1 Solution of Time-Invariant State-Space Equations

1.1 Continuous-Time State-Space Solutions

1.1.1 The Solution to $\dot{x} = ax + bu$

To solve the vector equation $\dot{x} = Ax + Bu$, we start with the scalar case when $x, a, b, u \in \mathbb{R}$. The solution can be easily derived using one fundamental property of exponential functions, that

$$\frac{d}{dt}e^{at} = ae^{at},$$

and

$$\frac{d}{dt}e^{-at} = -ae^{-at}.$$

Consider the ODE

$$\dot{x}(t) = ax(t) + bu(t),\ a \neq 0.$$

Since $e^{-at} \neq 0$, the above is equivalent to

$$e^{-at}\dot{x}(t) - e^{-at}ax(t) = e^{-at}bu(t),$$

namely,

$$\begin{aligned} \frac{d}{dt}\left\{e^{-at}x(t)\right\} &= e^{-at}bu(t), \\ \Leftrightarrow d\left\{e^{-at}x(t)\right\} &= e^{-at}bu(t)\,dt. \end{aligned}$$

Integrating both sides from t_0 to t_1 gives

$$e^{-at_1}x(t_1) = e^{-at_0}x(t_0) + \int_{t_0}^{t_1} e^{-at}bu(t)\,dt.$$

It does not matter whether we use t or τ in the integration $\int_{t_0}^{t_1} e^{-at}bu(t)\,dt$. Hence we can change notations and get

$$\begin{aligned} e^{-at}x(t) &= e^{-at_0}x(t_0) + \int_{t_0}^{t} e^{-a\tau}bu(\tau)\,d\tau, \\ \Leftrightarrow x(t) &= e^{a(t-t_o)}x(t_0) + \int_{t_0}^{t} e^{a(t-\tau)}bu(\tau)\,d\tau. \end{aligned}$$

Taking $t_0 = 0$ gives

$$\boxed{x(t) = \underbrace{e^{at}x(0)}_{\text{free response}} + \underbrace{\int_0^t e^{a(t-\tau)}bu(\tau)\,d\tau}_{\text{forced response}}} \tag{1}$$

where the free response is the part of the solution due only to initial conditions when no input is applied, and the forced response is the part due to the input alone.

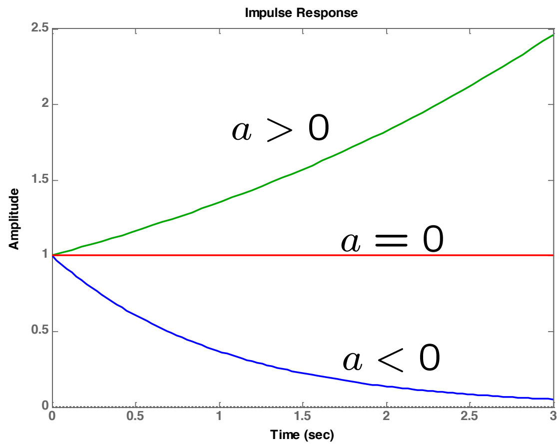
Solution Concepts.

Time Constant. When $a < 0$, e^{at} is a decaying function. For the free response $e^{at}x(0)$, the exponential function satisfies $e^{-1} \approx 37\%$, $e^{-2} \approx 14\%$, $e^{-3} \approx 5\%$, and $e^{-4} \approx 2\%$. The time constant is defined as

$$T = \frac{1}{|a|}.$$

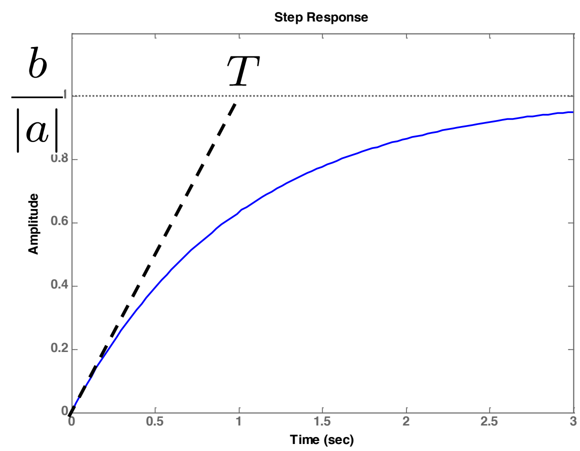
After three time constants, the free response reduces to 5% of its initial value. Roughly, we say the free response has died down.

