Outline

- Preliminaries
- Stability concepts
- Equilibrium point
- Stability around an equilibrium point
- Stability of LTI systems
- Method of eigenvalue/pole locations
- Routh Hurwitz criteria for CT and DT systems
- Lyapunov's direct approach

1 Definitions

1.1 Finite dimensional vector norms

Let $v \in \mathbb{R}^n$. A norm is:

- a metric in vector space: a function that assigns a real-valued length to each vector in a vector space, e.g.,
  - 2 (Euclidean) norm: $\|v\|_2 = \sqrt{v^T v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$
  - 1 norm: $\|v\|_1 = \sum_{i=1}^n |v_j|$ (absolute column sum)
  - infinity norm: $\|v\|_\infty = \max_i |v_i|$
  - p norm: $||v||_p = \left(\sum_{i=1}^n |v_i|^p\right)^{1/p},\ 1 \leq p < \infty$
- default in this set of notes: $\|\cdot\| = \|\cdot\|_2$

1.2 Equilibrium state

For n-th order unforced system

$$\dot{x} = f(x, t),\ x(t_0) = x_0$$

an equilibrium state/point x_e is one such that

$$\boxed{f(x_e, t) = 0,\ \forall t}$$

- the condition must be satisfied by all $t \geq 0$.
- if a system starts at equilibrium state, it stays there
- e.g., (inverted) pendulum resting at the verticle direction
- without loss of generality, we assume the origin is an equilibrium point

1.3 Equilibrium state of a linear system

For a linear system

$$\dot{x}(t) = A(t)x(t),\ x(t_0) = x_0$$

- origin $x_e = 0$ is always an equilibrium state
- when $A(t)$ is nonsingular, multiple equilibrium states exist

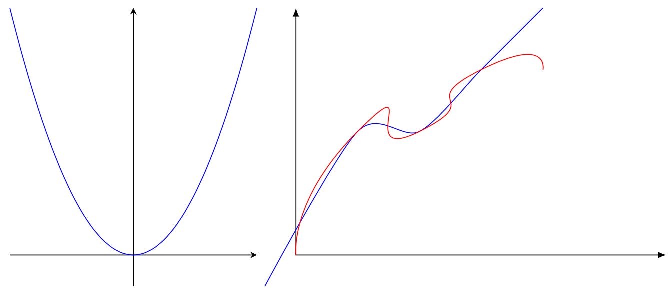


Figure 1: Continuous functions

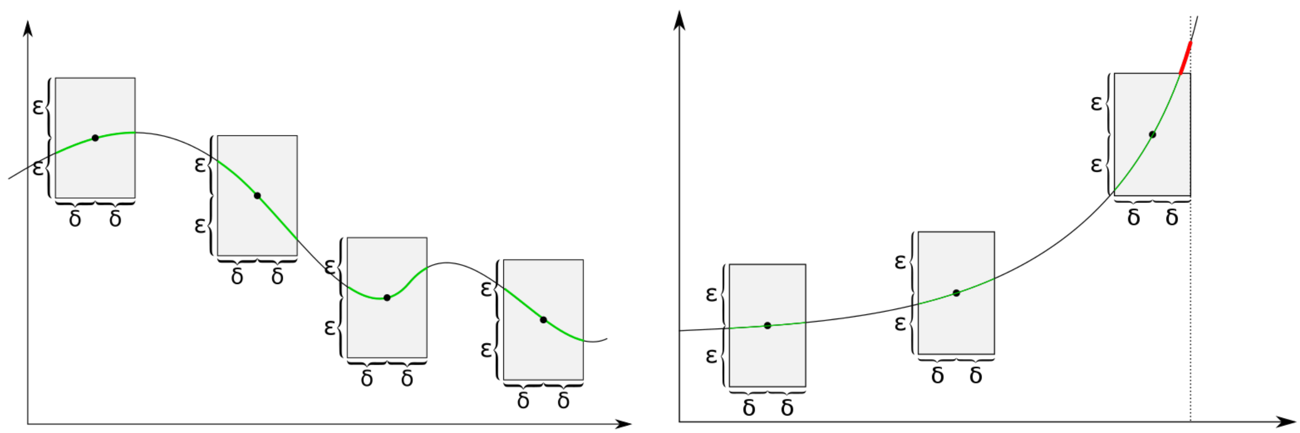


Figure 2: Uniformly continuous (left) and continuous but not uniformly continuous (right) functions

1.4 Continuous function

The function $f : \mathbb{R} \to \mathbb{R}$ is continuous at x_0 if $\forall \epsilon > 0$, there exists a $\delta(x_0, \epsilon) > 0$ such that

$$|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \epsilon$$

Graphically, continuous functions is a single unbroken curve:
e.g., $\sin x$, x^2, $\frac{1}{x-1}$, $\frac{x^2-1}{x-1}$, $\text{sign}(x - 1.5)$

1.5 Uniformly continuous function

The function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous if $\forall \epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that

$$|x - x_0| < \delta(\epsilon) \Longrightarrow |f(x) - f(x_0)| < \epsilon$$

- δ is a function of not x_0 but only ϵ

1.6 Lyapunov's definition of stability

- Lyapunov invented his stability theory in 1892 in Russia. Unfortunately, the elegant theory remained unknown to the West until approximately 1960.
- The equilibrium state 0 of $\dot{x} = f(x, t)$ is *stable in the sense of Lyapunov (s.i.L)* if for all $\epsilon > 0$, and t_0, there exists $\delta(\epsilon, t_0) > 0$ such that $\|x(t_0)\|_2 < \delta$ gives $\|x(t)\|_2 < \epsilon$ for all $t \geq t_0$

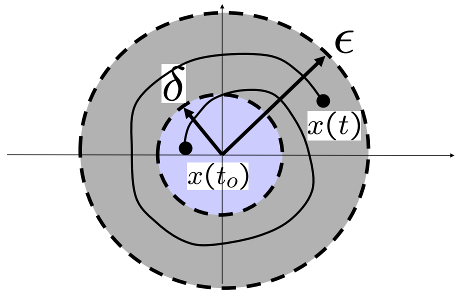


Figure 3: Stable in the sense of Lyapunov: $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon \ \forall t \geq t_0$.

