Lyapunov Stability

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Finite dimensional vector norms

Let $v \in \mathbb{R}^n$. A norm is:

- a metric in vector space: a function that assigns a real-valued length to each vector in a vector space
- e.g., 2 (Euclidean) norm: $\|v\|_2 = \sqrt{v^T v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

default in this set of notes: $\|\cdot\| = \|\cdot\|_2$

Equilibrium state

For an n-th order unforced system

$$\dot{x} = f(x, t), \; x(t_0) = x_0$$

an equilibrium state/point x_e is one such that

$$\boxed{f(x_e, t) = 0, \; \forall t}$$

- the condition must be satisfied by all $t \geq 0$
- if a system starts at equilibrium state, it stays there

Equilibrium state of a linear system

For a linear system

$$\dot{x}(t) = A(t)x(t), \; x(t_0) = x_0$$

- origin $x_e = 0$ is always an equilibrium state
- when $A(t)$ is singular, multiple equilibrium states exist

Lyapunov's definition of stability

- The equilibrium state 0 of $\dot{x} = f(x, t)$ is *stable in the sense of Lyapunov (s.i.L)* if for all $\epsilon > 0$, and t_0, there exists $\delta(\epsilon, t_0) > 0$ such that $\|x(t_0)\|_2 < \delta$ gives $\|x(t)\|_2 < \epsilon$ for all $t \geq t_0$

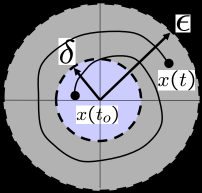


Figure: Stable s.i.L: $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon \; \forall t \geq t_0$.

Asymptotic stability

The equilibrium state 0 of $\dot{x} = f(x, t)$ is asymptotically stable if

- it is stable in the sense of Lyapunov, and
- for all $\epsilon > 0$ and t_0, there exists $\delta(\epsilon, t_0) > 0$ such that $\|x(t_0)\|_2 < \delta$ gives $x(t) \to 0$

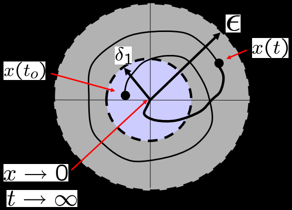


Figure: Asymptotically stable i.s.L: $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| \to 0$.

Stability of LTI systems: method of eigenvalue/pole locations

the stability of the equilibrium point 0 for $\dot{x} = Ax$ or $x(k+1) = Ax(k)$ can be concluded immediately based on $\lambda(A)$:

- the response $e^{At}x(t_0)$ involves modes such as $e^{\lambda t}$, $te^{\lambda t}$, $e^{\sigma t}\cos\omega t$, $e^{\sigma t}\sin\omega t$
- the response $A^k x(k_0)$ involves modes such as λ^k, $k\lambda^{k-1}$, $r^k\cos k\theta$, $r^k\sin k\theta$
- $e^{\sigma t} \to 0$ if $\sigma < 0$; $e^{\lambda t} \to 0$ if $\lambda < 0$
- $\lambda^k \to 0$ if $|\lambda| < 1$; $r^k \to 0$ if $|r| = \left|\sqrt{\sigma^2+\omega^2}\right| = |\lambda| < 1$

Lyapunov's approach to stability

The direct method of Lyapunov to stability problems:

- no need for explicit solutions to system responses
- an "energy" perspective
- fit for general dynamic systems (linear/nonlinear, time-invariant/time-varying)

Stability from an energy viewpoint: Example

Consider spring-mass-damper systems:

$$\dot{x}_1 = x_2 \qquad (x_1\text{: position; } x_2\text{: velocity})$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2,\ \ b>0 \qquad \text{(Newton's law)}$$

- $\lambda(A)$'s are in the left-half s-plane$\Rightarrow$ asymptotically stable
- total energy

$$\mathcal{E}(t) = \text{potential energy} + \text{kinetic energy} = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$$

- energy dissipates / is dissipative:

$$\dot{\mathcal{E}}(t) = kx_1\dot{x}_1 + mx_2\dot{x}_2 = -bx_2^2 \le 0$$

- $\dot{\mathcal{E}} = 0$ only when $x_2 = 0$. As $[x_1, x_2]^T = 0$ is the only equilibrium, the motion will not stop at $x_2 = 0$, $x_1 \neq 0$. Thus energy will keep decreasing toward 0 which is achieved at the origin.