Graphically, the exponential function looks like:



Unit Step Response. When $a < 0$ and $u(t) = 1(t)$ (the step function), the solution is

$$x(t) = \frac{b}{|a|}(1 - e^{at}).$$



1.1.2 * Fundamental Theorem of Differential Equations

The following theorem addresses the question of whether a dynamical system has a unique solution or not.

Theorem 1. *Consider* $\dot{x} = f(x, t)$, $x(t_0) = x_0$, *with:*

- $f(x, t)$ piecewise continuous in t
- $f(x, t)$ Lipschitz continuous in x

then there exists a *unique* function of time $\phi(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$ which is continuous almost everywhere and satisfies

- $\phi(t_0) = x_0$
- $\dot{\phi}(t) = f(\phi(t), t)$, $\forall t \in \mathbb{R}_+ \backslash D$, where D is the set of discontinuity points for f as a function of t.

Remark 1. Piecewise continuous functions are continuous except at finite points of discontinuity.

- example 1: $f(t) = |t|$
- example 2:

$$f(x, t) = \begin{cases} A_1 x, & t \le t_1 \\ A_2 x, & t > t_1 \end{cases}$$

Lipschitz continuous functions are those that satisfy the cone constraint:

$$\|f(x,t) - f(y,t)\| \leq k(t)\|x - y\|$$

where $k(t)$ is piecewise continuous.

- example: $f(x) = Ax + B$
- a graphical representation of a Lipschitz function is that it must stay within a cone in the space of $(x, f(x))$
- a function is Lipschitz continuous if it is continuously differentiable with its derivative bounded everywhere. This is a sufficient condition. Functions can be Lipschitz continuous but not differentiable: e.g., the saturation function and $f(x) = |x|$.
- A continuous function is not necessarily Lipschitz continuous at all: e.g., a function whose derivative at $x = 0$ is infinity.

1.1.3 The Solution to $n^{\mathbf{th}}$-order LTI System

Consider the general state-space equation

$$\Sigma : \begin{cases} \dot{x}(t) & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases} \qquad x(t_0) = x_0 \in \mathbb{R}^n, \ A \in \mathbb{R}^{n\times n}$$

Only the first equation here is a differential equation. Once we solve this equation for $x(t)$, we can find $y(t)$ very easily using the second equation. Also, $f(x,t) = Ax + Bu$ satisfies the conditions in Fundamental Theorem for Differential Equations. A unique solution thus exists. The solution of the state-space equations is given in closed form by

$$\boxed{x(t) = \underbrace{e^{A(t-t_0)}x_0}_{\text{free response}} + \underbrace{\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau}_{\text{forced response}}} \tag{2}$$

Derivation of the general state-space solution. Since $e^{-At} \neq 0$, $\dot{x}(t) = Ax(t) + Bu(t)$ is equivalent to

$$e^{-At}\dot{x}(t) - e^{-At}Ax(t) = e^{-At}Bu(t)$$

namely

$$\begin{aligned} \frac{d}{dt}\left(e^{-At}x(t)\right) &= e^{-At}Bu(t) \\ \Leftrightarrow d\left(e^{-At}x(t)\right) &= e^{-At}Bu(t)\,dt \end{aligned}$$

Integrating both sides from t_0 to t_1 gives

$$e^{-At_1}x(t_1) = e^{-At_0}x(t_0) + \int_{t_0}^{t_1} e^{-At}Bu(t)\,dt$$

Changing notations from t to τ in the integral yields

$$\begin{aligned} e^{-At}x(t) &= e^{-At_0}x(t_0) + \int_{t_0}^{t} e^{-A\tau}Bu(\tau)\,d\tau \\ \Leftrightarrow x(t) &= e^{A(t-t_o)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau \end{aligned}$$

In both the free and the forced responses, computing the matrix e^{At} is key. $e^{A(t-t_0)}$ is called the transition matrix, and can be computed using a few handy results in linear algebra.