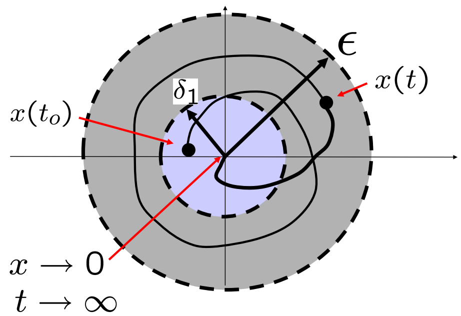


Figure 4: Asymptotically stable in the sense of Lyapunov: $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| \to 0$.

1.7 Uniform stability in the sense of Lyapunov

- The equilibrium state 0 of $\dot{x} = f(x,t)$ is *uniformly stable in the sense of Lyapunov* if for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $\|x(t_0)\|_2 < \delta$ gives $\|x(t)\|_2 < \epsilon$ for all $t \geq t_0$
- δ is not a function of t_0

1.8 Lyapunov's definition of asymptotic stability

The equilibrium state 0 of $\dot{x} = f(x,t)$ is asymptotically stable if

- it is stable in the sense of Lyapunov, and
- for all $\epsilon > 0$ and t_0, there exists $\delta(\epsilon, t_0) > 0$ such that $\|x(t_0)\|_2 < \delta$ gives $x(t) \to 0$

1.9 Uniform and global asymptotic stability

- the origin is uniformly and asymptotically stable if
  - it is asymptotically stable and
  - δ does not depend on time
  - e.g. $\dot{x} = -\frac{x}{1+t}$ is asymptotically stable but not uniformly stable
- the origin is globaly and asymptotically stable if
  - it is asymptotically stable, and
  - δ can be made arbitraily large (it does not matter how far the initial condition is from the origin)

2 Stability of LTI systems: method of eigenvalue/pole locations

The stability of the equilibrium point 0 for $\dot{x} = Ax$ or $x(k+1) = Ax(k)$ can be concluded immediatelly based on the eigenvalues, λ's, of A:

- the response $e^{At}x(t_0)$ involves modes such as $e^{\lambda t}$, $te^{\lambda t}$, $e^{\sigma t}\cos\omega t$, $e^{\sigma t}\sin\omega t$
- the response $A^k x(k_0)$ involves modes such as λ^k, $k\lambda^{k-1}$, $r^k\cos k\theta$, $r^k \sin k\theta$
- $e^{\sigma t} \to 0$ if $\sigma < 0$; $e^{\lambda t} \to 0$ if $\lambda < 0$
- $\lambda^k \to 0$ if $|\lambda| < 1$; $r^k \to 0$ if $|r| = \left|\sqrt{\sigma^2+\omega^2}\right| = |\lambda| < 1$

2.1 Stability of the origin for $\dot{x} = Ax$

stability of $\dot{x} = Ax$ at 0	$\lambda_i(A)$
unstable	$\mathrm{Re}\{\lambda_i\} > 0$ for some λ_i or $\mathrm{Re}\{\lambda_i\} \le 0$ for all λ_i's but for a repeated λ_m on the imaginary axis with multiplicity m, nullity $(A - \lambda_m I) < m$ (Jordan form)
stable i.s.L	$\mathrm{Re}\{\lambda_i\} \le 0$ for all λ_i's and for any repeated λ_m on the imaginary axis with multiplicity m, nullity $(A - \lambda_m I) = m$ (diagonal form)
asymptotically stable	$\mathrm{Re}\{\lambda_i\} < 0$ for all λ_i (such a matrix is called a Hurwitz matrix)

Example 1. Unstable system

$$\dot{x} = Ax,\ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- $\lambda_1 = \lambda_2 = 0$, $m = 2$, nullity $(A - \lambda_i I) =$ nullity $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 1 < m$
- i.e., two repeated eigenvalues but needs a generalized eigenvector $\Rightarrow$ Jordan form after similarity transform
- verify by checking $e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$: t grows unbounded

Example 2. Stable in the sense of Lyapunov

$$\dot{x} = Ax,\ A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- $\lambda_1 = \lambda_2 = 0$, $m = 2$, nullity $(A - \lambda_i I) =$ nullity $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 2 = m$
- verify by checking $e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2.2 Routh-Hurwitz criterion

- The asymptotic stability of the equilibrium point 0 for $\dot{x} = Ax$ can also be concluded based on the Routh-Hurwitz criterion.
  - The Routh Test (by E.J. Routh, in 1877) is a simple algebraic procedure to determine how many roots a given polynomial
  $$A(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$
  has in the closed right-half complex plane, without the need to explicitly solve the characteristic equation.
  - German mathematician Adolf Hurwitz independently proposed in 1895 to approach the problem from a matrix perspective .
  - popular if stability is the only concern and no details on eigenvalues (e.g., speed of response) are needed
- simply apply the Routh Test to $A(s) = \det(sI - A)$
- Recap: The poles of transfer function $G(s) = C(sI-A)^{-1}B + D$ come from $\det(sI - A)$ in computing the inverse $(sI - A)^{-1}$.

2.3 The Routh Array

For $A(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$, construct

$$\begin{array}{c|ccccc} s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\ s^{n-2} & q_{n-2} & q_{n-4} & q_{n-6} & \cdots & \\ s^{n-3} & q_{n-3} & q_{n-5} & q_{n-7} & \cdots & \\ \vdots & \vdots & \vdots & \vdots & & \\ s^1 & x_2 & x_0 & & & \\ s^0 & x_0 & & & & \end{array}$$

- first two rows contain the coefficients of $A(s)$
- third row of the Routh array is constructed from the previous two rows via

$$\begin{array}{c|cccc} \cdot & a & b & x & \cdot \\ \cdot & c & d & y & \cdot \\ \cdot & \dfrac{bc-ad}{c} & \dfrac{xc-ay}{c} & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

- **All roots of $A(s)$ are on the left half s-plane if and only if all elements of the first column of the Routh array are positive.**
- Special cases:
  - If the 1st element in any one row of Routh's array is zero, one can replace the zero with a small number ϵ and proceed further.
  - If the elements in one row of Routh's array are all zero, then the equation has at least one pair of real roots with equal magnitude but opposite signs, and/or the equation has one or more pairs of imaginary roots, and/or the equation has pairs of complex-conjugate roots forming symmetry about the origin of the s-plane.
  - There are other possible complications, which we will not pursue further. See, e.g. "Automatic Control Systems", by Kuo, 7th ed., pp. 339-340.

