Controllability and Observability

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The concept of controllability and observability

Controllability:

- inputs do not act directly on the states but via state dynamics:

$$\dot{x}(t) = Ax(t) + Bu(t) \text{ or } x(k+1) = Ax(k) + Bu(k) \quad (1)$$

- can the inputs drive the system to any value in the state space in finite time?

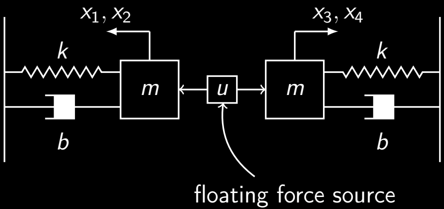
Observability:

- states are not all measured directly but instead impact the output via the output equation:

$$y = Cx + Du$$

- can we infer fully the initial state from the outputs and the inputs? (can then reveal the full state trajectory through (1))

The concept of controllability and observability



$$\dot{x}_1 = x_2$$
$$\dot{x}_3 = x_4$$

- assume $x(0) = 0$
- because of symmetry, we always have

$$x_1(t) = x_3(t), \; x_2(t) = x_4(t), \; \forall t \geq 0$$

- state cannot be arbitrarily steered $\Rightarrow$ uncontrollable

Controllability definition in discrete time

Definition

A discrete-time linear system $x(k+1) = A(k)x(k) + B(k)u(k)$ is called controllable at $k = 0$ if $\exists$ a finite time k_1 such that $\forall$ initial state $x(0)$ and target state x_1, there exists a control sequence $\{u(k); k = 0, 1, \ldots, k_1\}$ that will transfer the system from $x(0)$ at $k = 0$ to x_1 at $k = k_1$.

Controllability of LTI systems

$$x(k+1) = Ax(k) + Bu(k) \Rightarrow x(n) = A^n x(0) + \sum_{k=0}^{n-1} A^{n-1-k} Bu(k)$$

$$\Rightarrow \boxed{x(n) - A^n x(0) = \underbrace{\left[B, AB, A^2B, \dots, A^{n-1}B\right]}_{P_d} \underbrace{\begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix}}_{\mathrm{u}_n}}$$

- given any $x(n)$ and $x(0)$ in $\mathbb{R}^n$, u_n can be solved if the columns of P_d span $\mathbb{R}^n$
- equivalently, system is controllable if P_d has rank n (full row rank)

Controllability of LTI systems Cont'd

$x(k+1) = Ax(k) + Bu(k) \Rightarrow$

$$\boxed{x(n) - A^n x(0) = \underbrace{\left[B, AB, A^2B, \ldots, A^{n-1}B\right]}_{P_d} \underbrace{\begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix}}_{\mathrm{u}_n}}$$

- also, no need to go beyond n: adding A^nB, $A^{n+1}B$, $\ldots$ does not increase the rank of P_d (Cayley Halmilton Theorem):

$$x(k_1) - A^{k_1}x(0) = \underbrace{\left[\begin{array}{cccc|ccc} B & AB & \ldots & A^{n-1}B & \ldots & A^{k_1-1}B \end{array}\right]}_{\text{rank}=\text{rank}(P_d)} \begin{bmatrix} u(k_1-1) \\ u(k_1-2) \\ \vdots \\ u(0) \end{bmatrix}$$

Theorem (Cayley Halmilton Theorem)

Let $A \in \mathbb{R}^{n\times n}$. A^n is linearly dependent with $\{I, A, A^2, \cdots A^{n-1}\}$

Proof.