1.1.4 The State Transition Matrix e^{At}

For the scalar case with $a \in \mathbb{R}$, Tylor expansion gives

$$e^{at} = 1 + at + \frac{1}{2}(at)^2 + \cdots + \frac{1}{n!}(at)^n + \ldots \tag{3}$$

The transition scalar $\Phi(t, t_0) = e^{a(t-t_0)}$ satisfies

$$\begin{aligned} \Phi(t,t) &= 1 && \text{(transition to itself)} \\ \Phi(t_3,t_2)\Phi(t_2,t_1) &= \Phi(t_3,t_1) && \text{(consecutive transition)} \\ \Phi(t_2,t_1) &= \Phi^{-1}(t_1,t_2) && \text{(reverse transition)} \end{aligned}$$

For the matrix case with $A \in \mathbb{R}^{n\times n}$

$$\boxed{e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{n!}A^nt^n + \ldots} \tag{4}$$

As I and A^i are matrices of dimension $n \times n$, we confirm that $e^{At} \in \mathbb{R}^{n\times n}$.

The state transition matrix $\Phi(t, t_0) = e^{A(t-t_0)}$ satisfies

$$\begin{aligned} e^{A0} &= I_n \\ e^{At_1}e^{At_2} &= e^{A(t_1+t_2)} \\ e^{-At} &= \left[e^{At}\right]^{-1}. \end{aligned}$$

Similar to the scalar case, it can be shown that

$$\begin{aligned} \Phi(t,t) &= I \\ \Phi(t_3,t_2)\Phi(t_2,t_1) &= \Phi(t_3,t_1) \\ \Phi(t_2,t_1) &= \Phi^{-1}(t_1,t_2). \end{aligned}$$

Note, however, that $e^{At}e^{Bt} = e^{(A+B)t}$ if and only if $AB = BA$. (Check by using Tylor expansion.)

When A is a diagonal or Jordan matrix, the Tylor expansion formula readily generates e^{At}:

Diagonal matrix $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$**.** In this case $A^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$ is also diagonal and hence

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{n!}A^nt^n + \ldots && (5) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & 0 & 0 \\ 0 & \lambda_2 t & 0 \\ 0 & 0 & \lambda_3 t \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\lambda_1^2t^2 & 0 & 0 \\ 0 & \frac{1}{2}\lambda_2^2t^2 & 0 \\ 0 & 0 & \frac{1}{2}\lambda_3^2t^2 \end{bmatrix} + \ldots && (6) \\ &= \begin{bmatrix} 1+\lambda_1 t + \frac{1}{2}\lambda_1^2t^2 + \ldots & 0 & 0 \\ 0 & 1+\lambda_2 t + \frac{1}{2}\lambda_2^2t^2 + \ldots & 0 \\ 0 & 0 & 1+\lambda_3 t + \frac{1}{2}\lambda_3^2t^2 + \ldots \end{bmatrix} && (7) \\ &= \underline{\begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix}}. && (8) \end{aligned}$$

Jordan canonical form $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$**.** Decompose

$$A = \underbrace{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}}_{\lambda I_3} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{N}.$$

Then

$$e^{At} = e^{(\lambda I_3 t + Nt)}.$$

As $(\lambda It)(Nt) = \lambda N t^2 = (Nt)(\lambda It)$, we have $e^{At} = e^{\lambda It} e^{Nt} = e^{\lambda t} e^{Nt}$. Also, N has the special property of $N^3 = N^4 = \cdots = 0I_3$, yielding

$$e^{Nt} = I + Nt + \frac{1}{2}N^2 t^2 = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus

$$e^{At} = \underline{\begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}}. \tag{9}$$

Remark 2 (Nilpotent matrices). The matrix

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is a *nilpotent* matrix that equals to zero when raised to a positive integral power. (“nil” $\sim$ zero; “potent” $\sim$ taking powers.) When taking powers of N, the off-diagonal 1 elements march to the top right corner and finally vanish.