Stability from an energy viewpoint: Generalization

Consider unforced, time-varying, nonlinear systems

$$\dot{x}(t) = f\left(x(t), t\right), \ x\left(t_0\right) = x_0$$
$$x\left(k+1\right) = f\left(x(k), k\right), \ x\left(k_0\right) = x_0$$

- assume the origin is an equilibrium state
- energy function $\Rightarrow$ Lyapunov function: a scalar function of x and t (or x and k)
- goal is to relate properties of the state through the Lyapunov function
- main tool: matrix formulation, linear algebra, positive definite functions

Relevant tools

Quadratic functions

- intrinsic in energy-like analysis, e.g.

$$\frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2 = \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- convenience of matrix formulation:

$$\frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2 + x_1x_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} \frac{k}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{m}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2 + x_1x_2 + c = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{k}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{m}{2} & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$$

- general quadratic functions in matrix form

$$Q(x) = x^T P x, \; P^T = P$$

Relevant tools

Symmetric matrices

- recall: a real square matrix A is
  - *symmetric* if $A = A^T$
  - *skew-symmetric* if $A = -A^T$
- examples:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

- *Any* real square matrix can be decomposed as the sum of a *symmetric* matrix and a *skew-symmetric* matrix:

$$\text{e.g.} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2.5 \\ 2.5 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix}$$

$$\text{general case:} \quad P = \frac{P + P^T}{2} + \frac{P - P^T}{2}$$

Relevant tools

Symmetric matrices

- a real square matrix $A \in \mathbb{R}^{n\times n}$ is *orthogonal* if $A^TA = AA^T = I$
- meaning that the columns of A form a orthonormal basis of $\mathbb{R}^n$

$$A = \begin{bmatrix} | & | & | & | \\ a_1 & a_2 & \dots & a_n \\ | & | & | & | \end{bmatrix}$$

$$A^TA = \begin{bmatrix} a_1^Ta_1 & a_1^Ta_2 & \dots & a_1^Ta_n \\ a_2^Ta_1 & a_2^Ta_2 & \dots & a_2^Ta_n \\ \vdots & \vdots & \vdots & \vdots \\ a_n^Ta_1 & a_n^Ta_2 & \dots & a_n^Ta_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

namely, $a_j^Ta_j = 1$ and $a_j^Ta_m = 0\ \forall j \neq m$.

Theorem

The eigenvalues of symmetric matrices are all real.

Proof: $\forall : A \in \mathbb{R}^{n\times n}$ with $A^T = A$.

Eigenvalue-eigenvector pair: $Au = \lambda u \Rightarrow \overline{u}^T A u = \lambda \overline{u}^T u$, where $\overline{u}$ is the complex conjugate of u. $\overline{u}^T A u$ is a real number, as

$$\begin{aligned}
\overline{\overline{u}^T A u} &= u^T \overline{A} \overline{u} \\
&= u^T A \overline{u} \quad \because A \in \mathbb{R}^{n\times n} \\
&= u^T A^T \overline{u} \quad \because A = A^T \\
&= \lambda u^T \overline{u} \quad \because (Au)^T = (\lambda u)^T \\
&= \lambda \overline{u}^T u \quad \because u^T \overline{u} \in \mathbb{R} \\
&= \overline{u}^T A u \quad \because Au = \lambda u
\end{aligned}$$

Also, $\overline{u}^T u \in \mathbb{R}$. Thus $\lambda = \frac{\overline{u}^T A u}{\overline{u}^T u}$ must also be a real number. □

Example

- $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$: $\lambda = \pm 2$
- $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$: $\lambda = 1 \pm 2$

```
import numpy as np #larger-scale Python example
N = 100
P = np.random.randint(-200,200,size=(N,N))
P_symm = (P + P.T)/2
lambdas, _ = np.linalg.eig(P_symm)
print(lambdas)
```

Theorem

The eigenvalues of skew-symmetric matrices are all imaginary or zero.

- $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$: $\lambda = \pm 2j$

```
import numpy as np
N = 100
P = np.random.randint(-200,200,size=(N,N))
P_symm = (P - P.T)/2
lambdas, _ = np.linalg.eig(P_symm)
print(lambdas)
```

Theorem

All eigenvalues of an orthogonal matrix have a magnitude of 1.

- $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} : \lambda = 1 \pm 2j$

```
import numpy as np
from scipy.linalg import qr
n = 3
H = np.random.randn(n, n)
Q, _ = qr(H)
print (np.dot(Q,Q.T))
print (np.dot(Q.T,Q))
```

Important properties of symmetric matrices

Theorem

The eigenvalues of symmetric matrices are all real.

Theorem

The eigenvalues of skew-symmetric matrices are all imaginary or zero.

