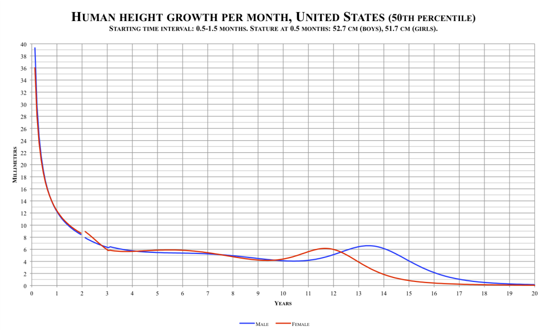
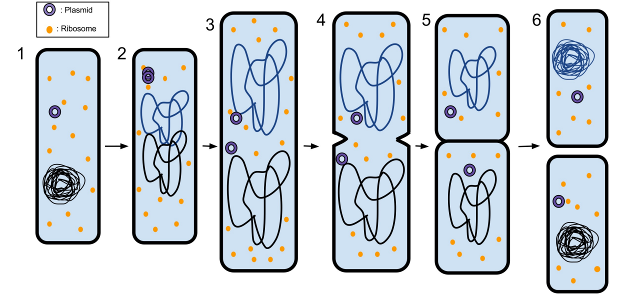
Solution of LTI State-Space Equations

Xu Chen

University of Washington



Population dynamics

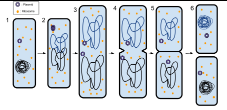


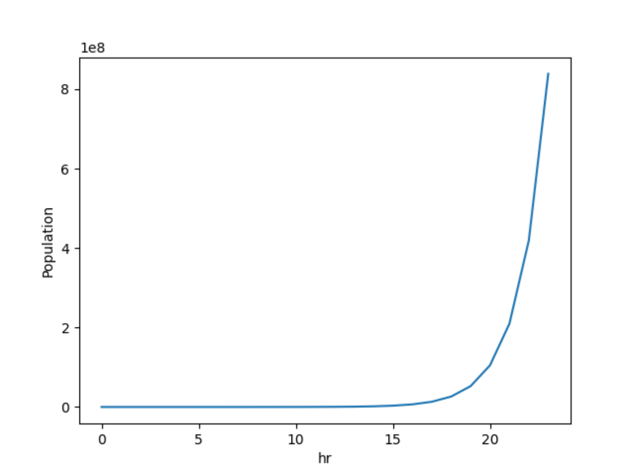
prokaryotic fission

- ~1 hour / division with infinite resource

$$100 \xrightarrow{1\text{hr}} 200 \xrightarrow{1\text{hr}} 400 \xrightarrow{1\text{hr}} 800 \xrightarrow{1\text{hr}} \dots$$

Population dynamics





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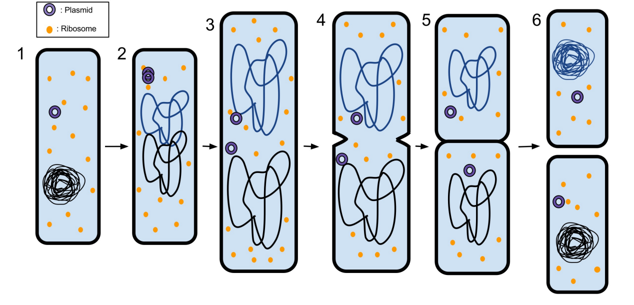


prokaryotic fission

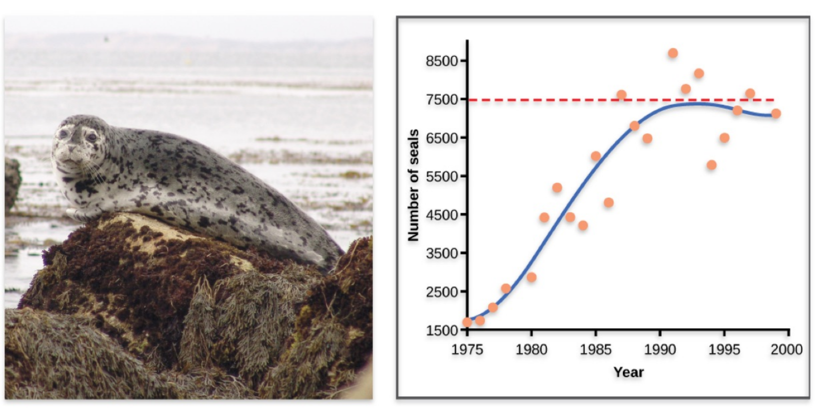
- ~1 hour / division with infinite resource

$$100 \xrightarrow{1\text{hr}} 200 \xrightarrow{1\text{hr}} 400 \xrightarrow{1\text{hr}} 800 \xrightarrow{1\text{hr}} \dots$$

- after 1 day:

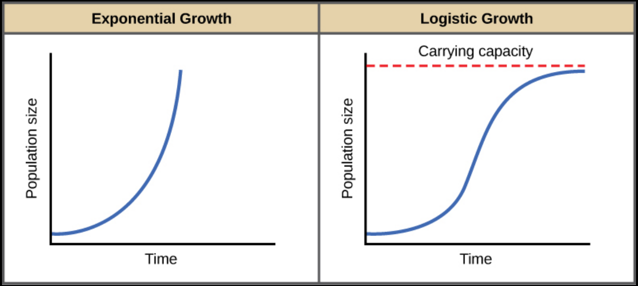
$$100 \xrightarrow[\frac{\Delta N}{N}=1]{1\text{hr}} 200 \xrightarrow{1\text{hr}} 400 \xrightarrow{1\text{hr}} \dots \longrightarrow 100 \times 2^{24} = 1.7\text{B!}$$

Population dynamics



"Environmental limits to population growth: Figure 1," by OpenStax College, Biology, CC BY 4.0.