Example 3. $A(s) = 2s^4 + s^3 + 3s^2 + 5s + 10$

$$\begin{array}{c|ccc} s^4 & 2 & 3 & 10 \\ s^3 & 1 & 5 & 0 \\ s^2 & 3 - \frac{2\times 5}{1} = -7 & 10 & 0 \\ s^1 & 5 - \frac{1\times 10}{-7} & 0 & 0 \\ s^0 & 10 & 0 & 0 \end{array}$$

- two sign changes in the first column
- unstable and two roots in the right half side of s-plane

2.4 Stability of the origin for $x(k+1) = f(x(k), k)$

- the theory of Lyapunov follows analogously for nonlinear time-varying discrete-times systems of the form

$$x(k+1) = f(x(k), k), \ x(k_0) = x_0$$

- equilibrium point x_e:

$$f(x_e, k) = x_e, \ \forall k$$

- without loss of generality, 0 is assumed an equilibrium point.

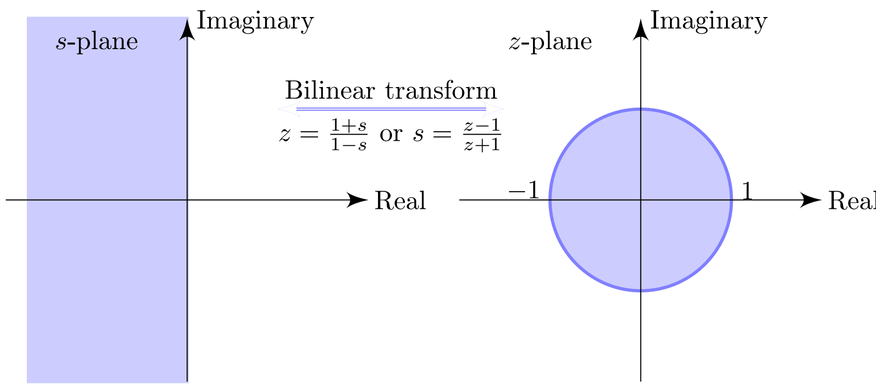
2.5 Stability of the origin for $x(k+1) = Ax(k)$

- 0 is always an equilibrium point, althoughly not necessarily the only one

stability of $x(k+1) = Ax(k)$ at 0	$\lambda_i(A)$
unstable	$\lvert\lambda_i\rvert > 1$ for some λ_i or $\lvert\lambda_i\rvert \leq 1$ for all λ_i's but for a repeated λ_m on the unit circle with multiplicity m, nullity $(A - \lambda_m I) < m$ (Jordan form)
stable i.s.L	$\lvert\lambda_i\rvert \leq 1$ for all λ_i's but for any repeated λ_m on the unit circle with multiplicity m, nullity $(A - \lambda_m I) = m$ (diagonal form)
asymptotically stable	$\lvert\lambda_i\rvert < 1$ for all λ_i (such a matrix is called a Schur matrix)

2.6 Routh-Hurwitz criterion for discrete-time LTI systems

- The stability domain $|\lambda_i| < 1$ for discrete-time systems is a unit disk.
- Routh array validates stability in the left-half complex plane.
- Bilinear transformation maps the closed left half s-plane to the closed unit disk in z-plane



- Given $A(z) = z^n + a_1 z^{n-1} + \cdots + a_n$, procedures of Routh-Hurwitz test:
  - apply bilinear transform $A(z)|_{z=\frac{1+s}{1-s}} = \left(\frac{1+s}{1-s}\right)^n + a_1\left(\frac{1+s}{1-s}\right)^{n-1} + \cdots + a_n = \frac{A^*(s)}{(1-s)^n}$
  - apply Routh test to $A^*(s) = a_n^* s^n + a_{n-1}^* s^{n-1} + \cdots + a_0^* = A(z)|_{z=\frac{1+s}{1-s}}(1-s)^n$

Example 4. $A(z) = z^3 + 0.8z^2 + 0.6z + 0.5$

- $A^*(s) = A(z)|_{z=\frac{1+s}{1-s}}(1-s)^3 = (1+s)^3 + 0.8(1+s)^2(1-s) + 0.6(1+s)(1-s)^2 + 0.5(1-s)^3 = 0.3s^3 + 3.1s^2 + 1.7s + 2.9$
- Routh array

$$\begin{array}{c|cc} s^3 & 0.3 & 1.7 \\ s^2 & 3.1 & 2.9 \\ s & 1.7 - \frac{0.3\times 2.9}{3.1} = 1.42 & 0 \\ s^0 & 2.9 & 0 \end{array} \; : \; \Rightarrow \; \begin{array}{c} \text{all elements in first column are positive} \\ \text{roots of } A(z) \text{ are all in the unit circle} \end{array}$$

3 Lyapunov's approach to stability

The direct method of Lyapunov to stability problems:

- no need for explicit solutions to system responses
- an "energy" perspective
- fit for general dynamic systems (linear/nonlinear, time-invariant/time-varying)

3.1 Stability from an energy viewpoint

Consider spring-mass-damper systems:

$$\dot{x}_1 = x_2 \qquad (x_1\text{: position; } x_2 \text{: velocity})$$
$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2,\ b > 0 \qquad \text{(Newton's law)}$$

- eigenvalues of the A matrix are in the left-half s-plane; system asymptotically stable.
- total energy
$$\mathcal{E}(t) = \text{potential energy} + \text{kinetic energy} = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$$
- dissipation of energy
$$\dot{\mathcal{E}}(t) = kx_1\dot{x}_1 + mx_2\dot{x}_2 = kx_1x_2 + mx_2\left(-\frac{k}{m}x_1 - \frac{b}{m}x_2\right) = -bx_2^2 \le 0$$
  - energy dissipates
  - $\dot{\mathcal{E}}$ is zero only when $x_2 = 0$. Since $[x_1, x_2]^T = 0$ is the only equilibrium state, the motion of the mass will not stop at $x_2 = 0$, $x_1 \neq 0$. Thus the energy will keep decreasing toward 0 which is achieved at the origin.
- the above is the direct[1] method of Lyapunov.