Consider characteristic polynomial

$$\begin{aligned} p(\lambda) &= \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 = \det(\lambda I - A) \\ &= (\lambda-\lambda_1)^{m_1}\dots(\lambda-\lambda_p)^{m_p} \end{aligned}$$

$$\begin{aligned} \Rightarrow p(A) &= A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I \\ &= (A-\lambda_1 I)^{m_1}\dots(A-\lambda_p I)^{m_p}, \quad m_1 + m_2 + \cdots + m_p = n \end{aligned}$$

$\forall$ eigenvector or generalized eigenvector t_i, say, associated to λ_i:
$p(A)\,t_i = (A-\lambda_1 I)^{m_1}\dots\underline{(A-\lambda_p I)^{m_p}\,t_i} =$
$(A-\lambda_1 I)^{m_1}\dots\underline{(A-\lambda_p I)^{m_p-1}(\lambda_i t_i - \lambda_p t_i)} = (\lambda_i-\lambda_1)^{m_1}\dots(\lambda_i-\lambda_p)^{m_p}\,t_i = 0$

- therefore $p(A)[t_1, t_2, \dots, t_n] = 0$
- but $T = [t_1, t_2, \dots, t_n]$ is invertible. Hence $p(A) = 0$
  $\Rightarrow A^n = -c_0 I - c_1 A - \cdots - c_{n-1}A^{n-1}$

□

Arthur Cayley: 1821-1895, British mathematician

- algebraic theory of curves and surfaces, group theory, linear algebra, graph theory, invariant theory, ...
- extraordinarily prolific career: ~1,000 math papers

William Hamilton: 1805-1865, Irish mathematician

- optics and classical mechanics in physics, dynamics, algebra, quaternions, ...
- quaternions: extending complex numbers to higher spatial dimensions: 4D case

$$i^2 = j^2 = k^2 = ijk = -1$$

now used in computer graphics, control theory, orbital mechanics, e.g., spacecraft attitude-control systems

Theorem (Controllability Theorem)

The n-dimensional r-input LTI system with $x(k+1) = Ax(k) + Bu(k)$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$ *is controllable if and only if either one of the following is satisfied:*

1. *the* $n \times nr$ *controllability matrix*

$$\boxed{P_d = \left[B, AB, A^2B, \ldots, A^{n-1}B\right]}$$

   has rank n

2. *the controllability gramian*

$$\boxed{W_{cd} = \sum_{k=0}^{k_1} A^k BB^T \left(A^T\right)^k}$$

   is nonsingular for some finite k_1

Proof: from controllability matrix to gramian

Recall

$$\boxed{x(n) - A^n x(0) = \underbrace{\left[B, AB, A^2B, \ldots, A^{n-1}B\right]}_{P_d} \mathrm{u}_n} \tag{2}$$

- P_d is full row rank$\Rightarrow P_d P_d^T = \underbrace{\sum_{k=0}^{n} A^k BB^T \left(A^T\right)^k}_{W_{cd} \text{ at } k_1 = n}$ is nonsingular
- a (least-square) solution to (2) is

$$\mathrm{u}_n = P_d^T \left(P_d P_d^T\right)^{-1} \left[x(n) - A^n x(0)\right]$$

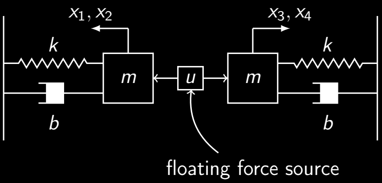
Example

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P_d = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \lambda_2 + \lambda_2 \\ 1 & \lambda_2 & \lambda_2^2 \end{bmatrix} \Rightarrow \text{rank}(P_d) = 2 < 3 \Rightarrow \text{uncontrollable}$$

Intuition: $\dot{x}_1 = \lambda_1 x_1$ is not impacted by the control input at all.

Example

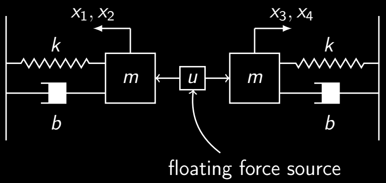


$$\dot{x}_1 = x_2$$
$$\dot{x}_3 = x_4$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} 0.4 & 0.4 & 0 & 0 \\ -0.9 & -0.07 & 0 & 0 \\ 0 & 0 & 0.4 & 0.4 \\ 0 & 0 & -0.9 & -0.07 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} 0.3 \\ 0.4 \\ 0.3 \\ 0.4 \end{bmatrix} u(k)$$