The exponential function and population dynamics



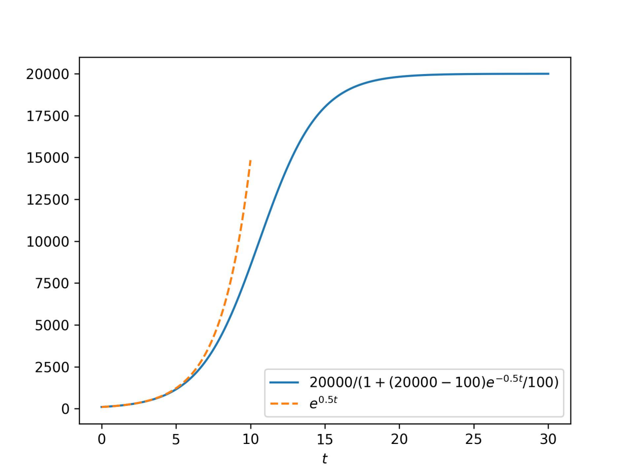
- more general population dynamics (w/ infinite resources)

$$\frac{dN}{dt} = \overbrace{(\text{birth rate} - \text{death rate})}^{r} N \Rightarrow N(t) = e^{rt} N(0)$$

- logistic growth (w/ limited resources in reality)

$$\frac{dN}{dt} = r\frac{K-N}{K} N \Rightarrow N(t) = \frac{KN_0 e^{rt}}{(K-N_0) + N_0 e^{rt}} = \frac{K}{1 + \frac{K-N_0}{N_0} e^{-rt}}$$

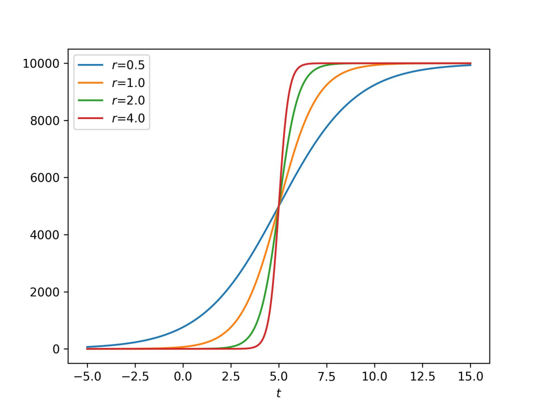
The exponential function and the logistic S curve: example



The logistic S curve

$\frac{K}{1+\frac{K-N_0}{N_0}e^{-rt}}$
can also be written as
$\frac{K}{1+e^{-r(t-t_o)}}$

- K: final value
- r: logistic growth rate
- t_o: midpoint



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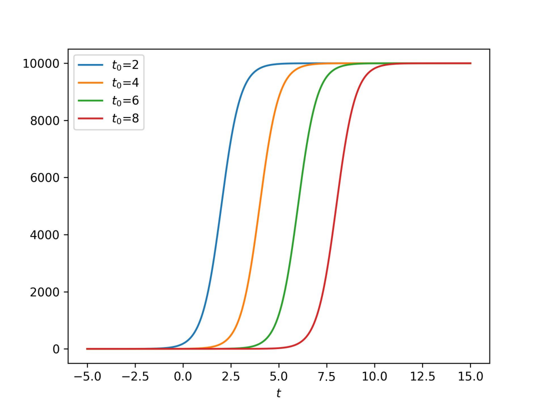

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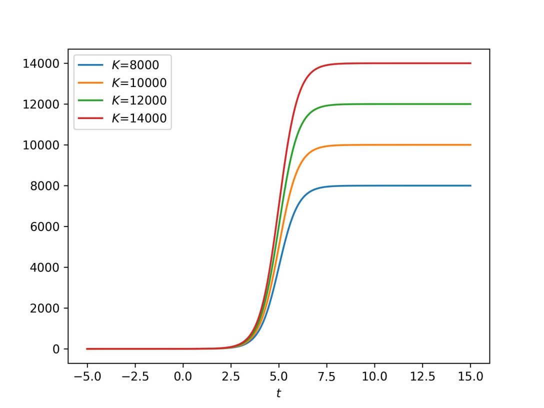
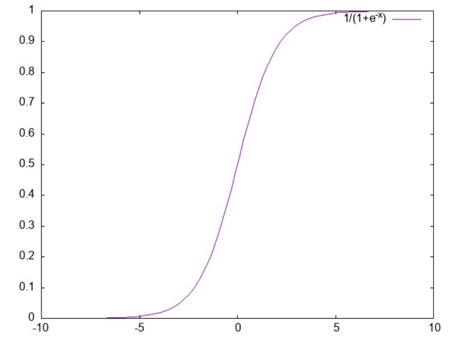


The logistic function in deep learning

$$\sigma(x) = \frac{1}{1+e^{-x}}$$



- transforms the input variables into a probability value between 0 and 1
- represents the likelihood of the dependent variable being 1 or 0

General LTI continuout-time state equation

$$\frac{dx}{dt} = Ax + Bu$$

$$\Sigma = \left[\begin{array}{c|c} A_{n\times n} & B_{n\times m} \\ \hline C_{n_y\times n} & D_{n_y\times m} \end{array}\right]$$

- to solve the vector equation $\dot{x} = Ax + Bu$, we start with the scalar case when $x, a, b, u \in \mathbb{R}$.

- fundamental property of exponential functions

$$\frac{d}{dt}e^{at} = ae^{at}, \quad \frac{d}{dt}e^{-at} = -ae^{-at}$$

- $\dot{x}(t) = ax(t) + bu(t),\ a \neq 0 \overset{\because e^{-at} \neq 0}{\Longrightarrow} e^{-at}\dot{x}(t) - e^{-at}ax(t) = e^{-at}bu(t)$
- namely,

$$\frac{d}{dt}\left\{e^{-at}x(t)\right\} = e^{-at}bu(t) \Leftrightarrow d\left\{e^{-at}x(t)\right\} = e^{-at}bu(t)\,dt$$

$$\Longrightarrow \boxed{e^{-at}x(t) = e^{-at_0}x(t_0) + \int_{t_0}^{t} e^{-a\tau}bu(\tau)\,d\tau}$$

The solution to $\dot{x} = ax + bu$

$$e^{-at}x(t) = e^{-at_0}x(t_0) + \int_{t_0}^{t} e^{-a\tau}bu(\tau)\,d\tau$$

when $t_0 = 0$, we have

$$\boxed{x(t) = \underbrace{e^{at}x(0)}_{\text{free response}} + \underbrace{\int_0^t e^{a(t-\tau)}bu(\tau)\,d\tau}_{\text{forced response}}}$$

