Linear Systems: Stability

1. Definitions in Lyapunov stability analysis

2. Stability of LTI systems: method of eigenvalue/pole locations

Finite dimensional vector norms

Let $v \in \mathbb{R}^n$. A norm is:

- a metric in vector space: a function that assigns a real-valued length to each vector in a vector space
- e.g., 2 (Euclidean) norm: $\|v\|_2 = \sqrt{v^T v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

default in this set of notes: $\|\cdot\| = \|\cdot\|_2$

Equilibrium state

For an *n*-th order unforced system

$$\dot{x} = f(x, t), \; x(t_0) = x_0$$

an equilibrium state/point x_e is one such that

$$\boxed{f(x_e, t) = 0, \; \forall t}$$

- the condition must be satisfied by all $t \geq 0$
- if a system starts at equilibrium state, it stays there

Equilibrium state of a linear system

For a linear system

$$\dot{x}(t) = A(t)x(t), \; x(t_0) = x_0$$

- origin $x_e = 0$ is always an equilibrium state
- when $A(t)$ is singular, multiple equilibrium states exist

Lyapunov's definition of stability

- The equilibrium state 0 of $\dot{x} = f(x, t)$ is *stable in the sense of Lyapunov (s.i.L)* if for all $\epsilon > 0$, and t_0, there exists $\delta(\epsilon, t_0) > 0$ such that $\|x(t_0)\|_2 < \delta$ gives $\|x(t)\|_2 < \epsilon$ for all $t \geq t_0$

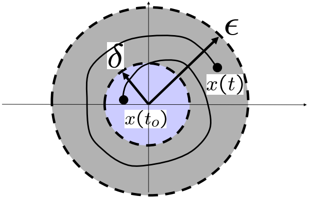


Figure: Stable s.i.L: $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon \; \forall t \geq t_0$.

Asymptotic stability

The equilibrium state 0 of $\dot{x} = f(x, t)$ is asymptotically stable if

- it is stable in the sense of Lyapunov, and
- for all $\epsilon > 0$ and t_0, there exists $\delta(\epsilon, t_0) > 0$ such that $\|x(t_0)\|_2 < \delta$ gives $x(t) \to 0$

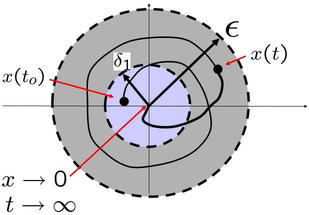


Figure: Asymptotically stable i.s.L: $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| \to 0$.

1. Definitions in Lyapunov stability analysis

2. Stability of LTI systems: method of eigenvalue/pole locations

Stability of LTI systems: method of eigenvalue/pole locations

the stability of the equilibrium point 0 for $\dot{x} = Ax$ or $x(k+1) = Ax(k)$ can be concluded immediately based on $\lambda(A)$:

- the response $e^{At}x(t_0)$ involves modes such as $e^{\lambda t}$, $te^{\lambda t}$, $e^{\sigma t}\cos\omega t$, $e^{\sigma t}\sin\omega t$
- the response $A^k x(k_0)$ involves modes such as λ^k, $k\lambda^{k-1}$, $r^k\cos k\theta$, $r^k\sin k\theta$
- $e^{\sigma t} \to 0$ if $\sigma < 0$; $e^{\lambda t} \to 0$ if $\lambda < 0$
- $\lambda^k \to 0$ if $|\lambda| < 1$; $r^k \to 0$ if $|r| = \left|\sqrt{\sigma^2 + \omega^2}\right| = |\lambda| < 1$

Stability of the origin for $\dot{x} = Ax$

stability at 0	$\lambda_i(A)$
unstable	$\mathrm{Re}\{\lambda_i\} > 0$ for some λ_i or $\mathrm{Re}\{\lambda_i\} \leq 0$ for all λ_i's but for a repeated λ_m on the imaginary axis with multiplicity m, nullity $(A - \lambda_m I) < m$ (Jordan form)
stable i.s.L	$\mathrm{Re}\{\lambda_i\} \leq 0$ for all λ_i's and $\forall$ repeated λ_m on the imaginary axis with multiplicity m, nullity $(A - \lambda_m I) = m$ (diagonal form)
asymptotically stable	$\mathrm{Re}\{\lambda_i\} < 0 \; \forall \lambda_i$ (A is then called Hurwitz stable)