$$\text{rank}\,(P_d) = \text{rank} \begin{bmatrix} \overbrace{\begin{matrix} 0.3 \\ 0.4 \\ 0.3 \\ 0.4 \end{matrix}}^{B} & \overbrace{\begin{matrix} 0.28 \\ -0.298 \\ 0.28 \\ -0.298 \end{matrix}}^{AB} & \overbrace{\begin{matrix} -0.0072 \\ -0.2311 \\ -0.0072 \\ -0.2311 \end{matrix}}^{A^2B} & \overbrace{\begin{matrix} -0.0953 \\ 0.0227 \\ -0.0953 \\ 0.0227 \end{matrix}}^{A^3B} \end{bmatrix} = 2 \Rightarrow \text{uncontrollable}$$

Example

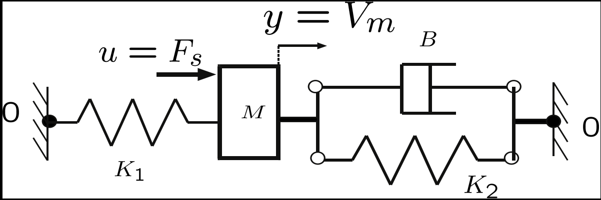


$$\dot{x}_1 = x_2$$
$$\dot{x}_3 = x_4$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} 0.4 & 0.4 & 0 & 0 \\ -0.9 & -0.07 & 0 & 0 \\ 0 & 0 & 0.4 & 0.4 \\ 0 & 0 & -0.9 & -0.07 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} 0.3 \\ 0.4 \\ 0.3 \\ 0.4 \end{bmatrix} u(k)$$

```
import numpy as np
import control as ct
A = np.array([[0.4, 0.4, 0, 0], [-0.9, -0.07, 0, 0], [0, 0, 0.4, 0.4], [0, 0, -0.9, -0.07]])
B = np.array([[0.3], [0.4], [0.3], [0.4]])
P = ct.ctrb(A,B)
print(np.linalg.matrix_rank(P))
```

Example



$$\frac{d}{dt}\begin{bmatrix} v_m \\ F_{k_1} \\ F_{k_2} \end{bmatrix} = \begin{bmatrix} -b/m & -1/m & -1/m \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_m \\ F_{k_1} \\ F_{k_2} \end{bmatrix} + \begin{bmatrix} 1/m \\ 0 \\ 0 \end{bmatrix} F$$

$$P = \begin{bmatrix} 1/m & -b/m^2 & b^2/m^3 - k_1/m^2 - k_2/m^2 \\ 0 & k_1/m & -bk_1/m^2 \\ 0 & k_2/m & -bk_2/m^2 \end{bmatrix} \Rightarrow \text{rank}(P) = 2$$

$\Rightarrow$uncontrollable

Analysis: controllability and controllable canonical form

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- controllability matrix

$$P_d = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & -a_1 + a_2^2 \end{bmatrix}$$

  has full row rank
- *system in controllable canonical form is controllable*

Recap

General LTI state-space models:

$$\dot{x}(t) = Ax(t) + Bu(t) \text{ or } x(k+1) = Ax(k) + Bu(k)$$
$$y = Cx + Du$$

	continuous time	discrete time
Lyapunov Eq.	$A^T P + PA = -Q$	$A^T PA - P = -Q$
unique sol. cond.	$\lambda_i(A) + \lambda_j(A) \neq 0$ $\forall\ i, j$	$\lvert\lambda_i(A)\rvert \lvert\lambda_j(A)\rvert < 1$ $\forall\ i, j$
solution	$P = \int_0^\infty e^{A^T t} Q e^{At} dt$ (if A is Hurwitz stable)	$P = \sum_{k=0}^\infty \left(A^T\right)^k Q A^k$ (if A is Schur stable)

Analysis: controllability gramian and Lyapunov Eq.