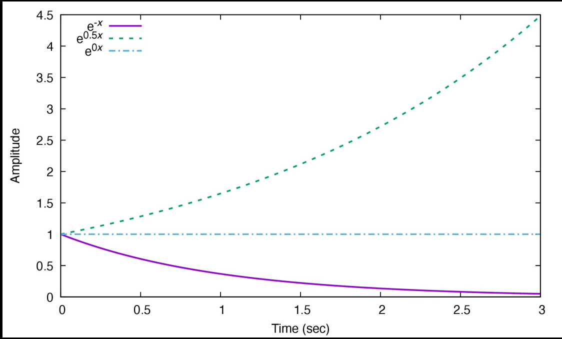
About e

- $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.71828\ldots$
- also $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$
- Python demonstration:

```
import math
math.e
for ii in range(10):
    print(sum(1/math.factorial(k) for k in range(ii)))
for ii in range(1,30):
    print((1+1/ii)**ii)
```

The solution to $\dot{x} = ax + bu$

Solution concepts of $e^{at}x(0)$



$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.71828\ldots$

$e^{-1} \approx 37\%$,

$e^{-2} \approx 14\%$,

$e^{-3} \approx 5\%$,

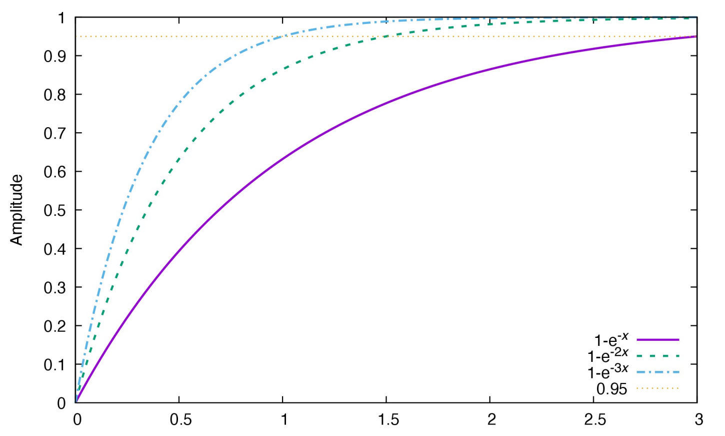
$e^{-4} \approx 2\%$

time constant $\tau \triangleq \frac{1}{|a|}$ when $a < 0$: after 3τ, $e^{at}x(0)$, the transient has approximately converged

The solution to $\dot{x} = ax + bu$

Unit step response

When $a < 0$ and $u(t) = 1(t)$ (the step function), the solution is $x(t) = \frac{b}{|a|}(1 - e^{at})$.



* Fundamental Theorem of Differential Equations

addresses the question of whether a dynamical system has a unique solution or not.

Theorem

Consider $\dot{x} = f(x,t)$, $x(t_0) = x_0$, *with:*

- $f(x,t)$ *piecewise continuous in* t *(continuous except at finite points of discontinuity)*
- $f(x,t)$ *Lipschitz continuous in* x *(satisfy the cone constraint:*$\|f(x,t) - f(y,t)\| \leq k(t)\|x - y\|$ *where* $k(t)$ *is piecewise continuous)*

then there exists a unique function of time $\phi(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$ *which is continuous almost everywhere and satisfies*

- $\phi(t_0) = x_0$
- $\dot{\phi}(t) = f(\phi(t), t)$, $\forall t \in \mathbb{R}_+ \backslash D$, *where* D *is the set of discontinuity points for* f *as a function of* t.

The solution to n^{th}-order LTI systems

- general state-space equation

$$\Sigma : \begin{cases} \dot{x}(t) & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases} \qquad x(t_0) = x_0 \in \mathbb{R}^n, \ A \in \mathbb{R}^{n\times n}$$

- solution

$$\boxed{x(t) = \underbrace{e^{A(t-t_0)}x_0}_{\text{free response}} + \underbrace{\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau}_{\text{forced response}}}$$

$$y(t) = Ce^{A(t-t_0)}x_0 + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

- in both the free and the forced responses, computing e^{At} is key
- $e^{A(t-t_0)}$: called the transition matrix

The state transition matrix e^{At}

scalar case with $a \in \mathbb{R}$: Taylor expansion gives

$$e^{at} = 1 + at + \frac{1}{2}(at)^2 + \cdots + \frac{1}{n!}(at)^n + \dots$$

the transition scalar $\Phi(t, t_0) = e^{a(t-t_0)}$ satisfies

$$\begin{aligned}
\Phi(t, t) &= 1 && \text{(transition to itself)} \\
\Phi(t_3, t_2)\Phi(t_2, t_1) &= \Phi(t_3, t_1) && \text{(consecutive transition)} \\
\Phi(t_2, t_1) &= \Phi^{-1}(t_1, t_2) && \text{(reverse transition)}
\end{aligned}$$

The state transition matrix e^{At}

matrix case with $A \in \mathbb{R}^{n\times n}$:

$$\boxed{e^{At} = I_n + At + \frac{1}{2}A^2t^2 + \dots + \frac{1}{n!}A^nt^n + \dots}$$

- as I_n and A^i are matrices of dimension $n \times n$, e^{At} must $\in \mathbb{R}^{n\times n}$
- the transition matrix $\Phi(t, t_0) = e^{A(t-t_0)}$ satisfies

$$\begin{aligned} e^{A0} &= I_n & \Phi(t,t) &= I_n \\ e^{At_1}e^{At_2} &= e^{A(t_1+t_2)} & \Phi(t_3,t_2)\Phi(t_2,t_1) &= \Phi(t_3,t_1) \\ e^{-At} &= \left[e^{At}\right]^{-1} & \Phi(t_2,t_1) &= \Phi^{-1}(t_1,t_2) \end{aligned}$$

- note, however, that $e^{At}e^{Bt} = e^{(A+B)t}$ if and only if $AB = BA$ (check by using Taylor expansion)