Theorem

All eigenvalues of an orthogonal matrix have a magnitude of 1.

matrix structure	analogy in complex plane
symmetric	real line
skew-symmetric	imaginary line
orthogonal	unit circle

The spectral theorem for symmetric matrices

When $A \in \mathbb{R}^{n\times n}$ has n distinct eigenvalues, we can do diagonalization $A = U\Lambda U^{-1}$. When A is symmetric, things are even better:

Theorem (Symmetric eigenvalue decomposition (SED))

$\forall: A \in \mathbb{R}^{n\times n},\ A^T = A$, there always exist $\lambda_i \in \mathbb{R}$ and $u_i \in \mathbb{R}^n$, s.t.

$$A = \sum_{i=1}^{n} \lambda_i u_i u_i^T = U\Lambda U^T \tag{1}$$

- λ_i's: eigenvalues of A
- u_i: eigenvector associated to λ_i, normalized to have unity norms
- $U = [u_1, u_2, \cdots, u_n]$ is orthogonal: $U^T U = UU^T = I$
- $\Lambda = diagonal(\lambda_1, \lambda_2, \ldots, \lambda_n)$

Elements of proof for SED

Theorem

$\forall : A \in \mathbb{R}^{n\times n}$ with $A^T = A$, *then eigenvectors of A, associated with different eigenvalues, are* ***orthogonal****.*

Proof.

Let $Au_i = \lambda_i u_i$ and $Au_j = \lambda_j u_j$. Then $u_i^T A u_j = u_i^T \lambda_j u_j = \lambda_j u_i^T u_j$. Also, $u_i^T A u_j = u_i^T A^T u_j = (Au_i)^T u_j = \lambda_i u_i^T u_j$. So $\lambda_i u_i^T u_j = \lambda_j u_i^T u_j$. But $\lambda_i \neq \lambda_j$. It must be that $u_i^T u_j = 0$. □

SED now follows:

- If A has distinct eigenvalues, then $U = [u_1, u_2, \cdots, u_n]$ is orthogonal after normalizing all the eigenvectors to unity norm.
- If A has $r(< n)$ distinct eigenvalues, we can *choose* multiple orthogonal eigenvectors for the eigenvalues with none-unity multiplicities.

Rethinking symmetric matrices

With the spectral theorem, next time we see a symmetric matrix A, we immediately know that

- λ_i is real for all i
- associated with λ_i, we can always find a real eigenvector
- $\exists$ an orthonormal basis $\{u_i\}_{i=1}^n$, which consists of the eigenvectors
- if $A \in \mathbb{R}^{2\times 2}$, then if you compute first λ_1, λ_2 and u_1, you won't need to go through the regular math to get u_2, but can simply solve for a u_2 that is orthogonal to u_1 with $\|u_2\| = 1$.

Example: $A = \begin{bmatrix} 5 & \sqrt{3} \\ \sqrt{3} & 7 \end{bmatrix}$

Computing the eigenvalues gives

$$\det \begin{bmatrix} 5-\lambda & \sqrt{3} \\ \sqrt{3} & 7-\lambda \end{bmatrix} = 35 - 12\lambda + \lambda^2 - 3 = (\lambda - 4)(\lambda - 8) = 0$$

$$\Rightarrow \lambda_1 = 4,\ \lambda_2 = 8$$

- first normalized eigenvector:

$$(A - \lambda_1 I)\, t_1 = 0 \Rightarrow \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix} t_1 = 0 \Rightarrow t_1 = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

- A is symmetric $\Rightarrow$ eigenvectors are orthogonal to each other: *choose* $t_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$. No need to solve $(A - \lambda_2 I)\, t_2 = 0$!

Theorem (Eigenvalues of symmetric matrices)

If $A = A^T \in \mathbb{R}^{n\times n}$, then the eigenvalues of A satisfy

$$\lambda_{\max} = \max_{x\in\mathbb{R}^n,\ x\neq 0} \frac{x^T A x}{\|x\|_2^2} \tag{2}$$

$$\lambda_{\min} = \min_{x\in\mathbb{R}^n,\ x\neq 0} \frac{x^T A x}{\|x\|_2^2} \tag{3}$$

Proof.

Perform SED to get $A = \sum_{i=1}^n \lambda_i u_i u_i^T$ where $\{u_i\}_{i=1}^n$ spans $\mathbb{R}^n$. Then any vector $x \in \mathbb{R}^n$ can be decomposed as $x = \sum_{i=1}^n \alpha_i u_i$. Thus

$$\max_{x\neq 0} \frac{x^T A x}{\|x\|_2^2} = \max_{\alpha_i} \frac{\left(\sum_i \alpha_i u_i\right)^T \sum_i \lambda_i \alpha_i u_i}{\sum_i \alpha_i^2} = \max_{\alpha_i} \frac{\sum_i \lambda_i \alpha_i^2}{\sum_i \alpha_i^2} = \lambda_{\max}$$

□

Positive definite matrices

- eigenvalues of symmetric matrices are real $\Rightarrow$ we can order the eigenvalues
- a symmetric matrix P is called positive-definite if all its eigenvalues are positive
- equivalently:

Definition (Positive Definite Matrices)

A symmetric matrix $P \in \mathbb{R}^{n \times n}$ is called **positive-definite**, written $P \succ 0$, if $x^T P x > 0$ for all $x \, (\neq 0) \in \mathbb{R}^n$.
P is called **positive-semidefinite,** written $P \succeq 0$, if $x^T P x \geq 0$ for all $x \in \mathbb{R}^n$

- $P \succ 0$ $(P \succeq 0) \Leftrightarrow P$ can be decomposed as $P = N^T N$ where N is nonsingular (singular)

Negative definite matrices

Definition

A symmetric matrix $Q \in \mathbb{R}^{n\times n}$ is called **negative-definite**, written $Q \prec 0$, if $-Q \succ 0$, i.e., $x^T Qx < 0$ for all $x\,(\neq 0) \in \mathbb{R}^n$.
Q is called **negative-semidefinite,** written $Q \preceq 0$, if $x^T Qx \leq 0$ for all $x \in \mathbb{R}^n$

Updated matrix analogies

matrix structure	eigenvalues	analogy in complex plane
symmetric	real	real axis
skew-symmetric	on imaginary axis	imaginary axis
orthogonal	magnitude 1	unit circle
positive definite	positive	$\mathbb{R}_+$ axis
negative definite	negative	$\mathbb{R}_-$ axis

Caution

- positive-definite matrices can have negative entries:

Example

$P = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ is positive-definite, as $P = P^T$ and take any $v = [x, y]^T$, we have

$$
\begin{aligned}
v^T P v &= \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2x^2 + 2y^2 - 2xy \\
&= x^2 + y^2 + (x - y)^2 \geq 0
\end{aligned}
$$

and the equality sign holds only when $x = y = 0$.

Caution

- conversely, matrices whose entries are all positive are not necessarily positive-definite:

Example

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is not positive-definite:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -2 < 0$$

Positive definite matrices

Theorem

For a symmetric matrix P, $P \succ 0$ if and only if all the eigenvalues of P are positive.

Proof.

Since P is symmetric, we have

$$\lambda_{\max}(P) = \max_{x \in \mathbb{R}^n,\ x \neq 0} \frac{x^T A x}{\|x\|_2^2} \tag{4}$$

$$\lambda_{\min}(P) = \min_{x \in \mathbb{R}^n,\ x \neq 0} \frac{x^T A x}{\|x\|_2^2} \tag{5}$$

which gives $x^T A x \in [\lambda_{\min}\|x\|_2^2,\ \lambda_{\max}\|x\|_2^2]$. Thus $x^T A x > 0,\ x \neq 0 \Leftrightarrow \lambda_{\min} > 0$. □

We often use the following necessary and sufficient conditions to check positive (semi-)definiteness:

- $P \succ 0$ ($P \succeq 0$) $\Leftrightarrow$ the leading principle minors defined below are positive (nonnegative)

Definition

The leading principle minors of $P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$ are defined as

p_{11}, $\det \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$, $\det P$.

Relevant tools

Checking positive definiteness of a matrix.

Example

None of the following matrices are positive definite:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Recap

energy

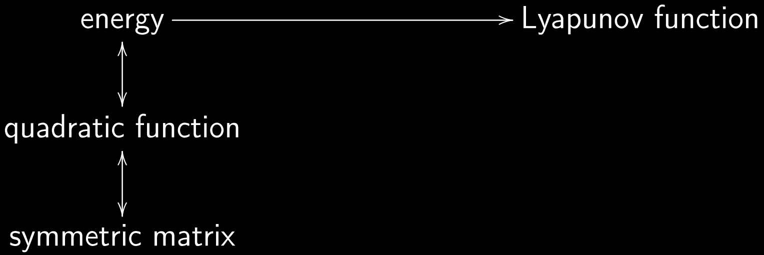
$\Updownarrow$

quadratic function

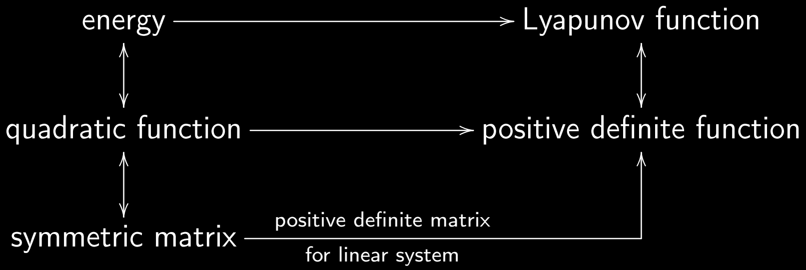
$\Updownarrow$

symmetric matrix

Recap


energy
Lyapunov function
quadratic function
symmetric matrix

Recap


energy
Lyapunov function
quadratic function
positive definite function
symmetric matrix
positive definite matrix
for linear system