$$W_{cd} = \sum_{k=0}^{k_1} A^k BB^T \left(A^T\right)^k$$

- If A is Schur, k_1 can be set to ∞

$$W_{cd} = \sum_{k=0}^{\infty} A^k \underbrace{BB^T}_{Q} \left(A^T\right)^k$$

which can be solved via the Lyapunov Eq.

$$\boxed{AW_{cd}A^T - W_{cd} = -BB^T}$$

Analysis: controllability and similarity transformation

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases} \overset{x=Tx^*}{\Longrightarrow} \begin{cases} x^*(k+1) = \overbrace{T^{-1}AT}^{\tilde{A}} x^*(k) + \overbrace{T^{-1}B}^{\tilde{B}} u(k) \\ y(k) = CTx^*(k) + Du(k) \end{cases}$$

- controllability matrix

$$\begin{aligned} P_d^* &= \left[\tilde{B}, \tilde{A}\tilde{B}, \dots, \tilde{A}^{n-1}\tilde{B}\right] \\ &= \left[T^{-1}B, T^{-1}AB, \dots, T^{-1}A^{n-1}B\right] = T^{-1}P_d \end{aligned}$$

hence $\underline{(A, B) \text{ controllable} \Leftrightarrow \left(T^{-1}AT, T^{-1}B\right) \text{ controllable}}$

- **The controllability property is invariant under any coordinate transformation.**

* Popov-Belevitch-Hautus (PBH) controllability test

- the full rank condition of the controllability matrix

$$\boxed{P_d = \left[B, AB, A^2B, \ldots, A^{n-1}B\right]}$$

is equivalent to: *the matrix* $[A - \lambda I,\ B]$ *having full row rank at every eigenvalue,* λ*, of* A

- to see this: if $[A - \lambda I,\ B]$ is not full row rank then there exists nonzero vector (a left eigenvector) such that

$$v^T[A - \lambda I\ B] = 0$$

$$\Leftrightarrow v^T A = \lambda v^T$$
$$v^T B = 0$$

i.e., the input vector B is orthogonal to a left eigenvector of A.

Example

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[A - \lambda_1 I, \quad B\right] =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_2 - \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_2 - \lambda_1 & 1 \end{bmatrix} \text{ not full row rank} \Rightarrow \text{uncontrollable}$$

Intuition: $\dot{x}_1 = \lambda_1 x_1$ is not impacted by the control input at all.

Observability of LTI systems

Definition

A discrete-time linear system

$$x(k+1) = A(k)x(k) + B(k)u(k)$$
$$y(k) = C(k)x(k) + D(k)u(k)$$

is called observable at $k = 0$ if $\exists$ a finite time k_1 such that $\forall$ initial state $x(0)$, the knowledge of input $\{u(k); k = 0, 1, \ldots, k_1\}$ and $\{y(k); k = 0, 1, \ldots, k_1\}$ suffice to determine the state $x(0)$. Otherwise, the system is said to be unobservable at time $k = 0$.

Observability of LTI systems

let us start with the unforced system

$$x(k+1) = Ax(k),\ A \in \mathbb{R}^n$$
$$y(k) = Cx(k),\ y \in \mathbb{R}^m$$

$x(k) = A^k x(0)$ and $y(k) = Cx(k)$ give

$$\underbrace{\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix}}_{\mathrm{y}_n} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{Q_d : nm \times n} x(0)$$

- if the linear matrix equation has a nonzero solution $x(0)$, the system is observable

Observability of LTI systems

generalizing to

$x(k+1) = Ax(k) + Bu(k),\ y(k) = Cx(k) + Du(k)$:

$$x(k) = A^k x(0) + \sum_{j=0}^{k-1} A^{k-1-j} Bu(j)$$

$$y(k) = \underbrace{CA^k x(0)}_{y_{\text{free}}(k)} + \underbrace{C\sum_{j=0}^{k-1} A^{k-1-j} Bu(j) + Du(k)}_{y_{\text{forced}}(k)}$$

$$\underbrace{\begin{bmatrix} y(0) - y_{\text{forced}}(0) \\ y(1) - y_{\text{forced}}(1) \\ \vdots \\ y(n-1) - y_{\text{forced}}(n-1) \end{bmatrix}}_{\bar{y}_n\text{: available from measurements and inputs}} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{Q_d : nm \times n} x(0)$$