Computing a structured e^{At} via Taylor expansion

convenient when A is a diagonal or Jordan matrix

the case with a diagonal matrix $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$:

- $A^2 = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}, \dots, A^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$
- all matrices on the right side of

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{n!}A^nt^n + \dots$$

are easy to compute

Computing a structured e^{At} via Taylor expansion

convenient when A is a diagonal or Jordan matrix

the case with a diagonal matrix $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$:

$$
\begin{aligned}
e^{At} &= I + At + \frac{1}{2}A^2t^2 + \dots + \frac{1}{n!}A^n t^n + \dots \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & 0 & 0 \\ 0 & \lambda_2 t & 0 \\ 0 & 0 & \lambda_3 t \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\lambda_1^2 t^2 & 0 & 0 \\ 0 & \frac{1}{2}\lambda_2^2 t^2 & 0 \\ 0 & 0 & \frac{1}{2}\lambda_3^2 t^2 \end{bmatrix} + \dots \\
&= \begin{bmatrix} 1 + \lambda_1 t + \frac{1}{2}\lambda_1^2 t^2 + \dots & 0 & 0 \\ 0 & 1 + \lambda_2 t + \frac{1}{2}\lambda_2^2 t^2 + \dots & 0 \\ 0 & 0 & 1 + \lambda_3 t + \frac{1}{2}\lambda_3^2 t^2 + \dots \end{bmatrix} \\
&\underline{= \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix}}.
\end{aligned}
$$

Computing a structured e^{At} via Taylor expansion

the case with a Jordan matrix $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$:

- decompose $A = \underbrace{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}}_{\lambda I_3} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{N} \Rightarrow$

  $$e^{At} = e^{(\lambda I_3 t + Nt)}$$

- also, $(\lambda I_3 t)(Nt) = \lambda N t^2 = (Nt)(\lambda I_3 t)$ and hence $e^{(\lambda I_3 t + Nt)} = e^{\lambda I t} e^{Nt}$

- thus

$$\underline{e^{At} = e^{(\lambda I_3 t + Nt)} = e^{\lambda I t} e^{Nt} \overset{\because e^{\lambda I t} = e^{\lambda t} I}{=\!=} e^{\lambda t} e^{Nt}}$$

Computing a structured e^{At} via Taylor expansion

$$A = \underbrace{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}}_{\lambda I_3} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{N}$$

$$e^{At} = e^{\lambda t} e^{Nt}$$

- N is *nilpotent*[1]*:* $N^3 = N^4 = \cdots = 0I_3$, yielding

$$e^{Nt} = I_3 + Nt + \frac{1}{2}N^2t^2 + \cancelto{0}{\frac{1}{3!}N^3t^3} + \cancelto{0}{\ldots} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

- thus

$$e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$

[1] "nil" $\sim$ zero; "potent" $\sim$ taking powers.

Example (mass moving on a straight line with zero friction and no external force)

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$x(t) = e^{At}x(0)$ where

$$e^{At} = I + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t + \frac{1}{2!}\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} t^2 + \dots = \underline{\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}}.$$

Computing low-order e^{At} via column solutions

an intuition of the matrix entries in e^{At}: consider:

$$\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x, \quad x(0) = x_0$$

$$\begin{aligned} x(t) = e^{At}x(0) &= \left[\begin{array}{c|c} \overbrace{a_1(t)}^{\text{1st column}} & \overbrace{a_2(t)}^{\text{2nd column}} \end{array} \right] \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} & (1) \\ &= a_1(t)x_1(0) + a_2(t)x_2(0) & (2) \end{aligned}$$

observation

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow x(t) = a_1(t),$$

$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow x(t) = a_2(t).$$

Computing low-order e^{At} via column solutions

$$\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x, \quad x(0) = x_0$$

hence, we can obtain e^{At} from:

- write out $\begin{aligned} \dot{x}_1(t) =& x_2(t) \\ \dot{x}_2(t) =& -x_2(t) \end{aligned} \Rightarrow \begin{aligned} x_1(t) =& e^{0t}x_1(0) + \int_0^t e^{0(t-\tau)}x_2(\tau)d\tau \\ x_2(t) =& e^{-t}x_2(0) \end{aligned}$
- let $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $\begin{aligned} x_1(t) \equiv 1 \\ x_2(t) \equiv 0 \end{aligned}$, namely $x(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- let $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $x_2(t) = e^{-t}$ and $x_1(t) = 1 - e^{-t}$, or more compactly, $x(t) = \begin{bmatrix} 1 - e^{-t} \\ e^{-t} \end{bmatrix}$
- using (1), write out directly $e^{At} = \begin{bmatrix} 1 & 1 - e^{-t} \\ 0 & e^{-t} \end{bmatrix}$

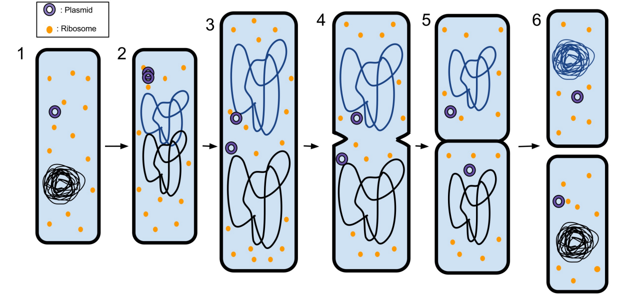
Computing low-order e^{At} via column solutions

Exercise

Compute e^{At} where

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

Recall: population dynamics



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- or: $N(k+1) = 2N(k) \Rightarrow N(k) = 2^k N(0)$

Discrete-time LTI case

discrete-time system:

$$x(k+1) = Ax(k) + Bu(k),\ x(0) = x_0,$$

iteration of the state-space equation gives:

$$x(k) = A^{k-k_0}x(k_o) + \left[A^{k-k_0-1}B, A^{k-k_0-2}B, \cdots, B\right]\begin{bmatrix} u(k_0) \\ u(k_0+1) \\ \vdots \\ u(k-1) \end{bmatrix}$$

$$\Leftrightarrow \boxed{x(k) = \underbrace{A^{k-k_0}x(k_o)}_{\text{free response}} + \underbrace{\sum_{j=k_0}^{k-1} A^{k-1-j}Bu(j)}_{\text{forced response}}}$$