3.2 Generalization

Consider unforced, time-varying, nonlinear systems

$$\dot{x}(t) = f(x(t), t),\ x(t_0) = x_0$$
$$x(k+1) = f(x(k), k),\ x(k_0) = x_0$$

- assume the origin is an equilibrium state
- energy function ⇒ Lyapunov function: a scalar function of x and t (or x and k in discrete-time case)
- goal is to relate properties of the state through the Lyapunov function
- main tool: matrix formulation, linear algebra, positive definite functions

3.3 Relevant tools

3.3.1 Quadratic functions

- Intrinsic in energy-like analysis, e.g.
$$\frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2 = \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
- Convenience of matrix formulation:
$$\frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2 + x_1x_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} \frac{k}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{m}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2 + x_1x_2 + c = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{k}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{m}{2} & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$$
- General quadratic functions in matrix form
$$Q(x) = x^TPx,\ P^T = P$$

[1] in the sense of concluding on stability without solving the state equations explicitly.

3.3.2 Symmetric matrices

- A real square matrix A is
  - *symmetric* if $A = A^T$
  - *skew-symmetric* if $A = -A^T$
- examples:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

- Any real square matrices can be decomposed as the sum of a symmetric matrix and a skew symmetric matrix:

$$\text{e.g.} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2.5 \\ 2.5 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix}$$

$$\text{formula: } P = \frac{P + P^T}{2} + \frac{P - P^T}{2}$$

- A real square matrix $A \in \mathbb{R}^{n\times n}$ is *orthogonal* if $A^T A = AA^T = I$, meaning that the columns of A form a orthonormal basis of $\mathbb{R}^n$. To see this, writing A in the column-vector notation

$$A = \begin{bmatrix} | & | & | & | \\ a_1 & a_2 & \dots & a_n \\ | & | & | & | \end{bmatrix}$$

we get

$$A^T A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix} \begin{bmatrix} a_1, & a_2, & \dots, & a_n \end{bmatrix} = \begin{bmatrix} a_1^T a_1 & a_1^T a_2 & \dots & a_1^T a_n \\ a_2^T a_1 & a_2^T a_2 & \dots & a_2^T a_n \\ \vdots & \vdots & \vdots & \vdots \\ a_n^T a_1 & a_n^T a_2 & \dots & a_n^T a_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

namely,

$$a_j^T a_j = 1$$
$$a_j^T a_m = 0 \ \forall j \neq m$$

- Extremely useful properties of the structured matrices

Theorem 5. *The eigenvalues of symmetric matrices are all real.*

Proof. $\forall : A \in \mathbb{R}^{n\times n}$ with $A^T = A$. Take eigenvalue-eigenvector pair: $Au = \lambda u \Rightarrow \overline{u}^T Au = \lambda \overline{u}^T u$, where $\overline{u}$ is the complex conjugate of u. $\overline{u}^T Au$ is a real number, as

$$\begin{aligned} \overline{\overline{u}^T Au} &= u^T \overline{A}\overline{u} \\ &= u^T A\overline{u} \quad \because A \in \mathbb{R}^{n\times n} \\ &= u^T A^T \overline{u} \quad \because A = A^T \\ &= \lambda u^T \overline{u} \quad \because (Au)^T = (\lambda u)^T \\ &= \lambda \overline{u}^T u \quad \because u^T \overline{u} \in \mathbb{R} \\ &= \overline{u}^T Au \quad \because Au = \lambda u \end{aligned}$$

By definition of complex conjugate numbers, $\overline{u}^T u \in \mathbb{R}$. Thus $\lambda = \frac{\overline{u}^T Au}{\overline{u}^T u}$ must also be a real number. □

Theorem 6. *The eigenvalues of skew-symmetric matrices are all imaginary or zero.*

Theorem 7. *The eigenvalues of an orthogonal matrix always have a magnitude of 1.*

matrix structure	analogy in complex plane
symmetric	real line
skew-symmetric	imaginary line
orthogonal	unit circle

3.3.3 Symmetric eigenvalue decomposition (SED)

When $A \in \mathbb{R}^{n\times n}$ has n distinct eigenvalues, we have seen the useful result of matrix diagonalization:

$$A = U\Lambda U^{-1} = [u_1, \dots, u_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} [u_1, \dots, u_n]^{-1} \tag{1}$$

where λ_i's are the distinct eigenvalues associated to the eigenvector u_i's. The inverse matrix in (1) can be cumbersome to compute though. The spectral theorem, aka symmetric eigenvalue decomposition theorem,[2] significantly simplifies the result when A is symmetric.

Theorem 8. *$\forall : A \in \mathbb{R}^{n\times n},\ A^T = A$, there always exist $\lambda_i \in \mathbb{R}$ and $u_i \in \mathbb{R}^n$, such that*

$$A = \sum_{i=1}^{n} \lambda_i u_i u_i^T = U\Lambda U^T \tag{2}$$

where:[3]

- *λ_i's: eigenvalues of A*
- *u_i: eigenvector associated to λ_i, normalized to have unity norms*
- *$U = [u_1, u_2, \cdots, u_n]^T$ is an orthogonal matrix, i.e., $U^T U = UU^T = I$*
- *$\{u_1, u_2, \cdots, u_n\}$ forms an orthonormal basis*
- $\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

To understand the result, we show an important theorem first.

Theorem 9. *$\forall : A \in \mathbb{R}^{n\times n}$* with *$A^T = A$, then eigenvectors of A, associated with different eigenvalues, are orthogonal.*

Proof. Let $Au_i = \lambda_i u_i$ and $Au_j = \lambda_j u_j$. Then $u_i^T A u_j = u_i^T \lambda_j u_j = \lambda_j u_i^T u_j$. In the meantime, $u_i^T A u_j = u_i^T A^T u_j = (Au_i)^T u_j = \lambda_i u_i^T u_j$. So $\lambda_i u_i^T u_j = \lambda_j u_i^T u_j$. But $\lambda_i \neq \lambda_j$. It must be that $u_i^T u_j = 0$. □

Theorem 8 now follows. If A has distinct eigenvalues, then $U = [u_1, u_2, \cdots, u_n]^T$ is orthogonal if we normalize all the eigenvectors to unity norm. If A has $r(< n)$ distinct eigenvalues, we can *choose* multiple orthogonal eigenvectors for the eigenvalues with none-unity multiplicities.

With the spectral theorem, next time we see a symmetric matrix A, we immediately know that

- λ_i is real for all i
- associated with λ_i, we can always find a real eigenvector
- $\exists$ an orthonormal basis $\{u_i\}_{i=1}^n$, which consists of the eigenvectors
- if $A \in \mathbb{R}^{2\times 2}$, then if you compute first λ_1, λ_2 and u_1, you won't need to go through the regular math to get u_2, but can simply solve for a u_2 that is orthogonal to u_1 with $\|u_2\| = 1$.