Observability of LTI systems

$$\underbrace{\begin{bmatrix} y(0) - y_{\text{forced}}(0) \\ y(1) - y_{\text{forced}}(1) \\ \vdots \\ y(n-1) - y_{\text{forced}}(n-1) \end{bmatrix}}_{\bar{y}_n} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{Q_d} x(0)$$

- $x(0)$ can be solved if Q_d has rank n (full column rank):
  - if Q_d is square, $x(0) = Q_d^{-1}\bar{y}_n$
  - if Q_d is a tall matrix, pick n linearly independent rows from Q_d

Observability of LTI systems Cont'd

$$\underbrace{\begin{bmatrix} y(0) - y_{\text{forced}}(0) \\ y(1) - y_{\text{forced}}(1) \\ \vdots \\ y(n-1) - y_{\text{forced}}(n-1) \end{bmatrix}}_{\bar{y}_n} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{Q_d} x(0)$$

- also, no need to go beyond n in Q_d: adding CA^n, CA^{n+1}, ... does not increase the column rank of Q_d (Cayley Halmilton Theorem)

Theorem (Observability Theorem)

System $x(k+1) = Ax(k) + Bu(k)$, $y(k) = Cx(k) + Du(k)$, $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{m \times n}$ *is observable if and only if either one of the following is satisfied:*

1. *the observability matrix* $Q_d = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{(mn) \times n}$ *has full column rank*
2. *the observability gramian*

$$\boxed{W_{od} = \sum_{k=0}^{k_1} \left(A^T\right)^k C^T C A^k}$$

is nonsingular for some finite k_1

3. ** PBF test: The matrix* $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ *has full column rank at every eigenvalue,* λ*, of* A.

Proof: from observability matrix to gramian

$$Q_d = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad W_{od} = \sum_{k=0}^{k_1} \left(A^T\right)^k C^T C A^k$$

- Q_d is full column rank$\Rightarrow Q_d^T Q_d = \underbrace{\sum_{k=0}^{n} \left(A^T\right)^k C^T C A^k}_{W_{od} \text{ at } k_1 = n}$ is nonsingular

Observability check

- Analogous to the case in controllability, the observability property is invariant under any coordinate transformation:

$$(A, C) \text{ is observable} \iff (T^{-1}AT, CT) \text{ is observable}$$

- If A is Schur, k_1 can be set to ∞ in the observability gramian

$$W_{od} = \sum_{k=0}^{\infty} \left(A^T\right)^k C^T C A^k$$

and we can compute by solving the Lyapunov equation

$$A^T W_{od} A - W_{od} = -C^T C$$

The solution is nonsingular if and only if the system is observable. In fact, $W_{od} \succeq 0$ by definition $\Rightarrow$ "nonsingular" can be replaced with "positive definite".

Observability and observable canonical form

$$A = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

- observability matrix

$$Q_d = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -a_2 & 1 & 0 \\ a_2^2 - a_1 & -a_2 & 1 \end{bmatrix}$$

  has full column rank
- *system in observable canonical form is observable*

* PBH test for observability

The matrix $\begin{bmatrix} A-\lambda I \\ C \end{bmatrix}$ has full column rank at every eigenvalue, λ, of A.

- if not full rank then there exists a nonzero eigenvector v:

$$\begin{aligned} Av &= \lambda v \\ Cv &= 0 \\ \Rightarrow CAv &= \lambda Cv = 0 \\ &\vdots \\ CA^{n-1}v &= 0 \end{aligned} \quad \Rightarrow \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} v = 0 \Rightarrow \text{unobservable}$$

- the reverse direction is analogous
- **interpretation**: some non-zero initial condition $x_0 = v$ will generate zero output, which is not distinguishable from the origin.