Example (Unstable moving mass)

$$\dot{x} = Ax,\ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- $\lambda_1 = \lambda_2 = 0,\ m = 2,$
  $\text{nullity}\,(A - \lambda_i I) = \text{nullity}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 1 < m$
- i.e., two repeated eigenvalues but needs a generalized eigenvector $\Rightarrow$ Jordan form after similarity transform
- verify by checking $e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$: t grows unbounded

Example (Stable in the sense of Lyapunov)

$$\dot{x} = Ax, \; A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- $\lambda_1 = \lambda_2 = 0, \; m = 2,$
  $\text{nullity}\,(A - \lambda_i I) = \text{nullity} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 2 = m$
- verify by checking $e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Routh-Hurwitz criterion

- the Routh Test (by E.J. Routh, in 1877): a simple algebraic procedure to determine how many roots a given polynomial

  $$A(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

  has in the closed right-half complex plane, without the need to explicitly solve for the roots
- German mathematician Adolf Hurwitz independently proposed in 1895 to approach the problem from a matrix perspective
- popular if stability is the only concern and no details on eigenvalues (e.g., speed of response) are needed

Routh-Hurwitz criterion

- the asymptotic stability of the equilibrium point 0 for $\dot{x} = Ax$ can also be concluded based on the Routh-Hurwitz criterion
- simply apply the Routh Test to $A(s) = \det(sI - A)$
- recap: the poles of transfer function $G(s) = C(sI - A)^{-1} B + D$ come from $\det(sI - A)$ in computing the inverse $(sI - A)^{-1}$

The Routh Array

for $A(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$, construct

$$
\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & q_{n-2} & q_{n-4} & q_{n-6} & \cdots & \\
s^{n-3} & q_{n-3} & q_{n-5} & q_{n-7} & \cdots & \\
\vdots & \vdots & \vdots & \vdots & & \\
s^1 & x_2 & x_0 & & & \\
s^0 & x_0 & & & &
\end{array}
$$

- first two rows contain the coefficients of $A(s)$
- third row constructed from the previous two rows via

$$
\begin{array}{c|cccc}
\cdot & a & b & x & \cdot \\
\cdot & c & d & y & \cdot \\
\cdot & \dfrac{bc-ad}{c} & \dfrac{xc-ay}{c} & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

The Routh Array

for $A(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$, construct

$$
\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & q_{n-2} & q_{n-4} & q_{n-6} & \cdots & \\
s^{n-3} & q_{n-3} & q_{n-5} & q_{n-7} & \cdots & \\
\vdots & \vdots & \vdots & \vdots & & \\
s^1 & x_2 & x_0 & & & \\
s^0 & x_0 & & & &
\end{array}
$$

- **All roots of $A(s)$ are on the left half s-plane if and only if all elements of the first column of the Routh array are positive.**

The Routh Array

Example ($A(s) = 2s^4 + s^3 + 3s^2 + 5s + 10$)

$$
\begin{array}{c|ccc}
s^4 & 2 & 3 & 10 \\
s^3 & 1 & 5 & 0 \\
s^2 & 3 - \frac{2\times 5}{1} = -7 & 10 & 0 \\
s^1 & 5 - \frac{1\times 10}{-7} & 0 & 0 \\
s^0 & 10 & 0 & 0
\end{array}
$$

- two sign changes in the first column
- unstable and two roots in the right half side of s-plane

The Routh Array

special cases:

- If the 1st element in any one row of Routh's array is zero, one can replace the zero with a small number ϵ and proceed further.
- There are other possible complications, which we will not pursue further. See, e.g. "Automatic Control Systems", by Kuo, 7th ed., pp. 339-340.