[2] Recall that the set of all the eigenvalues of A is called the spectrum of A. The largest of the absolute values of the eigenvalues of A is called the spectral radius of A.

[3] $u_i u_i^T \in \mathbb{R}^{n\times n}$ is a symmetric dyad, the so-called outerproduct of u_i and u_i. It has the following properties:

- $\forall\ v \in \mathbb{R}^{n\times 1}$, $(vv^T)_{ij} = v_i v_j$. (Proof: $(vv^T)_{ij} = e_i^T (vv^T) e_j = v_i v_j$, where e_i is the unit vector with all but the i_{th} elements being zero.)
- link with quadratic functions: $q(x) = x^T (vv^T) x = (v^T x)^2$

Example 10. Consider the matrix $A = \begin{bmatrix} 5 & \sqrt{3} \\ \sqrt{3} & 7 \end{bmatrix}$. Computing the eigenvalues gives

$$\det \begin{bmatrix} 5-\lambda & \sqrt{3} \\ \sqrt{3} & 7-\lambda \end{bmatrix} = 35 - 12\lambda + \lambda^2 - 3 = (\lambda - 4)(\lambda - 8) = 0$$
$$\Rightarrow \lambda_1 = 4,\ \lambda_2 = 8$$

And we can know one of the eigenvectors from

$$(A - \lambda_1 I)\, t_1 = 0 \Rightarrow \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix} t_1 = 0 \Rightarrow t_1 = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

Note here we normalized t_1 such that $||t_1||_2 = 1$. With the above computation, we no more need to do $(A - \lambda_2 I)\, t_2 = 0$ for getting t_2. Keep in mind that A here is symmetric, so has eigenvectors orthogonal to each other. By direct observation, we can see that

$$x = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

is orthogonal to t_1 and $||x||_2 = 1$. So $t_2 = x$.

Theorem 11 (Eigenvalues of symmetric matrices)**.** *If $A = A^T \in \mathbb{R}^{n\times n}$, then the eigenvalues of A satisfy*

$$\lambda_{\max} = \max_{x\in\mathbb{R}^n,\ x\neq 0} \frac{x^T A x}{\|x\|_2^2} \tag{3}$$

$$\lambda_{\min} = \min_{x\in\mathbb{R}^n,\ x\neq 0} \frac{x^T A x}{\|x\|_2^2} \tag{4}$$

Proof. Perform SED to get

$$A = \sum_{i=1}^{n} \lambda_i u_i^T u_i$$

where $\{u_i\}_{i=1}^n$ form a basis of $\mathbb{R}^n$. Then any vector $x \in \mathbb{R}^n$ can be decomposed as

$$x = \sum_{i=1}^{n} \alpha_i u_i$$

Thus

$$\max_{x\neq 0} \frac{x^T A x}{\|x\|_2^2} = \max_{\alpha_i} \frac{\left(\sum_i \alpha_i u_i\right)^T \sum_i \lambda_i \alpha_i u_i}{\sum_i \alpha_i^2} = \max_{\alpha_i} \frac{\sum_i \lambda_i \alpha_i^2}{\sum_i \alpha_i^2} = \lambda_{\max}$$

The proof for (4) is analogous and omitted. □

3.3.4 Positive definite matrices

Since the eigenvalues of symmetric matrices are real, we can order them. The symmetric matrices whose eigenvalues are all positive have excellent properties

Definition 12 (Positive Definite Matrices)**.** A symmetric matrix $P \in \mathbb{R}^{n\times n}$ is called **positive-definite**, written $P \succ 0$, if $x^T P x > 0$ for all $x\,(\neq 0) \in \mathbb{R}^n$. P is called **positive-semidefinite,** written $P \succeq 0$, if $x^T P x \geq 0$ for all $x \in \mathbb{R}^n$

Definition 13. A symmetric matrix $Q \in \mathbb{R}^{n\times n}$ is called **negative-definite**, written $Q \prec 0$, if $-Q \succ 0$, i.e., $x^T Q x < 0$ for all $x\,(\neq 0) \in \mathbb{R}^n$. Q is called **negative-semidefinite,** written $Q \preceq 0$, if $x^T Q x \leq 0$ for all $x \in \mathbb{R}^n$

- When A and B have compatible dimensions, $A \succ B$ means $A - B \succ 0$.
- Positive-definite matrices can have negative entries:

Example 14. $P=\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ is positive-definite, as $P=P^T$ and take any $v=[x,y]^T$, we have

$$v^T P v=\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2x^2+2y^2-2xy = x^2+y^2+(x-y)^2 \geq 0$$

and the equality sign holds only when $x=y=0$.

- Conversely, matrices whose entries are all positive are not necessarily positive-definite.

Example 15. $A=\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is not positive-definite as

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -2 < 0$$

Theorem 16. *For a symmetric matrix P, $P \succ 0$ if and only if all the eigenvalues of P are positive.*

Proof. Since P is symmetric, we have

$$\lambda_{\max}(P) = \max_{x\in\mathbb{R}^n,\ x\neq 0} \frac{x^T A x}{\|x\|_2^2} \tag{5}$$

$$\lambda_{\min}(P) = \min_{x\in\mathbb{R}^n,\ x\neq 0} \frac{x^T A x}{\|x\|_2^2} \tag{6}$$

which gives

$$x^T A x \in \left[\lambda_{\min}\|x\|_2^2,\ \lambda_{\max}\|x\|_2^2\right]$$

For $x\neq 0$, $\|x\|_2^2$ is always positive. It can thus be seen that $x^T A x > 0,\ x\neq 0 \Leftrightarrow \lambda_{\min} > 0$. □

Checking positive definiteness of a matrix. We often use the following necessary and sufficient conditions to check if a symmetric matrix P is positive (semi-)definite or not:

- $P \succ 0$ ($P \succeq 0$) $\Leftrightarrow$ the leading principle minors defined below are positive (nonnegative)
- $P \succ 0$ ($P \succeq 0$) $\Leftrightarrow$ P can be decomposed as $P=N^T N$ where N is nonsingular (singular)

Definition 17. The leading principle minors of

$$P=\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

are defined as p_{11}, $\det \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$, $\det P$.