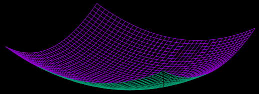
Relevant tools

Definition (Positive Definite Functions)

A continuous time function $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$, called to be PD, satisfying

- $W(x) > 0$ for all $x \neq 0$
- $W(0) = 0$
- $W(x) \rightarrow \infty$ as $|x| \rightarrow \infty$ uniformly in x

In the 3D space, positive definite functions are "bowl-shaped", e.g., $W(x_1, x_2) = x_1^2 + x_2^2$.



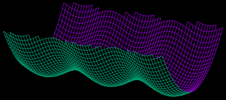
Relevant tools

Definition (Locally Positive Definite Functions)

A continuous time function $W : \mathbb{R}^n \to \mathbb{R}_+$, called to be LPD, satisfying

- $W(x) > 0$ for all $x \neq 0$ and $|x| < r$
- $W(0) = 0$

In the 3D space, locally positive definite functions are "bowl-shaped" locally, e.g., $W(x_1, x_2) = x_1^2 + \sin^2 x_2$ for $x_1 \in \mathbb{R}$ and $|x_2| < \pi$



Relevant tools

Exercise

Let $x = [x_1, x_2, x_3]^T$. Check the positive definiteness of the following functions

1. $V(x) = x_1^4 + x_2^2 + x_3^4$ (PD)
2. $V(x) = x_1^2 + x_2^2 + 3x_3^2 - x_3^4$ (LPD for $|x_3| < \sqrt{3}$)

1. Definitions in Lyapunov stability analysis

2. Lyapunov's approach to stability
    - Relevant tools
    - Lyapunov stability theorems
    - Instability theorem
    - Discrete-time case

3. Recap

Lyapunov stability theorems

- recall the spring mass damper example in matrix form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- energy function is PD:
  $\mathcal{E}(t) =$ potential energy + kinetic energy $= \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$
  and its derivative is NSD:

$$\dot{\mathcal{E}}(t) = \left[\frac{\partial \mathcal{E}}{\partial x_1}, \frac{\partial \mathcal{E}}{\partial x_2}\right]\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = k_1 x_1 \dot{x}_1 + m x_2 \dot{x}_2 \tag{6}$$

$$= k_1 x_1 x_2 + m x_2 \left(-\frac{k}{m}x_1 - \frac{b}{m}x_2\right) = \left[\frac{\partial \mathcal{E}}{\partial x_1}, \frac{\partial \mathcal{E}}{\partial x_2}\right] Ax \tag{7}$$

$$= -bx_2^2$$

Theorem

The equilibrium point 0 of $\dot{x}(t) = f(x(t), t)$, $x(t_0) = x_0$ *is* ***stable in the sense of Lyapunov*** *if there exists a locally positive definite function* $V(x,t)$ *such that* $\dot{V}(x,t) \le 0$ *for all* $t \ge t_0$ *and all* x *in a local region* $x : |x| < r$ *for some* $r > 0$.

- such a $V(x,t)$ is called a Lyapunov function
- i.e., $V(x)$ is PD and $\dot{V}(x)$ is negative semidefinite in a local region $|x| < r$

Theorem

The equilibrium point 0 of $\dot{x}(t) = f(x(t), t)$, $x(t_0) = x_0$ *is* ***locally asymptotically stable*** *if there exists a Lyapunov function* $V(x)$ *such that* $\dot{V}(x)$ *is locally negative definite.*

Theorem

The equilibrium point 0 of $\dot{x}(t) = f\left(x(t), t\right), \ x\left(t_0\right) = x_0$ *is globally asymptotically stable if there exists a Lyapunov function* $V(x)$ *such that* $V(x)$ *is positive definite and* $\dot{V}(x)$ *is negative definite.*

Lyapunov stability concept for linear systems

- for linear system $\dot{x} = Ax$, a good Lyapunov candidate is the quadratic function $V(x) = x^T P x$ where $P = P^T$ and $P \succ 0$
- the derivative along the state trajectory is then

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax)^T P x + x^T P A x \\ &= x^T \left(A^T P + PA\right) x \end{aligned}$$

- such a $V(x) = x^T P x$ is a Lyapunov function for $\dot{x} = Ax$ when $A^T P + PA \preceq 0$
- and the origin is stable in the sense of Lyapunov

Theorem (Lyapunov stability theorem for linear systems)

For $\dot{x} = Ax$ with $A \in \mathbb{R}^{n \times n}$, the origin is asymptotically stable if and only if for any symmetric positive definite matrix $Q \succ 0$, the Lyapunov equation

$$\boxed{A^T P + PA = -Q}$$

has a unique positive definite solution $P \succ 0$, $P^T = P$.

