Discretization of Continuous-time Transfer-function Models

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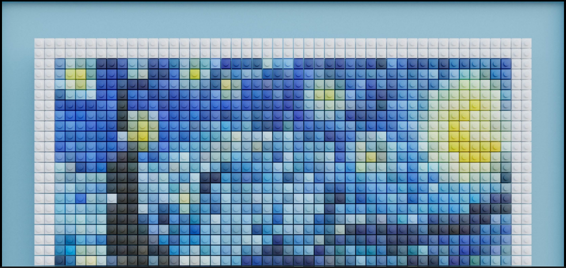
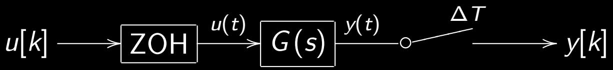


Overview

- consider the discrete-time controller implementation scheme

where $u[k]$ and $y[k]$ have the same sampling time
- for this note, we use $[k]$ to distinguish DT signals from their CT counter parts
- goal: to derive the transfer function from $u[k]$ to $y[k]$
- solution concept: let $u[k]$ be a discrete-time unit impulse (whose Z transform is 1) and obtain the Z transform of $y[k]$

Solution



- $u[k]$ is a DT impulse $\Rightarrow$ after ZOH

$$u(t) = \begin{cases} 1, & 0 \le t < \Delta T \\ 0, & \text{otherwise} \end{cases} = 1(t) - 1(t - \Delta T) \Longrightarrow U(s) = \frac{1 - e^{-s\Delta T}}{s}$$

- hence

$$y(t) = \mathcal{L}^{-1}\left[G(s)\frac{1 - e^{-s\Delta T}}{s}\right] = \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right] - \mathcal{L}^{-1}\left[G(s)\frac{e^{-s\Delta T}}{s}\right]$$

Solution

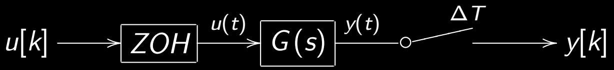
$$y(t) = \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right] - \mathcal{L}^{-1}\left[G(s)\frac{e^{-s\Delta T}}{s}\right]$$

- sampling $y(t)$ at ΔT and performing Z transform give:

$$\underbrace{Y(z)}_{=G(z)\times 1} = \mathcal{Z}\left\{\underbrace{\overbrace{\mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right]}^{\tilde{y}(t)}\Bigg|_{t=k\Delta T}}_{\triangleq \tilde{y}[k]} - \underbrace{\overbrace{\mathcal{L}^{-1}\left[G(s)\frac{e^{-s\Delta T}}{s}\right]}^{\tilde{y}(t-\Delta T)}\Bigg|_{t=k\Delta T}}_{=\tilde{y}[k-1]!!!}\right\}$$

$$= \mathcal{Z}\left\{\mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right]\Bigg|_{t=k\Delta T}\right\} - z^{-1}\mathcal{Z}\left\{\mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right]\Bigg|_{t=k\Delta T}\right\}$$

Solution



- $u[k] = \delta[k]$ $(U(z) = 1)$
- $Y(z) = \mathcal{Z}\left\{\mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right]\Big|_{t=k\Delta T}\right\} - z^{-1}\mathcal{Z}\left\{\mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right]\Big|_{t=k\Delta T}\right\}$
- $\Rightarrow$the zero order hold equivalent of $G(s)$ is

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right]\Big|_{t=k\Delta T}\right\}$$

where ΔT is the sampling time

Example: obtain the ZOH equivalent of

$$G(s)=\frac{a}{s+a}$$

- $\frac{G(s)}{s}=\frac{a}{s(s+a)}=\frac{1}{s}-\frac{1}{s+a}$
- hence
$$\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}=1(t)-e^{-at}1(t)$$
- sampling at ΔT gives $1[k]-e^{-ak\Delta T}1[k]$, whose Z transform is
$$\frac{z}{z-1}-\frac{z}{z-e^{-a\Delta T}}=\frac{z(1-e^{-a\Delta T})}{(z-1)(z-e^{-a\Delta T})}$$
- hence the ZOH equivalent is
$$(1-z^{-1})\frac{z(1-e^{-a\Delta T})}{(z-1)(z-e^{-a\Delta T})}=\frac{1-e^{-a\Delta T}}{z-e^{-a\Delta T}}$$

Matlab and Python commands

- *c2d(G,dt)*
- e.g., $G(s) = \frac{1}{s^2}$ and $\Delta T = 1$

```matlab
% matlab
dt = 1;
num = 1;
den = [1,0,0];
G = tf(num,den);
Gd = c2d(G,dt);
```

```python
#Python
import control as ct
dt = 1
num = [1]
den = [1,0,0]
G = ct.tf(num,den)
Gd = ct.c2d(G,dt)
print(Gd)
```

Exercise

Find the zero order hold equivalent of $G(s) = e^{-Ls}$, $2\Delta T < L < 3\Delta T$, where ΔT is the sampling time.