Example 18. None of the following matrices are positive definite:

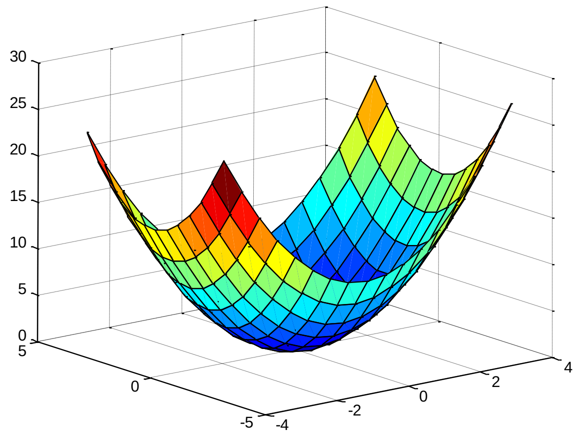
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

3.3.5 Positive definite functions

Definition 19 (Positive Definite Functions)**.** A continuous time function $W:\mathbb{R}^n\to\mathbb{R}_+$ satisfying

- $W(x)>0$ for all $x\neq 0$
- $W(0)=0$
- $W(x)\to\infty$ as $|x|\to\infty$ uniformly in x

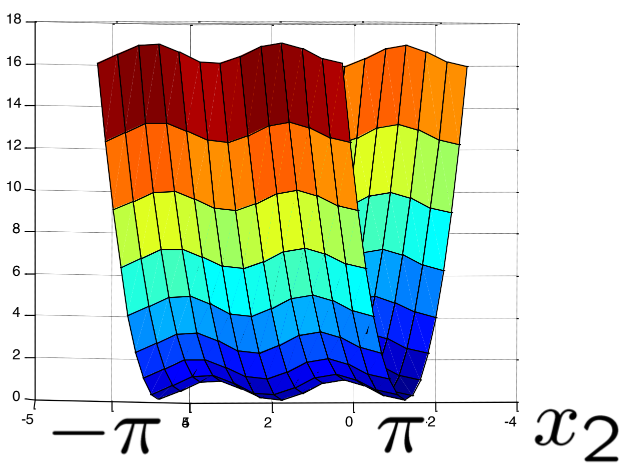
In the three dimensional space, positive definite functions are "bowl-shaped", e.g., $W(x_1, x_2) = x_1^2 + x_2^2$.



Definition 20 (Locally Positive Definite Functions)**.** A continuous time function $W : \mathbb{R}^n \to \mathbb{R}_+$ satisfying

- $W(x) > 0$ for all $x \neq 0$ and $|x| < r$
- $W(0) = 0$

In the three dimensional space, locally positive definite functions are "bowl-shaped" locally, e.g., $W(x_1, x_2) = x_1^2 + \sin^2 x_2$ for $x_1 \in \mathbb{R}$ and $|x_2| < \pi$



Definition 21 (Positive Semi-Definite Functions)**.** A continuous time function $W : \mathbb{R}^n \to \mathbb{R}_+$ satisfying

- $W(x) \geq 0$ for all x
- $W(0) = 0$

e.g., $W(x_1, x_2) = x_1^2 + \sin^2 x_2$ for all x

Exercise 22. Let $x = [x_1, x_2, x_3]^T$. Check the positive definiteness of the following functions

1. $V(x) = x_1^4 + x_2^2 + x_3^4$
2. $V(x) = x_1^2 + x_2^2 + 3x_3^2 - x_3^4$

3.4 Lyapunov stability theorems

Recall the spring mass damper example, this time in matrix form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The energy function

$$\mathcal{E}(t) = \text{potential energy} + \text{kinetic energy} = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$$

is positive definite and its derivative

$$\begin{aligned} \dot{\mathcal{E}}(t) &= k_1x_1\dot{x}_1 + mx_2\dot{x}_2 = \left[\frac{\partial \mathcal{E}}{\partial x_1}, \frac{\partial \mathcal{E}}{\partial x_2}\right][\dot{x}_1, \dot{x}_2]^T && (7) \\ &= k_1x_1x_2 + mx_2\left(-\frac{k}{m}x_1 - \frac{b}{m}x_2\right) = \left[\frac{\partial \mathcal{E}}{\partial x_1}, \frac{\partial \mathcal{E}}{\partial x_2}\right]Ax && (8) \\ &= -bx_2^2 \end{aligned}$$

is negative semi-definite.

- $\dot{\mathcal{E}}(t)$ is a derivative along the state trajectory: (7) takes the derivative of $\mathcal{E}$ w.r.t. $x = [x_1, x_2]^T$; (8) is the time derivative along the trajectory of the state.
- Generalizing the concept to system $\dot{x} = f(x)$: let $V(x)$ be a general energy function, the energy dissipation w.r.t. time is

$$\frac{dV(x)}{dt} = \nabla V^T(x) f(x) = \underbrace{\left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \dots, \frac{\partial V}{\partial x_n}\right]}_{\nabla V^T(x):\ \nabla V(x) \text{ is the gradient w.r.t. } x}\begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

  also denoted as $L_f V(x)$, the Lie derivative of $V(x)$ w.r.t. $f(x)$.
- We concluded stability of the system by analyzing how energy will dissipate to zero along the trajectory of the state.

Theorem 23. *The equilibrium point* 0 *of* $\dot{x}(t) = f(x(t), t)$, $x(t_0) = x_0$ *is* <u>*stable in the sense of Lyapunov*</u> *if there exists a locally positive definite function* $V(x,t)$ *such that* $\dot{V}(x,t) \le 0$ *for all* $t \ge t_0$ *and all* x *in a local region* $x : |x| < r$ *for some* $r > 0$.

- $V(x,t)$ satisfying the properties in the theorem is called the Lyapunov function
- i.e., $V(x)$ is positive definite and $\dot{V}(x)$ is negative semidefinite for all states in a local region $|x| < r$

Theorem 24. *The equilibrium point* 0 *of* $\dot{x}(t) = f(x(t), t)$, $x(t_0) = x_0$ *is* <u>*locally asymptotically stable*</u> *if there exists a Lyapunov function* $V(x)$ *such that for some* $r > 0$, $\dot{V}(x)$ *is negative definite* $\forall |x| < r$.

