace{CR^{-1}}_{\tilde{C}}\tilde{x}
\]

- solving goals 1 and 2 yields
  \[
r_1 = C
  \]
  \[
r_2 = r_1A + a_{n-1}r_1
  \]
  \[
r_3 = r_2A + a_{n-2}r_1
  \]
  \[
  r_{i+1} = r_iA + a_{n-i}r_1, \quad i = 1, \ldots, n-1
  \]
  
- when implementing, obtain \( a_0, a_1, \ldots, a_{n-1} \) first by calculating \( \det(sI - A) \)
Transforming SO observable system into ocf

Example: \( x(k+1) = \begin{bmatrix} 1 & 0.01 \\ 0 & 0 \end{bmatrix} x(k) \)  
\( y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \)

\[
\text{det}(A - \lambda I) = \lambda^2 - \lambda \Rightarrow a_1 = -1, \ a_0 = 0
\]

\[
r_1 = C = [1, 0]
\]
\[
r_2 = r_1 C + a_1 r_1 = [1, 0] A + (-1) [1, 0]
\]
\[
R = \begin{bmatrix} 1 & 0 \\ 0 & 0.01 \end{bmatrix}, \ R^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix}
\]
\[
\tilde{C} = CR^{-1} = [1, 0] \iff \text{ocf!}
\]
\[
\tilde{A} = RAR^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \iff \text{ocf!}
\]