Essense of the Lyapunov Eq.

Observations:

- $A^T P + PA$ is a linear operation on P: e.g.,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad Q = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix}, \quad P = \begin{bmatrix} | & | \\ p_1 & p_2 \\ | & | \end{bmatrix}$$

$$A^T \begin{bmatrix} | & | \\ p_1 & p_2 \\ | & | \end{bmatrix} + \begin{bmatrix} | & | \\ p_1 & p_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = - \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix}$$

$$A^T p_1 + a_{11} p_1 + a_{21} p_2 = -q_1$$
$$A^T p_2 + a_{12} p_1 + a_{22} p_2 = -q_2$$

Essense of the Lyapunov Eq.

Observations: with now

$$A^T P + PA = Q \Leftrightarrow \begin{cases} A^T p_1 + a_{11} p_1 + a_{21} p_2 & = -q_1 \\ A^T p_2 + a_{12} p_1 + a_{22} p_2 & = -q_2 \end{cases}$$

- can stack the columns of $A^T P + PA$ and Q to yield

$$\begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} a_{11} I & a_{21} I \\ a_{12} I & a_{22} I \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = - \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\underbrace{\left\{ \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} + \begin{bmatrix} a_{11} I & a_{21} I \\ a_{12} I & a_{22} I \end{bmatrix} \right\}}_{L_A} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = - \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

The Lyapunov Eq.: Existence of solution

$$L_A(P) = A^T P + PA$$

- L_A is invertible if and only if $\lambda_i + \lambda_j \neq 0$ for all eigenvalues of A:
  - let $A^T u_i = \lambda_i u_i$ and $A^T u_j = \lambda_j u_j$
  - $L_A\left(u_i u_j^T\right) = u_i u_j^T A + A^T u_i u_j^T = u_i\left(\lambda_j u_j\right)^T + \lambda_i u_i u_j^T = \left(\lambda_i + \lambda_j\right) u_i u_j^T$
  - so $\lambda_i + \lambda_j$ is an eigenvalue of the operator $L_A(\cdot)$
  - if $\lambda_i + \lambda_j \neq 0$, the operator is invertible

The Lyapunov operator: eigenvalues

$$L_A = \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} + \begin{bmatrix} a_{11}I & a_{21}I \\ a_{12}I & a_{22}I \end{bmatrix}$$

- can simply write $L_A = \underbrace{I \otimes A^T + A^T \otimes I}_{\text{mirror symmetric}}$ using the Kronecker product notation $B \otimes C = \begin{bmatrix} b_{11}C & b_{11}C & \dots & b_{11}C \\ b_{21}C & b_{22}C & \dots & b_{2n}C \\ \vdots & \vdots & \dots & \vdots \\ b_{m1}C & b_{m2}C & \dots & b_{mn}C \end{bmatrix}$

The Lyapunov operator: eigenvalues

$$L_A = \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} + \begin{bmatrix} a_{11}I & a_{21}I \\ a_{12}I & a_{22}I \end{bmatrix}$$

- e.g., $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

$$L_A = I \otimes A^T + A^T \otimes I = \begin{bmatrix} A^T + a_{11}I & a_{21}I \\ a_{12}I & A^T + a_{22}I \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} -1-1 & -1 & -1 & 0 \\ 1 & 0-1 & 0 & -1 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right] = \left[\begin{array}{cc|cc} -2 & -1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Example: $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, $\lambda_{1,2} = -0.5 \pm i\sqrt{3}/2$

$$L_A = I \otimes A^T + A^T \otimes I = \left[\begin{array}{cc|cc} -2 & -1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

The eigenvalues of L_A are -1, -1, $-1-\sqrt{3}$, $-1+\sqrt{3}$, which are precisely $\lambda_1+\lambda_1$, $\lambda_1+\lambda_2$, $\lambda_2+\lambda_1$, $\lambda_2+\lambda_2$.

```
import numpy as np
A = [[-1,1],[-1,0]]; I2=np.eye(2); AT=np.transpose(A)
L_A=np.kron(I2,AT)+np.kron(AT,I2)
eigLA,_=np.linalg.eig(L_A)
eigA,_=np.linalg.eig(A)
print(eigLA)
print(eigA)
```

Theorem (Lyapunov stability theorem for linear systems)

For $\dot{x} = Ax$ with $A \in \mathbb{R}^{n \times n}$, the origin is asymptotically stable if and only if for any symmetric positive definite matrix $Q \succ 0$, the Lyapunov equation

$$\boxed{A^T P + PA = -Q}$$

has a unique positive definite solution $P \succ 0$, $P^T = P$.