1. Concepts

2. DT controllability
   - Controllability and controllable canonical form
   - Controllability and Lyapunov Eq.

3. DT observability
   - Observability and observable canonical form

4. CT cases

5. The degrees of controllability and observability

6. Transforming controllable systems into controllable canonical forms

7. Transforming observable systems into observable canonical forms

Theorem (Controllability of continuous-time systems)

The n-dimensional r-input LTI system with $\dot{x} = Ax + Bu$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$ is controllable if and only if either one of the following is satisfied

1. *the $n \times nr$ controllability matrix*

$$P = \left[B, AB, A^2B, \ldots, A^{n-1}B\right]$$

   has rank n

2. *the controllability gramian*

$$\boxed{W_{cc} = \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau}$$

   is nonsingular for any $t > 0$

Theorem (Observability of continuous-time systems)

System $\dot{x} = Ax + Bu, \; y = Cx + Du$, $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{m\times n}$ *is observable if and only if either one of the following is satisfied*

1. *the* $(mn) \times n$ *observability matrix*

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \text{ has rank } n \text{ (full column rank)}$$

2. *the observability gramian*

$$\boxed{W_{oc} = \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau} \text{ is nonsingular for any } t > 0$$

Summary: computing the gramians

	Controllability Gramian	Observability Gramian
continuous time	$\int_0^t e^{A\tau}BB^T\left(e^{A\tau}\right)^T d\tau$	$\int_0^t \left(e^{A\tau}\right)^T C^T C e^{A\tau} d\tau$
Lyapunov eq. if $t \to \infty$ & A is Hurwitz stable	$AW_c + W_c A^T = -BB^T$	$A^T W_o + W_o A = -C^T C$
discrete time	$\sum_{k=0}^{k_1} A^k BB^T \left(A^T\right)^k$	$\sum_{k=0}^{k_1} (A^T)^k C^T C A^k$
Lyapunov eq. if $k_1 \to \infty$ & A is Schur stable	$AW_{cd}A^T - W_{cd} = -BB^T$	$A^T W_{od} A - W_{od} = -C^T C$

- duality: (A, B) is controllable if and only if $\left(\overline{A}, \overline{C}\right) = \left(A^T, B^T\right)$ is observable

Exercise

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

- exercise: show that the system is not observable.
- in fact, by similarity transform $\bar{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x$, we get

$$\bar{A} = \left[\begin{array}{cc|c} -2 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 2 & 0 \end{array}\right], \; \bar{B} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$
$$\bar{C} = \left[\begin{array}{cc|c} 1 & 1 & 0 \end{array}\right]$$

where the third state is not observable

The degree of controllability

consider two systems

$$S_1: \; x(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$S_2: \; x(k+1) = \begin{bmatrix} 0 & 0.01 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

- both systems are controllable:

$$P_{d_1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad P_{d_2} = \begin{bmatrix} 0 & 0.01 \\ 1 & 1 \end{bmatrix}$$

- however, P_{d_2} is nearly singular $\Rightarrow$ S_2 not "easy" to control
- e.g., to move from $x(0) = [0, 0]^T$ to $x(1) = [1, 1]^T$ in two steps:

$$S_1: \{u(0), u(1)\} = \{1, 1\} \qquad S_2: \{u(0), u(1)\} = \{100, -99\}$$

$\Rightarrow$ more energy for S_2!

The degree of controllability: multi-input case

consider two systems

$$S_1: \; x(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u(k)$$

$$S_2: \; x(k+1) = \begin{bmatrix} 0 & 0.01 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u(k)$$

- both systems are controllable:

$$P_{d_1} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad P_{d_2} = \begin{bmatrix} 0 & 0.01 & 0.01 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

- degree of controllability reflected in the controllability Gramian:

$$W_{cd1} = P_{d1} P_{d1}^T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \; W_{cd2} = \begin{bmatrix} 2 \times 0.01^2 & 0.02 \\ 0.02 & 3 \end{bmatrix}$$