Discrete-time LTI case

$$\boxed{x(k) = \underbrace{A^{k-k_0}x(k_o)}_{\text{free response}} + \underbrace{\sum_{j=k_0}^{k-1} A^{k-1-j}Bu(j)}_{\text{forced response}}}$$

$\Phi(k,j) = A^{k-j}$: the transition matrix:

$$\Phi(k,k) = 1$$

$$\Phi(k_3,k_2)\Phi(k_2,k_1) = \Phi(k_3,k_1) \qquad k_3 \geq k_2 \geq k_1$$

$$\Phi(k_2,k_1) = \Phi^{-1}(k_1,k_2) \quad \text{if and only if } A \text{ is nonsingular}$$

The state transition matrix A^k

similar to the continuous-time case, when A is a diagonal or Jordan matrix, A^k is easy

- diagonal matrix $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$: $A^k = \begin{bmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{bmatrix}$

Computing a structured A^k via Taylor expansion

- Jordan canonical form

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}}_{\lambda I_3} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{N}:$$

$$\begin{aligned} A^k &= (\lambda I_3 + N)^k \\ &= (\lambda I_3)^k + k(\lambda I_3)^{k-1} N + \underbrace{\begin{pmatrix} k \\ 2 \end{pmatrix}}_{\text{2 combination}} (\lambda I_3)^{k-2} N^2 + \underbrace{\begin{pmatrix} k \\ 3 \end{pmatrix} (\lambda I_3)^{k-3} N^3 + \dots}_{N^3 = N^4 = \dots = 0 I_3} \\ &= \begin{bmatrix} \lambda^k & 0 & 0 \\ 0 & \lambda^k & 0 \\ 0 & 0 & \lambda^k \end{bmatrix} + k\lambda^{k-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{k(k-1)}{2} \lambda^{k-2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2!} k(k-1) \lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix} \end{aligned}$$

Computing a structured A^k via Taylor expansion

Exercise

Recall that $\begin{pmatrix} k \\ 3 \end{pmatrix} = \frac{1}{3!}k(k-1)(k-2)$. Show

$$A = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$$\Rightarrow A^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2!}k(k-1)\lambda^{k-2} & \frac{1}{3!}k(k-1)(k-2)\lambda^{k-3} \\ 0 & \lambda^k & k\lambda^{k-1} & \frac{1}{2!}k(k-1)\lambda^{k-2} \\ 0 & 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & 0 & \lambda^k \end{bmatrix}$$

Explicit computation of a general e^{At}

- why another method: general matrices may not be diagonal or Jordan
- approach: transform a general matrix to a diagonal or Jordan form, via similarity transformation

Computing e^{At} via similarity transformation

principle concept:

- given

$$\dot{x}(t) = Ax(t) + Bu(t), \ x(t_0) = x_0 \in \mathbb{R}^n, \ A \in \mathbb{R}^{n\times n}$$

- find a nonsingular $T \in \mathbb{R}^{n\times n}$ such that a coordinate transformation defined by $x(t) = Tx^*(t)$ yields

$$\frac{d}{dt}\left(Tx^*(t)\right) = ATx^*(t) + Bu(t)$$

$$\frac{d}{dt}x^*(t) = \underbrace{T^{-1}AT}_{\triangleq \Lambda\text{: diagonal or Jordan}} x^*(t) + \underbrace{T^{-1}B}_{B^*} u(t)$$

$$x^*(0) = T^{-1}x_0$$

Computing e^{At} via similarity transformation

- when $u(t) = 0$

$$\dot{x}(t) = Ax(t) \overset{x=Tx^*}{\Longrightarrow} \frac{d}{dt}x^*(t) = \underbrace{T^{-1}AT}_{\triangleq \Lambda\text{: diagonal or Jordan}} x^*(t)$$

- now $x^*(t)$ can be solved easily: e.g., if $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, then

$$x^*(t) = e^{\Lambda t}x^*(0) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} x_1^*(0) \\ x_2^*(0) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t}x_1^*(0) \\ e^{\lambda_2 t}x_2^*(0) \end{bmatrix}.$$

- $x(t) = Tx^*(t)$ then yields

$$x(t) = Te^{\Lambda t}x^*(0) = Te^{\Lambda t}T^{-1}x_0$$

- on the other hand, $x(t) = e^{At}x_0 \Rightarrow$

$$\boxed{e^{At} = Te^{\Lambda t}T^{-1}}$$

Similarity transformation

- existence of solutions: T comes from the theory of eigenvalues and eigenvectors in linear algebra
- if A and $B \in \mathbb{C}^{n\times n}$ are similar: $A = TBT^{-1}$, $T \in \mathbb{C}^{n\times n}$, then
  - their A^n and B^n are also similar: e.g.,

$$A^2 = TBT^{-1}TBT^{-1} = TB^2T^{-1}$$

  - their exponential matrices are also similar

$$e^{At} = Te^{Bt}T^{-1}$$

as

$$\begin{aligned} Te^{Bt}T^{-1} &= T(I_n + Bt + \frac{1}{2}B^2t^2 + \dots)T^{-1} \\ &= TI_nT^{-1} + TBtT^{-1} + \frac{1}{2}TB^2t^2T^{-1} + \dots \\ &= I + At + \frac{1}{2}A^2t^2 + \dots = e^{At} \end{aligned}$$

Similarity transformation

- for $A \in \mathbb{R}^{n\times n}$, an eigenvalue $\lambda \in \mathcal{C}$ of A is the solution to the characteristic equation

$$\boxed{\det\left(A-\lambda I\right)=0} \tag{3}$$

- the corresponding eigenvectors are the nonzero solutions to

$$At=\lambda t \Leftrightarrow \left(A-\lambda I\right)t=0 \tag{4}$$

Similarity transformation

The case with distinct eigenvalues (diagonalization)

recall: when $A \in \mathbb{R}^{n\times n}$ has n distinct eigenvalues such that

$$Ax_1 = \lambda_1 x_1$$
$$\vdots$$
$$Ax_n = \lambda_n x_n$$

or equivalently

$$A \underbrace{[x_1, x_2, \dots, x_n]}_{\triangleq T} = [x_1, x_2, \dots, x_n] \underbrace{\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}}_{\Lambda}$$