Example. Consider a mass moving on a straight line with zero friction and no external force. Let x_1 and x_2 be be the position and the velocity of the mass, respectively. The state-space description of the system is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then $x(t) = e^{At}x(0)$ and

$$e^{At} = I + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t + \frac{1}{2!}\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} t^2 + \ldots = \underline{\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}}.$$

Columns of the state-transition matrix. We discuss an intuition of the matrix entries in the e^{At} matrix. Consider the system equation

$$\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x, \quad x(0) = x_0,$$

with the solution

$$x(t) = e^{At}x(0) = \left[\begin{array}{c|c} | & | \\ a_1(t) & a_2(t) \\ | & | \end{array}\right]\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = a_1(t)x_1(0) + a_2(t)x_2(0), \tag{10}$$

where $a_1(t)$ and $a_2(t)$ are columns of e^{At}. They satisfy

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow x(t) = a_1(t),$$
$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow x(t) = a_2(t).$$

Hence, we can obtain e^{At} from the following, without using explicitly the Tylor expansion,

1. write out $\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_2(t) \end{aligned} \Rightarrow \begin{aligned} x_1(t) &= e^{0t}x_1(0) + \int_0^t e^{0(t-\tau)}x_2(\tau)d\tau = e^{0t}x_1(0) + \int_0^t e^{-\tau}x_2(0)d\tau \\ x_2(t) &= e^{-t}x_2(0) \end{aligned}$

2. let $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $\begin{array}{c} x_1(t) \equiv 1 \\ x_2(t) \equiv 0 \end{array}$, namely $x(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

3. let $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $x_2(t) = e^{-t}$ and $x_1(t) = 1 - e^{-t}$, or more compactly, $x(t) = \begin{bmatrix} 1 - e^{-t} \\ e^{-t} \end{bmatrix}$

4. using the property of (10), write out directly

$$e^{At} = \begin{bmatrix} 1 & 1 - e^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

Exercise. Use the above method to compute e^{At} where

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

1.2 Discrete-Time LTI State-Space Solutions

For the discrete-time system

$$x(k+1) = Ax(k) + Bu(k),\ x(0) = x_0,$$

iteration of the state-space equation gives

$$x\left(k+1\right) = Ax\left(k\right) + Bu\left(k\right) \tag{11}$$

$$\Rightarrow x\left(k\right) = A^{k-k_0} x\left(k_o\right) + \begin{bmatrix} A^{k-k_0-1}B & A^{k-k_0-2}B & \cdots & B \end{bmatrix} \begin{bmatrix} u\left(k_0\right) \\ u\left(k_0+1\right) \\ \vdots \\ u\left(k-1\right) \end{bmatrix} \tag{12}$$

$$\Leftrightarrow \boxed{x\left(k\right) = \underbrace{A^{k-k_0} x\left(k_o\right)}_{\text{free response}} + \underbrace{\sum_{j=k_0}^{k-1} A^{k-1-j} Bu\left(j\right)}_{\text{forced response}}} \tag{13}$$

where the transition matrix is defined by $\Phi(k,j) = A^{k-j}$ and satisfies

$$\begin{aligned} \Phi(k,k) &= 1 & & \\ \Phi(k_3,k_2)\Phi(k_2,k_1) &= \Phi(k_3,k_1) & & k_3 \geq k_2 \geq k_1 \\ \Phi(k_2,k_1) &= \Phi^{-1}(t_1,t_2) & & \text{if and only if } A \text{ is nonsingular} \end{aligned}$$