Theorem 25. *The equilibrium point* 0 *of* $\dot{x}(t) = f(x(t), t)$, $x(t_0) = x_0$ *is* <u>*globally asymptotically stable*</u> *if there exists a Lyapunov function* $V(x)$ *such that* $V(x)$ *is positive definite and* $\dot{V}(x)$ *is negative definite.*

- for linear system $\dot{x} = Ax$, a good Lyapunov candidate is the quadratic function $V(x) = x^T P x$ where $P = P^T$ and $P \succ 0$
- the derivative along the state trajectory is then

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax)^T Px + x^T PAx \\ &= x^T\left(A^T P + PA\right)x \end{aligned}$$

- such a $V(x) = x^T P x$ is a Lyapunov function for $\dot{x} = Ax$ when $A^T P + PA \preceq 0$

- and the origin is stable in the sense of Lyapunov

Theorem 26. *For system $\dot{x} = Ax$ with $A \in \mathbb{R}^{n\times n}$, the origin is asymptotically stable if and only if $\exists\ Q \succ 0$, such that* *the Lyapunov equation*

$$\boxed{A^T P + PA = -Q}$$

has a unique positive definite solution $P \succ 0$, $P^T = P$.

Proof. $\Rightarrow$: from Theorem 11,

$$\frac{\dot{V}}{V} = -\frac{x^T Qx}{x^T Px} \leq -\underbrace{\frac{(\lambda_Q)_{\min}}{(\lambda_P)_{\max}}}_{\triangleq \alpha} \Longrightarrow V(t) \leq e^{-\alpha t} V(0)$$

Since Q and P are positive definite, $(\lambda_Q)_{\min} > 0$ and $(\lambda_P)_{\max} > 0$. Thus $\alpha > 0$ and $V(t)$ decays exponentially to zero. Because $V(x)$ is a positive definite function of x, $V(x) = 0$ only at $x = 0$. Therefore, the response x of system $\dot{x} = Ax$ will go to 0 as $t \to \infty$, regardless of the initial condition.[4]

$\Leftarrow$: if the zero state of $\dot{x} = Ax$ is asymptotically stable, then all eigenvalues of A have negative real parts. For any Q, the Lyapunov equation has a unique solution P. Note $x(t) = e^{At}x_0 \to 0$ as $t \to \infty$. We have

$$\begin{aligned}\cancelto{0}{x^T(\infty)Px(\infty)} - x^T(0)Px(0) &= \int_0^\infty \frac{d}{dt} x^T(t) Px(t)\, dt = -\int_0^\infty x^T(t)\left(A^T P + TA\right)x(t)\, dt \\ \Rightarrow x(0)^T Px(0) &= \int_0^\infty x^T(t) Qx(t)\, dt = \int_0^\infty x(0) e^{A^T t} Q e^{At} x(0)\, dt\end{aligned}$$

If Q is positive definite, there exists a nonsingular N matrix such that $Q = N^T N$. Thus

$$\begin{aligned} x(0)^T Px(0) &= \int_0^\infty \|Ne^{At}x(0)\|^2 dt \geq 0 \\ x(0)^T Px(0) &= 0 \text{ only if } x_0 = 0\end{aligned}$$

Thus P is positive definite. Also, P has the following closed form solution:

$$\boxed{P = \int_0^\infty e^{A^T t} Q e^{At} dt}$$

$\square$

Example 27. Given system model $\dot{x} = Ax$, $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, consider solving the Lyapunov equation

$$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}^T \underbrace{\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}}_{P} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = -\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{Q}$$

namely

$$\begin{bmatrix} -p_{11} - p_{12} & -p_{12} - p_{22} \\ p_{11} & p_{12} \end{bmatrix} + \begin{bmatrix} -p_{11} - p_{12} & p_{11} \\ -p_{12} - p_{22} & p_{12} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

We need

$$\begin{cases} -2p_{11} - 2p_{12} = -1 \\ -p_{12} - p_{22} + p_{11} = 0 \\ 2p_{12} = -1 \end{cases} \Rightarrow \begin{cases} p_{11} = 1 \\ p_{22} = 3/2 \\ p_{12} = -1/2 \end{cases}$$

To check whether P is positive definite or not, we check the leading principle minors: $p_{11} > 0$, $p_{11}p_{22} - p_{12}^2 > 0$ $\Rightarrow P \succ 0$. Thus, asymptotic stability holds.

Observations:

[4] In fact, as $x^T Px \geq (\lambda_P)_{\min} \|x\|^2$, we have exponential convergence: $(\lambda_P)_{\min} \|x\|^2 \leq e^{-\alpha t} V(0) \Rightarrow \|x\|^2 \leq e^{-\alpha t} V(0) / (\lambda_P)_{\min} \leq e^{-\alpha t} (\lambda_P)_{\max} \|x(0)\|^2 / (\lambda_P)_{\min}$.

- A^TP+PA is a linear operation on P: e.g.,

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\ Q=\begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix},\ P=\begin{bmatrix} | & | \\ p_1 & p_2 \\ | & | \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T}_{A^T}\underbrace{\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}}_{P}+\underbrace{\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}}_{P}\underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A}=-\underbrace{\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}}_{Q}$$

$$\Downarrow$$

$$A^T\begin{bmatrix} | & | \\ p_1 & p_2 \\ | & | \end{bmatrix}+\begin{bmatrix} | & | \\ p_1 & p_2 \\ | & | \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}=-\begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix}$$

$$A^Tp_1+a_{11}p_1+a_{21}p_2=-q_1$$
$$A^Tp_2+a_{12}p_1+a_{22}p_2=-q_2$$

- can stack the columns of A^TP+PA and Q to yield, e.g.

$$\begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}+\begin{bmatrix} a_{11}I & a_{21}I \\ a_{12}I & a_{22}I \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}=-\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$
$$\underbrace{\left\{\begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix}+\begin{bmatrix} a_{11}I & a_{21}I \\ a_{12}I & a_{22}I \end{bmatrix}\right\}}_{L_A}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}=-\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

- can simply write $L_A=I\otimes A^T+A^T\otimes I$ using the Kronecker product notation

$$B\otimes C=\begin{bmatrix} b_{11}C & b_{11}C & \dots & b_{11}C \\ b_{21}C & b_{22}C & \dots & b_{2n}C \\ \vdots & \vdots & \dots & \vdots \\ b_{m1}C & b_{m2}C & \dots & b_{mn}C \end{bmatrix}$$

- can show that L_A is invertible if and only if $\lambda_i+\lambda_j\neq 0$ for all eigenvalues of A. (see Section 3.7 of Linear System Theory and Design by Chen.)
- To check, let $A^Tu_i=\lambda_iu_i$ and $A^Tu_j=\lambda_ju_j$ be eigenvalue eigenvector pairs of A^T. Note that $u_iu_j^TA+A^Tu_iu_j^T=u_i\left(\lambda_ju_j\right)^T+\lambda_iu_iu_j^T=\left(\lambda_i+\lambda_j\right)u_iu_j^T$. So $\lambda_i+\lambda_j$ is an eigenvalue of the operator $L_A(P)$. If $\lambda_i+\lambda_j\neq 0$, the operator is invertible.