$[x_1, x_2, \dots, x_n]$ is square and invertible. Hence

$$A = T\Lambda T^{-1},\ \Lambda = T^{-1}AT$$

Example (Mechanical system with strong damping)

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- find eigenvalues: $\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -2 & -\lambda - 3 \end{bmatrix} = (\lambda + 2)(\lambda + 1) \Rightarrow \lambda_1 = -2, \lambda_2 = -1$
- find associate eigenvectors:
  - $\lambda_1 = -2$: $(A - \lambda_1 I)\, t_1 = 0 \Rightarrow t_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
  - $\lambda_1 = -1$: $(A - \lambda_2 I)\, t_2 = 0 \Rightarrow t_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- define T and Λ: $T = \begin{bmatrix} t_1 & t_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$,

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

Example (Mechanical system with strong damping)

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- $T = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$
- compute $T^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$
- compute $e^{At} = Te^{\Lambda t}T^{-1} = T\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-1t} \end{bmatrix}T^{-1} =$
$\begin{bmatrix} -e^{-2t} + 2e^{-t} & -e^{-2t} + e^{-t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}$

Similarity transform: diagonalization

Physical interpretations

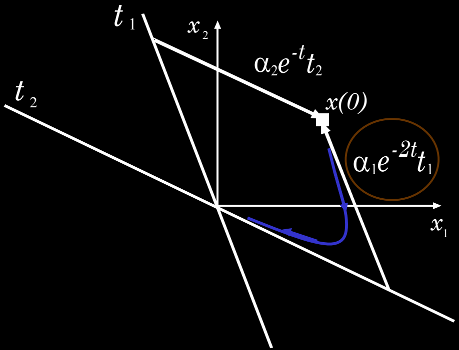
- diagonalized system:
$$x^*(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} x_1^*(0) \\ x_2^*(0) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} x_1^*(0) \\ e^{\lambda_2 t} x_2^*(0) \end{bmatrix}$$
- $x(t) = Tx^*(t) = e^{\lambda_1 t} x_1^*(0) t_1 + e^{\lambda_2 t} x_2^*(0) t_2$ then decomposes the state trajectory into two modes parallel to the two eigenvectors.

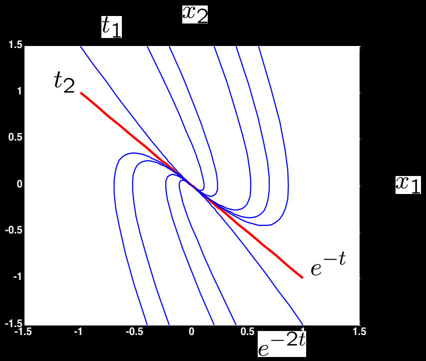
Similarity transform: diagonalization

Physical interpretations

- if $x(0)$ is aligned with one eigenvector, say, t_1, then $x_2^*(0) = 0$ and $x(t) = e^{\lambda_1 t} x_1^*(0) t_1 + e^{\lambda_2 t} x_2^*(0) t_2$ dictates that $x(t)$ will stay in the direction of t_1
- i.e., if the state initiates along the direction of one eigenvector, then the free response will stay in that direction without "making turns"
- if $\lambda_1 < 0$, then $x(t)$ will move towards the origin of the state space; if $\lambda_1 = 0$, $x(t)$ will stay at the initial point; and if positive, $x(t)$ will move away from the origin along t_1
- furthermore, the magnitude of λ_1 determines the speed of response

Physical interpretations: example





Similarity transformation

The case with complex eigenvalues

consider the undamped spring-mass system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ \det(A-\lambda I) = \lambda^2+1 = 0 \Rightarrow \lambda_{1,2,} = \pm j.$$

the eigenvectors are

$$\lambda_1 = j: \ (A - jI)t_1 = 0 \Rightarrow t_1 = \begin{bmatrix} 1 \\ j \end{bmatrix}$$

$$\lambda_2 = -j: \ (A + jI)t_2 = 0 \Rightarrow t_2 = \begin{bmatrix} 1 \\ -j \end{bmatrix} \text{ (complex conjugate of } t_1\text{)}.$$

hence

$$T = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}, \ T^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix}$$

Similarity transformation

The case with complex eigenvalues

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- $\lambda_{1,2,} = \pm j$
- $T = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}, \; T^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix}$
- we have

$$e^{At} = Te^{\Lambda t}T^{-1} = T\begin{bmatrix} e^{jt} & 0 \\ 0 & e^{-jt} \end{bmatrix}T^{-1} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

Similarity transformation
The case with complex eigenvalues

for a general $A \in \mathbb{R}^{2\times 2}$ with complex eigenvalues $\sigma \pm j\omega$, by using $T = [t_R, t_I]$, where t_R and t_I are the real and the imaginary parts of t_1, an eigenvector associated with $\lambda_1 = \sigma + j\omega$, $x = Tx^*$ transforms $\dot{x} = Ax$ to

$$\dot{x}^*(t) = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} x^*(t)$$

and

$$e^{\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} t} = \begin{bmatrix} e^{\sigma t}\cos\omega t & e^{\sigma t}\sin\omega t \\ -e^{\sigma t}\sin\omega t & e^{\sigma t}\cos\omega t \end{bmatrix}.$$

Similarity transformation

The case with repeated eigenvalues via generalized eigenvectors

consider $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$: two repeated eigenvalues $\lambda(A) = 1$, and

$$(A - \lambda I)\, t_1 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} t_1 = 0 \Rightarrow t_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

- No other linearly independent eigenvectors exist. What next?
- A is already very similar to the Jordan form. Try instead

$$A \begin{bmatrix} t_1 & t_2 \end{bmatrix} = \begin{bmatrix} t_1 & t_2 \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},$$

which requires $At_2 = t_1 + \lambda t_2$, i.e.,

$$(A - \lambda I)\, t_2 = t_1 \Leftrightarrow \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} t_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow t_2 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

t_2 is linearly independent from $t_1 \Rightarrow t_1$ and t_2 span $\mathbb{R}^2$. (t_2 is called a generalized eigenvector.)