Proof.

"$\Rightarrow$": $\frac{\dot{V}}{V} = -\frac{x^T Q x}{x^T P x} \leq -\underbrace{\frac{(\lambda_Q)_{\min}}{(\lambda_P)_{\max}}}_{\triangleq \alpha} \Longrightarrow V(t) \leq e^{-\alpha t} V(0)$. $Q \succ 0$ and $P \succ 0 \Rightarrow (\lambda_Q)_{\min} > 0$ and $(\lambda_P)_{\max} > 0$. Thus $\alpha > 0$; $V(t)$ decays exponentially to zero. $V(x) \succ 0 \Rightarrow V(x) = 0$ only at $x = 0$. Therefore, $x \to 0$ as $t \to \infty$, regardless of the initial condition. □

Proof.

"$\Leftarrow$": if 0 of $\dot{x} = Ax$ is asymptotically stable, then all eigenvalues of A have negative real parts. For any Q, the Lyapunov equation has a unique solution P. Note $x(t) = e^{At}x_0 \to 0$ as $t \to \infty$. We have

$$\underbrace{x^T(\infty)Px(\infty)}_{\to 0} - x^T(0)Px(0) = \int_0^\infty \frac{d}{dt}x^T(t)Px(t)\,dt = \int_0^\infty x^T(t)\left(A^TP + PA\right)x(t)\,dt$$

$$\Rightarrow x(0)^T Px(0) = \int_0^\infty x^T(t)Qx(t)\,dt = \int_0^\infty x(0)\,e^{A^Tt}Qe^{At}x(0)\,dt$$

If $Q \succ 0$, there exists a nonsingular N matrix: $Q = N^TN$. Thus

$$x(0)^T Px(0) = \int_0^\infty \|Ne^{At}x(0)\|^2 dt \geq 0$$

$$x(0)^T Px(0) = 0 \text{ only if } x_0 = 0$$

Thus $P \succ 0$. Furthermore

$$\boxed{P = \int_0^\infty e^{A^Tt}Qe^{At}dt}$$

□

Procedures of Lyapunov's direct method

1. Given A, select an arbitrary positive-definite symmetric matrix Q (e.g., I).
2. Find the solution matrix P to the Lyapunov equation $A^T P + PA = -Q$.
3. If a solution P cannot be found, the origin is not asymptotically stable.
4. If a solution is found:
   - if P is positive-definite, then A is Hurwitz stable and the origin is asymptotically stable;
   - if P is not positive-definite, then A has at least one eigenvalue with a positive real part and the origin is an unstable equilibrium.

Lyapunov stability theorems

Example

$\dot{x} = Ax$, $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$. The Lyapunov equation is

$$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}^T \underbrace{\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}}_{P} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = -\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{Q}$$

We need

$$\begin{cases} -2p_{11} - 2p_{12} = -1 \\ -p_{12} - p_{22} + p_{11} = 0 \\ 2p_{12} = -1 \end{cases} \Rightarrow \begin{cases} p_{11} = 1 \\ p_{22} = 3/2 \\ p_{12} = -1/2 \end{cases}$$

Leading principle minors: $p_{11} > 0$, $p_{11}p_{22} - p_{12}^2 > 0$
$\Rightarrow P \succ 0 \Rightarrow$asymptotically stable

Lyapunov analysis with Matlab

$$\dot{x} = Ax,\ A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

```
A = [-1,1;-1,0]
Q = eye(2)
P = lyap(A',Q)
w = eig(P)
```

Lyapunov analysis with Python

$\dot{x} = Ax$, $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$.

```
import control as ct
import numpy as np
A = np.array([[-1,1],[-1,0]])
Q = np.identity(2)
P = ct.lyap(A.transpose(),Q)
print(P)
w = np.linalg.eigvals(P)
print(w)
```

It suffices to select $Q = I$

For linear systems we can let $Q = I$ and check whether the resulting P is positive definite. If it is, then we can assert the asymptotic stability:

- take any $Q \succ 0$. there exists $Q = N^T N$, where N is invertible, yielding

$$A^T P + PA = -I$$
$$\Updownarrow$$
$$\underbrace{N^T A^T N^{-T}}_{\tilde{A}^T} \underbrace{N^T P N}_{\tilde{P}} + \underbrace{N^T P N}_{\tilde{P}} \underbrace{N^{-1} A N}_{\tilde{A}} = -N^T N$$

  - $\tilde{A} = N^{-1} A N$ and A are similar matrices and have the same eigenvalues.
  - $\tilde{P} = N^T P N$ and P have the same definiteness. If we can find a positive definite solution P then the $\tilde{P}$ will also be positive definite. Vise versa.