Example 28. $A=\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, $\lambda_{1,2}=-0.5\pm i\sqrt{3}/2$

$$L_A=I\otimes A^T+A^T\otimes I=\begin{bmatrix} A^T+a_{11}I & a_{21}I \\ a_{12}I & A^T+a_{22}I \end{bmatrix}$$
$$=\left[\begin{array}{cc|cc} -1-1 & -1 & -1 & 0 \\ 1 & 0-1 & 0 & -1 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right]=\left[\begin{array}{cc|cc} -2 & -1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

The eigenvalues of L_A are, e.g., by Matlab, -1, -1, $-1-\sqrt{3}$, $-1+\sqrt{3}$, which are precisely $\lambda_1+\lambda_1$, $\lambda_1+\lambda_2$, $\lambda_2+\lambda_1$, $\lambda_2+\lambda_2$.

Procedures of Lyapunov's direct method

- given a matrix A, select an arbitrary positive definite symmetric matrix Q (e.g., I)
- find the solution matrix P to the Lyapunov equation $A^T P + PA = -Q$
- if a solution P cannot be found, A is not Hurwitz
- if a solution is found
  - if $P \succ 0$ then A is Hurwitz
  - if P is not positive definite, then A has at least one eigenvalue with a positive real part

3.5 Instability theorem

- the previous theorems only provide sufficient but not necessary conditions
- failure to find a Lyapunov function does not imply instability

Theorem 29. *The equilibrium state* 0 *of* $\dot{x} = f(x)$ *is unstable if there exists a function* $W(x)$ *such that*

- $\dot{W}(x)$ *is positive definite locally:* $\dot{W}(x) > 0 \ \forall |x| < r$ *for some* r *and* $\dot{W}(0) = 0$
- $W(0) = 0$
- *there exist states* x *arbitrarily close to the origin such that* $W(x) > 0$

3.6 Discrete-time case

For the discrete-time system

$$x(k+1) = Ax(k)$$

we consider a quadratic Lyapunov function candidate

$$V(x) = x^T Px, \ P = P^T \succ 0$$

and compute $\Delta V(x)$ along the trajectory of the state

$$V(x(k+1)) - V(x(k)) = x^T(k) \underbrace{\left[A^T PA - P\right]}_{\triangleq -Q} x(k)$$

The above gives the discrete-time Lyapunov stability theorem for LTI systems.

Theorem 30. *For system* $x(k+1) = Ax(k)$ *with* $A \in \mathbb{R}^{n\times n}$*, the origin is asymptotically stable if and only if* $\exists\ Q \succ 0$, *such that the discrete-time Lyapunov equation*

$$\boxed{A^T PA - P = -Q}$$

has a unique positive definite solution $P \succ 0$, $P^T = P$.

- Solution to the discrete-time Lyapunov equation, when asymptotic stability holds (A is Schur), is from the following:

$$\cancelto{0}{V(x(\infty))} - V(x(0)) = \sum_{k=0}^{\infty} x^T(k)\left[A^T PA - P\right]x(k) = -\sum_{k=0}^{\infty} x^T(0)\left(A^T\right)^k QA^k x(0)$$
$$\Rightarrow P = \sum_{k=0}^{\infty}\left(A^T\right)^k QA^k$$

  where $\cancelto{0}{V(x(\infty))}$ because $x \to 0$ as $t \to \infty$ under asymptotic stability.

- Can show that the discrete-time Lyapunov operator $L_A = A^T PA - P$ is invertible if and only if for all i, j $(\lambda_A)_i (\lambda_A)_j \neq 1$

4 Recap

- Internal stability
  - Stability in the sense of Lyapunov: ε, δ conditions
  - Asymptotic stability
- Stability analysis of linear time invariant systems ($\dot{x} = Ax$ or $x(k+1) = Ax(k)$)
  - Based on the eigenvalues of A
    * Time response modes
    * Repeated eigenvalues on the imaginary axis
  - Routh's criterion
    * No need to solve the characteristic equation
    * Routh's array
    * Discrete time case: bilinear transform ($z = \frac{1+s}{1-s}$)
  - Lyapunov equations

    Theorem: All the eigenvalues of A have negative real parts if and only if for any given $Q \succ 0$, the Lyapunov equation
    $$A^T P + PA = -Q$$
    has a unique solution P and $P \succ 0$.

    Note: Given Q, the Lyapunov equation $A^T P + PA = -Q$ has a unique solution, when $\lambda_{A,i} + \lambda_{A,j} \neq 0$ for all i and j.

    Theorem: All the eigenvalues of A are inside the unit circle if and only if for any given $Q \succ 0$, the Lyapunov equation
    $$A^T PA - P = -Q$$
    has a unique solution P and $P \succ 0$.

    Note: Given Q, the Lyapunov equation $A^T PA - P = -Q$ has a unique solution, when $\lambda_{A,i}\lambda_{A,j} \neq 1$ for all i and j.
  - P is positive definite if and only if any one of the following conditions holds:
    1. All the eigenvalues of P are positive.
    2. All the leading principle minors of P are positive.
    3. There exists a nonsingular matrix N such that $P = N^T N$.

	continuous time $\dot{x} = Ax$	discrete time $x(k+1) = Ax(k)$
key equations	$\dot{V} = x^T \left(A^T P + PA\right) x$	$V(k+1) - V(k) = x^T(k)\left(A^T PA - P\right) x(k)$
Lyapunov equation	$A^T P + PA = -Q$	$A^T PA - P = -Q$
unique solution condition	$\lambda_i + \lambda_j \neq 0$ for all i and j	$\lvert\lambda_i\rvert\lvert\lambda_j\rvert < 1$ for all i and j
solution	$P = \int_0^\infty e^{A^T t} Q e^{At} dt$ when A is Hurwitz	$P = \sum_{k=0}^\infty \left(A^T\right)^k Q A^k$ when A is Schur