Similarity transformation

The case with repeated eigenvalues via generalized eigenvectors

for general 3×3 matrices with $\det(\lambda I - A) = (\lambda - \lambda_m)^3$, i.e., $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_m$, we look for T such that

$$A = TJT^{-1}$$

where J has three canonical forms:

$$i), \begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix}, \quad iii), \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 1 \\ 0 & 0 & \lambda_m \end{bmatrix}$$

$$ii), \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix} \text{ or } \begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \lambda_m & 1 \\ 0 & 0 & \lambda_m \end{bmatrix}$$

Similarity transformation

The case with repeated eigenvalues via generalized eigenvectors

$$i),\ A = TJT^{-1},\ J = \begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix}$$

this happens

- when A has three linearly independent eigenvectors, i.e., $(A - \lambda_m I)t = 0$ yields t_1, t_2, and t_3 that span $\mathbb{R}^3$.
- mathematically: when $\text{nullity}\,(A - \lambda_m I) = 3$, namely, $\text{rank}(A - \lambda_m I) = 3 - \text{nullity}\,(A - \lambda_m I) = 0$.

Similarity transformation

The case with repeated eigenvalues via generalized eigenvectors

$$ii),\ A = TJT^{-1},\ J = \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix} \text{ or } \begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \lambda_m & 1 \\ 0 & 0 & \lambda_m \end{bmatrix}$$

- this happens when $(A - \lambda_m I)t = 0$ yields two linearly independent solutions, i.e., when nullity $(A - \lambda_m I) = 2$
- we then have, e.g.,

$$A[t_1, t_2, t_3] = [t_1, t_2, t_3] \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix}$$

$$\Leftrightarrow [\lambda_m t_1, t_1 + \lambda_m t_2, \lambda_m t_3] = [At_1, At_2, At_3] \tag{5}$$

- t_1 and t_3 are the directly computed eigenvectors.
- For t_2, the second column of (5) gives

$$(A - \lambda_m I)\, t_2 = t_1$$

Similarity transformation

The case with repeated eigenvalues via generalized eigenvectors

$$iii),\ A = TJT^{-1},\ J = \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 1 \\ 0 & 0 & \lambda_m \end{bmatrix}$$

- this is for the case when $(A - \lambda_m I)t = 0$ yields only one linearly independent solution, i.e., when nullity$(A - \lambda_m I) = 1$
- We then have

$$A[t_1, t_2, t_3] = [t_1, t_2, t_3] \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 1 \\ 0 & 0 & \lambda_m \end{bmatrix}$$

$$\Leftrightarrow [\lambda_m t_1, t_1 + \lambda_m t_2, t_2 + \lambda_m t_3] = [At_1, At_2, At_3]$$

yielding

$$\begin{aligned} (A - \lambda_m I)\, t_1 &= 0 \\ (A - \lambda_m I)\, t_2 &= t_1,\ (t_2 : \text{ generalized eigenvector}) \\ (A - \lambda_m I)\, t_3 &= t_2,\ (t_3 : \text{ generalized eigenvector}) \end{aligned}$$

Example

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \ \det(A - \lambda I) = \lambda^2 \Rightarrow \lambda_1 = \lambda_2 = 0, \ J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- two repeated eigenvalues with rank$(A - 0I) = 1 \Rightarrow$ only one linearly independent eigenvector:$(A - 0I)\, t_1 = 0 \Rightarrow t_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- generalized eigenvector:$(A - 0I)\, t_2 = t_1 \Rightarrow t_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- coordinate transform matrix:
$$T = [t_1, t_2] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \ T^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$e^{At} = Te^{Jt}T^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{0t} & te^{0t} \\ 0 & e^{0t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 - t & t \\ -t & 1 + t \end{bmatrix}$$

Example

$A=\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix}$, $\det(A-\lambda I)=\lambda^2 \Rightarrow \lambda_1=\lambda_2=0.$

observation:

- $\lambda_1=0,\ t_1=\begin{bmatrix}1\\1\end{bmatrix}$ implies that if $x_1(0)=x_2(0)$ then the response is characterized by $e^{0t}=1$
- i.e., $x_1(t)=x_1(0)=x_2(0)=x_2(t)$. This makes sense because $\dot{x}_1=-x_1+x_2$ from the state equation.

Example (Multiple eigenvectors)

Obtain the eigenvectors of

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \ (\lambda_1 = 5,\ \lambda_2 = \lambda_3 = -3).$$

Generalized eigenvectors

Physical interpretation.

when $\dot{x} = Ax$, $A = TJT^{-1}$ with $J = \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix}$, we have

$$x(t) = e^{At}x(0) = T \begin{bmatrix} e^{\lambda_m t} & te^{\lambda_m t} & 0 \\ 0 & e^{\lambda_m t} & 0 \\ 0 & 0 & e^{\lambda_m t} \end{bmatrix} T^{-1}x(0)$$

$$= T \begin{bmatrix} e^{\lambda_m t} & te^{\lambda_m t} & 0 \\ 0 & e^{\lambda_m t} & 0 \\ 0 & 0 & e^{\lambda_m t} \end{bmatrix} \overbrace{\cancel{T^{-1}T}}^{I} x^*(0)$$

- if the initial condition is in the direction of t_1, i.e., $x^*(0) = [x_1^*(0), 0, 0]^T$ and $x_1^*(0) \neq 0$, the above equation yields $x(t) = x_1^*(0) t_1 e^{\lambda_m t}$

Generalized eigenvectors

Physical interpretation Cont'd.

when $\dot{x} = Ax$, $A = TJT^{-1}$ with $J = \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix}$, we have

$$x(t) = e^{At}x(0) = T \begin{bmatrix} e^{\lambda_m t} & te^{\lambda_m t} & 0 \\ 0 & e^{\lambda_m t} & 0 \\ 0 & 0 & e^{\lambda_m t} \end{bmatrix} T^{-1}x(0)$$

$$= T \begin{bmatrix} e^{\lambda_m t} & te^{\lambda_m t} & 0 \\ 0 & e^{\lambda_m t} & 0 \\ 0 & 0 & e^{\lambda_m t} \end{bmatrix} \overset{I}{\cancel{T^{-1}T}} x^*(0)$$