Instability theorem

- for nonlinear systems, Lyapunov function can be nontrivial to find
- failure to find a Lyapunov function does not imply instability

Theorem

The equilibrium state 0 *of* $\dot{x} = f(x)$ *is unstable if there exists a function* $W(x)$ *such that*

- $\dot{W}(x)$ *is PD locally:* $\dot{W}(x) > 0 \; \forall \, |x| < r$ *for some* r *and* $\dot{W}(0) = 0$
- $W(0) = 0$
- *there exist states* x *arbitrarily close to the origin such that* $W(x) > 0$

Discrete-time case: key concept of Lyapunov

For the discrete-time system

$$x(k+1) = Ax(k)$$

we consider a quadratic Lyapunov function candidate

$$V(x) = x^T P x,\ \ P = P^T \succ 0$$

and compute $\Delta V(x)$ along the trajectory of the state

$$V(x(k+1)) - V(x(k)) = x^T(k)\underbrace{\left(A^T P A - P\right)}_{\triangleq -Q} x(k)$$

Asymptotic stability desires $\Delta V(x)$ to be negative.

DT Lyapunov stability theorem for linear systems

Theorem

For system $x(k+1) = Ax(k)$ *with* $A \in \mathbb{R}^{n\times n}$, *the origin is asymptotically stable if and only if* $\exists\ Q \succ 0$, *such that the discrete-time Lyapunov equation*

$$\boxed{A^T PA - P = -Q}$$

has a unique positive definite solution $P \succ 0$, $P^T = P$.

The DT Lyapunov Eq.

$$\boxed{A^T PA - P = -Q}$$

- Solution to the DT Lyapunov equation, when asymptotic stability holds (A is Schur stable), comes from:

$$\cancelto{0}{V(x(\infty))} - V(x(0)) = \sum_{k=0}^{\infty} x^T(k)\left[A^T PA - P\right]x(k)$$
$$= -\sum_{k=0}^{\infty} x^T(0)\left(A^T\right)^k QA^k x(0)$$
$$\Rightarrow P = \sum_{k=0}^{\infty}\left(A^T\right)^k QA^k$$

- can show that the DT Lyapunov operator $L_A = A^T PA - P$ is invertible if and only if $\forall i, j\ (\lambda_A)_i (\lambda_A)_j \neq 1$

DT Lyapunov analysis with MATLAB

Example

$$x(k+1) = Ax(k), \;\; A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.275 & -0.225 & -0.1 \end{bmatrix}$$

```
% MATLAB
A=[ 0 1 0; 0 0 1; 0.275 -0.225 -0.1]
Q = eye(3)
P = dlyap(A',Q) % check function definition in Matlab help
eig(P)
```

DT Lyapunov analysis with Python

Example

$$x(k+1) = Ax(k),\ \ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.275 & -0.225 & -0.1 \end{bmatrix}$$

```
#Python
import control as ct
import numpy as np
from numpy.linalg import eig
A = np.array([[0,1,0],[0,0,1],[0.275,-0.225,-0.1]])
Q = np.identity(3)
P = ct.dlyap(A.transpose(),Q)
w,v = eig(P)
print(w)
```

Recap

- Internal stability
  - Stability in the sense of Lyapunov: ε, δ conditions
  - Asymptotic stability
- Stability analysis of linear time invariant systems ($\dot{x} = Ax$ or $x(k+1) = Ax(k)$)
  - Based on the eigenvalues of A
    - Time response modes
    - Repeated eigenvalues on the imaginary axis
  - Routh's criterion
    - No need to solve the characteristic equation
    - Discrete time case: bilinear transform ($z = \frac{1+s}{1-s}$)

Recap

- Lyapunov equations
  Theorem: All eigenvalues of A have negative real parts iff for any given $Q \succ 0$, the Lyapunov equation

  $$A^T P + PA = -Q$$

  has a unique solution P and $P \succ 0$.
  Given Q, the Lyapunov equation $A^T P + PA = -Q$ has a unique solution when $\lambda_{A,i} + \lambda_{A,j} \neq 0$ for all i and j.
  Theorem: All eigenvalues of A are inside the unit circle iff for any given $Q \succ 0$, the Lyapunov equation

  $$A^T PA - P = -Q$$

  has a unique solution P and $P \succ 0$.
  Given Q, the Lyapunov equation $A^T PA - P = -Q$ has a unique solution when $\lambda_{A,i}\lambda_{A,j} \neq 1$ for all i and j.

Recap

- P is positive definite if and only if any one of the following conditions holds:
  1. All the eigenvalues of P are positive.
  2. All the leading principle minors of P are positive.
  3. There exists a nonsingular matrix N such that $P = N^T N$.