1.2.1 The State Transition Matrix A^k

Similar to the continuous-time case, when A is a diagonal or Jordan matrix, the Tylor expansion formula readily generates A^k. We have

- Diagonal matrix $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$: $A^k = \begin{bmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{bmatrix}$

- Jordan canonical form $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}}_{\lambda I_3} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{N}$: With the nilpotent N and the commutative property $(\lambda I_3)\, N = N\,(\lambda I_3)$, we have

$$A^k = (\lambda I_3 + N)^k = (\lambda I_3)^k + k\,(\lambda I_3)^{k-1} N + \underbrace{\begin{pmatrix} k \\ 2 \end{pmatrix}}_{2 \text{ combination}} (\lambda I_3)^{k-2} N^2 + \underbrace{\begin{pmatrix} k \\ 3 \end{pmatrix} (\lambda I_3)^{k-3} N^3 + \dots}_{N^3 = N^4 = \dots = 0 I_3}$$

$$= \begin{bmatrix} \lambda^k & 0 & 0 \\ 0 & \lambda^k & 0 \\ 0 & 0 & \lambda^k \end{bmatrix} + k\lambda^{k-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{k(k-1)}{2}\lambda^{k-2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \underline{\begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2!}k\,(k-1)\,\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}}$$

Exercise. Show that

$$A = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \Rightarrow A^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2!}k\,(k-1)\,\lambda^{k-2} & \frac{1}{3!}k\,(k-1)\,(k-2)\,\lambda^{k-3} \\ 0 & \lambda^k & k\lambda^{k-1} & \frac{1}{2!}k\,(k-1)\,\lambda^{k-2} \\ 0 & 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & 0 & \lambda^k \end{bmatrix}$$

1.3 Explicit Computation of the State Transition Matrix e^{At}

General matrices may have structures other than the diagonal and Jordan canonical forms. However, via similar transformation, we can readily transform a general matrix to a diagonal or Jordan form under a different choice of state vectors.

Principle Concept.

1. Given

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(t_0) = x_0 \in \mathbb{R}^n,\ A \in \mathbb{R}^{n\times n}$$

we will find a nonsingular matrix $T \in \mathbb{R}^{n\times n}$ such that a coordinate transformation defined by $x(t) = Tx^*(t)$ yields

$$\frac{d}{dt}\left(Tx^*(t)\right) = ATx^*(t) + Bu(t)$$

$$\underline{\frac{d}{dt}x^*(t) = \underbrace{T^{-1}AT}_{\triangleq\Lambda}\, x^*(t) + \underbrace{T^{-1}B}_{B^*}\, u(t),\ x^*(0) = T^{-1}x_0}$$

where Λ is diagonal or in Jordan form.

2. Now $x^*(t)$ can be solved easily, and the free response is $x^*(t) = e^{\Lambda t}x^*(0)$. For example, when $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, we would readily obtain $x^*(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} x_1^*(0) \\ x_2^*(0) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t}x_1^*(0) \\ e^{\lambda_2 t}x_2^*(0) \end{bmatrix}$.

3. As $x(t) = Tx^*(t)$, the above implies

$$x(t) = Te^{\Lambda t}T^{-1}x_0$$

4. From the original state-space description, $x(t) = e^{At}x_0$. Hence

$$\boxed{e^{At} = Te^{\Lambda t}T^{-1}}$$

Existence of Solutions. The solution of T comes from the theory of eigenvalues and eigenvectors in linear algebra.