- if $x(0)$ starts in the direction of t_2, i.e., $x^*(0) = [0, x_2^*(0), 0]^T$, then $x(t) = x_2^*(0)(t_1 te^{\lambda_m t} + t_2 e^{\lambda_m t})$. In this case, the response does not remain in the direction of t_2 but is confined in the subspace spanned by t_1 and t_2

Example

Obtain eigenvalues of J and e^{Jt} by inspection:

$$J = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}.$$

Explicit computation of A^k

everything in getting the similarity transform applies to the DT case:

$$A^k = T\Lambda^k T^{-1} \text{ or } A^k = TJ^k T^{-1}.$$

J	J^k
$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$	$\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}$
$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$	$\begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2!}k\left(k-1\right)\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}$
$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$	$\begin{bmatrix} \lambda^k & k\lambda^{k-1} & 0 \\ 0 & \lambda^k & 0 \\ 0 & 0 & \lambda_3^k \end{bmatrix}$
$\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$	$r^k \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$ $r = \sqrt{\sigma^2 + \omega^2}$ $\theta = \tan^{-1} \frac{\omega}{\sigma}$

Example

Write down J^k for $J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ and

$$J = \begin{bmatrix} -10 & 1 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -100 & 1 \\ 0 & 0 & 0 & -1 & -100 \end{bmatrix}.$$

1. Introduction

2. State-space solution formula
    - Continuous-time case
        - The solution to $\dot{x} = ax + bu$
        - The solution to n^{th}-order LTI systems
        - The state transition matrix e^{At}
        - Computing e^{At} when A is diagonal or in Jordan form
    - Discrete-time LTI case
        - The state transition matrix A^k
        - Computing A^k when A is diagonal or in Jordan form

3. Explicit computation of the state transition matrix e^{At}

4. Explicit Computation of the State Transition Matrix A^k

5. Transition Matrix via Inverse Transformation

Transition matrix via inverse transformation

	Continuous-time system
state eq.	$\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0$
solution	$x(t) = \underbrace{e^{At}x(0)}_{\text{free response}} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}_{\text{forced response}}$
transition matrix	e^{At}

On the other hand, from Laplace transform:

$$\dot{x}(t) = Ax(t) + Bu(t) \Rightarrow X(s) = \underbrace{(sI - A)^{-1}x(0)}_{\text{free response}} + \underbrace{(sI - A)^{-1}BU(s)}_{\text{forced response}}$$

Comparing $x(t)$ and $X(s)$ gives

$$\boxed{e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}} \tag{6}$$

Example

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

$$\begin{aligned} e^{At} &= \mathcal{L}^{-1} \begin{bmatrix} s-\sigma & -\omega \\ \omega & s-\sigma \end{bmatrix}^{-1} \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{(s-\sigma)^2+\omega^2} \begin{bmatrix} s-\sigma & \omega \\ -\omega & s-\sigma \end{bmatrix} \right\} \\ &= e^{\sigma t} \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} \end{aligned}$$

Transition matrix via inverse transformation (DT case)

	Discrete-time system
state eq.	$x(k+1) = Ax(k) + Bu(k)$, $x(0) = x_0$
solution	$x(k) = \underbrace{A^k x(0)}_{\text{free response}} + \underbrace{\sum_{j=0}^{(k-1)} A^{(k-1-j)} Bu(j)}_{\text{forced response}}$
transition matrix	transition matrix A^k

On the other hand, from Z transform:

$$X(z) = (zI - A)^{-1} zx(0) + (zI - A)^{-1} BU(s)$$

Hence

$$\boxed{A^k = \mathcal{Z}^{-1}\left\{(zI - A)^{-1} z\right\}} \tag{7}$$

Example

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

$$\begin{aligned} A^k &= \mathcal{Z}^{-1}\left\{ z \begin{bmatrix} z-\sigma & -\omega \\ \omega & z-\sigma \end{bmatrix}^{-1} \right\} \\ &= \mathcal{Z}^{-1}\left\{ \frac{z}{(z-\sigma)^2+\omega^2} \begin{bmatrix} z-\sigma & \omega \\ -\omega & z-\sigma \end{bmatrix} \right\} \\ &= \mathcal{Z}^{-1}\left\{ \frac{z}{z^2-2r\cos\theta z+r^2} \begin{bmatrix} z-r\cos\theta & r\sin\theta \\ -r\sin\theta & z-r\cos\theta \end{bmatrix} \right\} \\ &\quad,\ r=\sqrt{\sigma^2+\omega^2},\ \theta=\tan^{-1}\frac{\omega}{\sigma} \\ &= r^k \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix} \end{aligned}$$

Example

Consider $A = \begin{bmatrix} 0.7 & 0.3 \\ 0.1 & 0.5 \end{bmatrix}$. We have

$$
\begin{aligned}
& (zI - A)^{-1} z \\
&= \begin{bmatrix} \frac{z(z-0.5)}{(z-0.8)(z-0.4)} & \frac{0.3z}{(z-0.8)(z-0.4)} \\ \frac{0.1z}{(z-0.8)(z-0.4)} & \frac{z(z-0.7)}{(z-0.8)(z-0.4)} \end{bmatrix} \\
&= \begin{bmatrix} \frac{0.75z}{z-0.8} + \frac{0.25z}{z-0.4} & \frac{0.75z}{z-0.8} - \frac{0.75z}{z-0.4} \\ \frac{0.25z}{z-0.8} - \frac{0.25z}{z-0.4} & \frac{0.25z}{z-0.8} + \frac{0.75z}{z-0.4} \end{bmatrix} \\
\Rightarrow A^k &= \begin{bmatrix} 0.75\,(0.8)^k + 0.25\,(0.4)^k & 0.75\,(0.8)^k - 0.75\,(0.4)^k \\ 0.25\,(0.8)^k - 0.25\,(0.4)^k & 0.25\,(0.8)^k + 0.75\,(0.4)^k \end{bmatrix}
\end{aligned}
$$