Stability of the origin for $x(k+1) = f(x(k), k)$

- stability analysis follows analogously for nonlinear time-varying discrete-time systems of the form

$$x(k+1) = f(x(k), k), \; x(k_0) = x_0$$

- equilibrium point x_e:

$$f(x_e, k) = x_e, \; \forall k$$

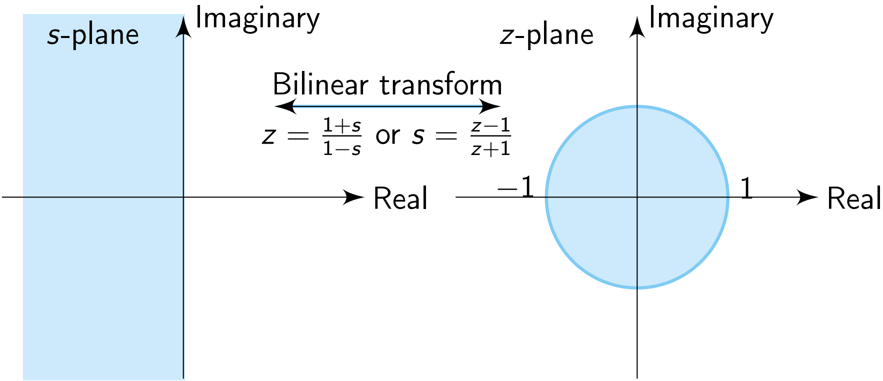
- without loss of generality, 0 is assumed an equilibrium point

Stability of the origin for $x(k+1) = Ax(k)$

stability at 0	$\lambda_i(A)$
unstable	$\lvert\lambda_i\rvert > 1$ for some λ_i or $\lvert\lambda_i\rvert \leq 1$ for all λ_i's but for a repeated λ_m on the unit circle with multiplicity m, nullity $(A - \lambda_m I) < m$ (Jordan form)
stable i.s.L	$\lvert\lambda_i\rvert \leq 1$ for all λ_i's but for any repeated λ_m on the unit circle with multiplicity m, nullity $(A - \lambda_m I) = m$ (diagonal form)
asymptotically stable	$\lvert\lambda_i\rvert < 1\ \forall \lambda_i$ (such a matrix is called Schur stable)

Routh-Hurwitz criterion for DT LTI systems

- the stability domain $|\lambda_i| < 1$ is a unit disk
- Routh array validates stability in the left-half plane
- bilinear transformation maps the closed left half s-plane to the closed unit disk in z-plane



Routh-Hurwitz criterion for DT LTI systems

- Given $A(z) = z^n + a_1 z^{n-1} + \cdots + a_n$, procedures of Routh-Hurwitz test:
  - apply bilinear transform
    $A(z)|_{z=\frac{1+s}{1-s}} = \left(\frac{1+s}{1-s}\right)^n + a_1 \left(\frac{1+s}{1-s}\right)^{n-1} + \cdots + a_n =: \frac{A^*(s)}{(1-s)^n}$
  - apply Routh test to
    $A^*(s) = a_n^* s^n + a_{n-1}^* s^{n-1} + \cdots + a_0^* = A(z)|_{z=\frac{1+s}{1-s}} (1-s)^n$

Routh-Hurwitz criterion for DT LTI systems

Example $(A(z) = z^3 + 0.8z^2 + 0.6z + 0.5)$

- $A^*(s) = A(z)|_{z=\frac{1+s}{1-s}} (1-s)^3 = (1+s)^3 + 0.8(1+s)^2(1-s) + 0.6(1+s)(1-s)^2 + 0.5(1-s)^3 = 0.3s^3 + 3.1s^2 + 1.7s + 2.9$
- Routh array

$$
\begin{array}{c|cc}
s^3 & 0.3 & 1.7 \\
s^2 & 3.1 & 2.9 \\
s & 1.7 - \frac{0.3\times 2.9}{3.1} = 1.42 & 0 \\
s^0 & 2.9 & 0
\end{array}
$$

- all elements in first column are positive $\Rightarrow$ roots of $A(z)$ are all in the unit circle