More generally If two matrices A, $B \in \mathbb{C}^{n\times n}$ are similar: $A = TBT^{-1},\ T \in \mathbb{C}^{n\times n}$, then

- their A^n and B^n are also similar: e.g., $AA = TBT^{-1}TBT^{-1} = TB^2T^{-1}$
- their exponential matrices are also similar
$$e^{At} = Te^{Bt}T^{-1}$$
as
$$\begin{aligned} Te^{Bt}T^{-1} &= T(I + Bt + \frac{1}{2}B^2t^2 + \dots)T^{-1} = TIT^{-1} + TBtT^{-1} + \frac{1}{2}TB^2t^2T^{-1} + \dots \\ &= I + At + \frac{1}{2}A^2t^2 + \cdots = e^{At} \end{aligned}$$

Eigenvalues and Eigenvectors. The principle concept of computing e^{At} in this section relies on the similarity transform $\Lambda = T^{-1}AT$, where Λ is structurally simple: i.e., in diagonal or Jordan form. We already observed the resulting convenience in computing $x^*(t) = e^{\Lambda t}x^*(0) \overset{\text{e.g.}}{=} \begin{bmatrix} e^{\lambda_1 t}x_1^*(0) \\ e^{\lambda_2 t}x_2^*(0) \end{bmatrix}$. Under the coordinate transformation defined by $x(t) = Tx^*(t)$, we then have

$$x(t) = Te^{\Lambda t}x^*(0) \overset{\text{e.g.}}{=} \underbrace{[t_1, t_2]}_{T} \begin{bmatrix} e^{\lambda_1 t}x_1^*(0) \\ e^{\lambda_2 t}x_2^*(0) \end{bmatrix} = e^{\lambda_1 t}x_1^*(0)t_1 + e^{\lambda_2 t}x_2^*(0)t_2$$

in other words, the state trajectory is conveniently decomposed into two modes along the directions defined by t_1 and t_2, the column vectors of T.

In practice, Λ and T are obtained using the tools of eigenvalues and eigenvectors.

For $A \in \mathbb{R}^{n\times n}$, an eigenvalue $\lambda \in \mathcal{C}$ of A is the solution to the characteristic equation

$$\boxed{\det(A - \lambda I) = 0} \tag{14}$$

The corresponding eigenvectors are the nonzero solutions to

$$At = \lambda t \Leftrightarrow (A - \lambda I)\,t = 0 \tag{15}$$

The case with distinct eigenvalues (diagonalization). When $A \in \mathbb{R}^{n\times n}$ has n distinct eigenvalues such that

$$\begin{aligned} Ax_1 &= \lambda_1 x_1 \\ Ax_2 &= \lambda_2 x_2 \\ &\vdots \\ Ax_n &= \lambda_n x_n \end{aligned}$$

we can write the above as

$$A\underbrace{[x_1, x_2, \dots, x_n]}_{\triangleq T} = [\lambda_1 x_1, \lambda_2 x_2, \dots, \lambda_n x_n] = [x_1, x_2, \dots, x_n] \underbrace{\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}}_{\Lambda}$$

The matrix $[x_1, x_2, \dots, x_n]$ is square. From linear algebra, the eigenvectors are linearly independent and $[x_1, x_2, \dots, x_n]$ is invertible. Hence

$$A = T\Lambda T^{-1},\ \Lambda = T^{-1}AT$$

Example 1. Mechanical system with strong damping

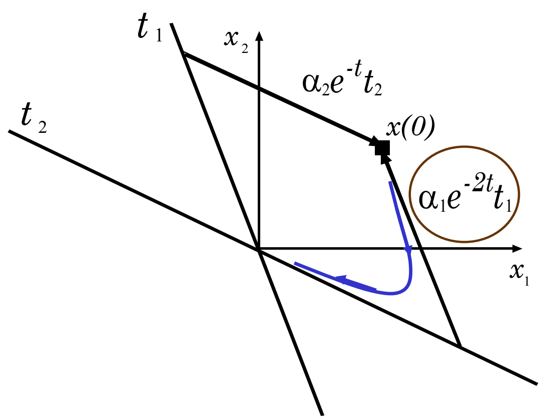
Consider a spring-mass-damper system with $m = 1$, $k = 2$, $b = 3$. Let x_1 and x_2 be the position and velocity of the mass, respectively. We have