W_{cd2} is almost singular (eigenvalues at 0.0001 and 3.0001)

The degree of controllability: multi-input case

- for general stable and controllable systems $\Sigma = (A, B, C, D)$, W_{cd} is computed from the Lyapunov Equation $AW_{cd}A^T - W_{cd} = -BB^T$
- if W_{cd} have eigenvalues close to zero, then the system is more difficult to control in the sense that it requires more energy in the input to steer the states in the state space

The degree of observability

consider two systems

$$S_1 : x(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(k) \qquad y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

$$S_2 : x(k+1) = \begin{bmatrix} 1 & 0.01 \\ 0 & 0 \end{bmatrix} x(k) \qquad y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

- both systems are observable:

$$Q_{d_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad Q_{d_2} = \begin{bmatrix} 1 & 0 \\ 1 & 0.01 \end{bmatrix}$$

- however, Q_{d_2} is nearly singular, hinting that S_2 is not "easy" to observe
- e.g., to infer $x(0) = [2, 1]^T$, the two measurements $y(0) = 2$ and $y(1) = CAx(0) = 2.001$ are nearly identical in S_2!

The degree of observability: multi-output case

- for general stable and controllable systems $\Sigma = (A, B, C, D)$, the observability matrix Q_d is not square
- the degree of observability is reflected in the eigenvalues of the observability Gramian W_{od}
- for stable systems, W_{od} is computed from the Lyapunov Equation $A^T W_{od} A - W_{od} = -C^T C$
- if W_{od} have eigenvalues close to zero, then the system is more difficult to observe

Balanced state-space realizations

we know now

- the controllability and observability Gramians represent the degrees of controllability and observability
- easily controllable systems may not be easily observable
- easily observable systems may not be easily controllable

$\Rightarrow$ there exists realizations that balance the two degrees of controllability and observability

Balanced state-space realizations

consider a stable system $\Sigma = (A, B, C, D)$ in a minimal[1] realization

- minimal realization $\Rightarrow$ Σ is controllable and observable
- stable $\Rightarrow$ can compute the Gramians from Lyapunov Equations
- if W_{cd} and W_{od} are equal and diagonal, then Σ is called a balanced realization
- i.e., there exists a diagonal matrix $M = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ such that

$$M = AMA^T + BB^T$$
$$M = A^T MA + C^T C$$

[1] i.e., $\dim A$ is the minimal order of the system

Transforming single-input controllable system into ccf

Let $x = M\tilde{x}$, where $M = \begin{bmatrix} | & | & & | \\ m_1 & m_2 & \dots & m_n \\ | & | & & | \end{bmatrix}$, then

$$\dot{\tilde{x}} = M^{-1}\dot{x} = M^{-1}(Ax + Bu) = M^{-1}AM\tilde{x} + \underbrace{M^{-1}B}_{\tilde{B}}u$$

If system is controllable, we show how to transform the state equation into the controllable canonical form.

- goal 1: $\tilde{B}$ be in controllable canonical form$\Leftrightarrow$

$$M^{-1}B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \Rightarrow B = [m_1, m_2, \dots, m_n] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = m_n$$

Transforming SI controllable system into ccf

Let $x = M\tilde{x}$, where $M = [m_1, m_2, \ldots, m_n]$, then

$$\dot{\tilde{x}} = M^{-1}\dot{x} = M^{-1}(Ax + Bu) = \underbrace{M^{-1}AM}_{\tilde{A}}\tilde{x} + M^{-1}Bu$$

- goal 2: $\tilde{A}$ be in controllable canonical form$\Leftrightarrow$

$$A[m_1, m_2, \ldots, m_n] =$$

$$[m_1, m_2, \ldots, m_n]\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \ldots & 0 & 0 & 1 \\ -a_0 & -a_1 & \ldots & \ldots & -a_{n-1} \end{bmatrix}$$

Transforming SI controllable system into ccf

Let $x = M\tilde{x}$, where $M = [m_1, m_2, \ldots, m_n]$, then

$$\dot{\tilde{x}} = M^{-1}\dot{x} = M^{-1}(Ax + Bu) = M^{-1}AM\tilde{x} + M^{-1}Bu$$

- solving goals 1 and 2 yields

$$\begin{aligned} m_n &= B \\ m_{n-1} &= Am_n + a_{n-1}m_n \\ m_{n-2} &= Am_{n-1} + a_{n-2}m_n \\ m_{i-1} &= Am_i + a_{i-1}m_n, \;\; i = n, \ldots, 2 \\ &\vdots \end{aligned}$$

- when implementing, obtain a_0, a_1, …, a_{n-1} first by calculating $\det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$

Transforming single-output (SO) observable system into ocf

Let $x = R^{-1}\tilde{x}$, where $R = \left[r_1^T, r_2^T, \dots, r_n^T\right]^T$ (r_i is a row vector).

$$\dot{\tilde{x}} = R\dot{x} = R\left(Ax + Bu\right) = \underbrace{RAR^{-1}}_{\tilde{A}}\tilde{x} + RBu$$

$$y = Cx = \underbrace{CR^{-1}}_{\tilde{C}}\tilde{x}$$

If system is observable, we show how to transform the state equation into the observable canonical form.

- goal 1: $\tilde{C}$ be in observable canonical form$\Leftrightarrow$

$$CR^{-1} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \Rightarrow C = r_1$$

Transforming SO observable system into ocf

Let $x = R^{-1}\tilde{x}$, where $R = \left[r_1^T, r_2^T, \dots, r_n^T\right]^T$ (r_i is a row vector).

$$\dot{\tilde{x}} = R\dot{x} = R\left(Ax + Bu\right) = \underbrace{RAR^{-1}}_{\tilde{A}}\tilde{x} + RBu$$

$$y = Cx = \underbrace{CR^{-1}}_{\tilde{C}}\tilde{x}$$

- goal 2: $\tilde{A}$ be in observable canonical form$\Leftrightarrow$

$$\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} A = \begin{bmatrix} -a_{n-1} & 1 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ & 0 & \ddots & \ddots & 0 \\ -a_1 & \vdots & \ddots & \ddots & 1 \\ -a_0 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$

Transforming SO observable system into ocf

Let $x = R^{-1}\tilde{x}$, where $R = \left[r_1^T, r_2^T, \ldots, r_n^T\right]^T$ (r_i is a row vector).

$$\dot{\tilde{x}} = R\dot{x} = R\left(Ax + Bu\right) = \underbrace{RAR^{-1}}_{\tilde{A}}\tilde{x} + RBu$$

$$y = Cx = \underbrace{CR^{-1}}_{\tilde{C}}\tilde{x}$$

- solving goals 1 and 2 yields

$$\begin{aligned} r_1 &= C \\ r_2 &= r_1 A + a_{n-1} r_1 \\ r_3 &= r_2 A + a_{n-2} r_1 \\ r_{i+1} &= r_i A + a_{n-i} r_1, \ i = 1, \ldots, n-1 \\ &\vdots \end{aligned}$$

- when implementing, obtain a_0, a_1, ..., a_{n-1} first by calculating $\det\left(sI - A\right)$

Transforming SO observable system into ocf

Example: $x(k+1) = \begin{bmatrix} 1 & 0.01 \\ 0 & 0 \end{bmatrix} x(k)\, y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$

$$\det(A - \lambda I) = \lambda^2 - \lambda \Rightarrow a_1 = -1,\ \ a_0 = 0$$

$$\begin{aligned}
r_1 &= C = [1, 0] \\
r_2 &= r_1 C + a_1 r_1 = [1, 0] A + (-1)[1, 0] \\
R &= \begin{bmatrix} 1 & 0 \\ 0 & 0.01 \end{bmatrix}, R^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix} \\
\tilde{C} &= CR^{-1} = [1, 0] \Longleftarrow \text{ocf!} \\
\tilde{A} &= RAR^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Longleftarrow \text{ocf!}
\end{aligned}$$