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 + 2x_1 + 3x_2 &= 0 \end{cases} \qquad \Longrightarrow \qquad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

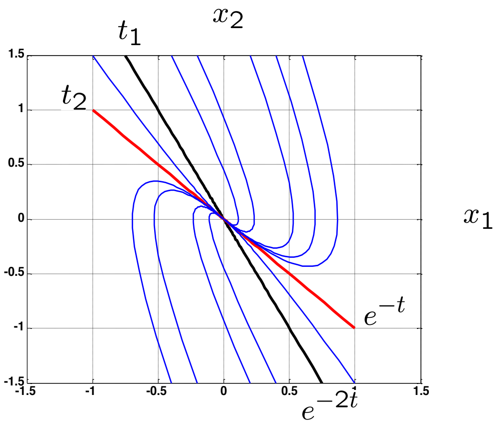
- Find eigenvalues: $\det(A-\lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -2 & -\lambda-3 \end{bmatrix} = (\lambda+2)(\lambda+1) \Rightarrow \lambda_1 = -2, \lambda_2 = -1$
- Find associate eigenvectors:
  - $\lambda_1 = -2$: $(A-\lambda_1 I)\, t_1 = 0 \Rightarrow t_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
  - $\lambda_1 = -1$: $(A-\lambda_2 I)\, t_2 = 0 \Rightarrow t_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- Define T and Λ: $T = \begin{bmatrix} t_1 & t_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$
- Compute $T^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$
- Compute $e^{At} = Te^{\Lambda t}T^{-1} = T\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-1t} \end{bmatrix}T^{-1} = \begin{bmatrix} -e^{-2t}+2e^{-t} & -e^{-2t}+e^{-t} \\ 2e^{-2t}-2e^{-t} & 2e^{-2t}-e^{-t} \end{bmatrix}$

Physical interpretations Let us revisit the intuition at the beginning of this subsection:

- $x(t) = e^{\lambda_1 t}x_1^*(0)t_1 + e^{\lambda_2 t}x_2^*(0)t_2$ decomposes the state trajectory into two modes along the direction of the two eigenvectors t_1 and t_2.
- The two modes are scaled by $x_1^*(0)$ and $x_2^*(0)$ defined from $x(0) = Tx^*(0)$, or more explicitly, $x(0) = [t_1, t_2][x_1^*(0), x_2^*(0)]^T = x_1^*(0)t_1 + x_2^*(0)t_2$. This is nothing but decomposing $x(0)$ into the sum of two vectors along the directions of the eigenvectors; and $x_1^*(0)$ and $x_2^*(0)$ are the coefficients of the decomposition!



- If the initial condition $x(0)$ is aligned with one eigenvector, say, t_1, then $x_2^*(0) = 0$. The decomposition $x(t) = e^{\lambda_1 t}x_1^*(0)t_1 + e^{\lambda_2 t}x_2^*(0)t_2$ then dictates that $x(t)$ will stay in the direction of t_1. In other words, if the state initiates along the direction of one eigenvector, then the free response will stay in that direction without "making turns". If $\lambda_1 < 0$, then $x(t)$ will move towards the origin of the state space; if $\lambda_1 = 0$, $x(t)$ will stay at the initial point; and if positive, $x(t)$ will move away from the origin along t_1. Furthermore, the magnitude of λ_1 determines the speed of response.



The case with complex eigenvalues Consider the undamped spring-mass system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ \det(A-\lambda I) = \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2,} = \pm j.$$

The eigenvectors are

$$\lambda_1 = j: \ (A - jI)t_1 = 0 \Rightarrow t_1 = \begin{bmatrix} 1 \\ j \end{bmatrix}$$
$$\lambda_2 = -j: \ (A + jI)t_2 = 0 \Rightarrow t_2 = \begin{bmatrix} 1 \\ -j \end{bmatrix} \text{ (complex conjugate of } t_1\text{)}.$$

Hence

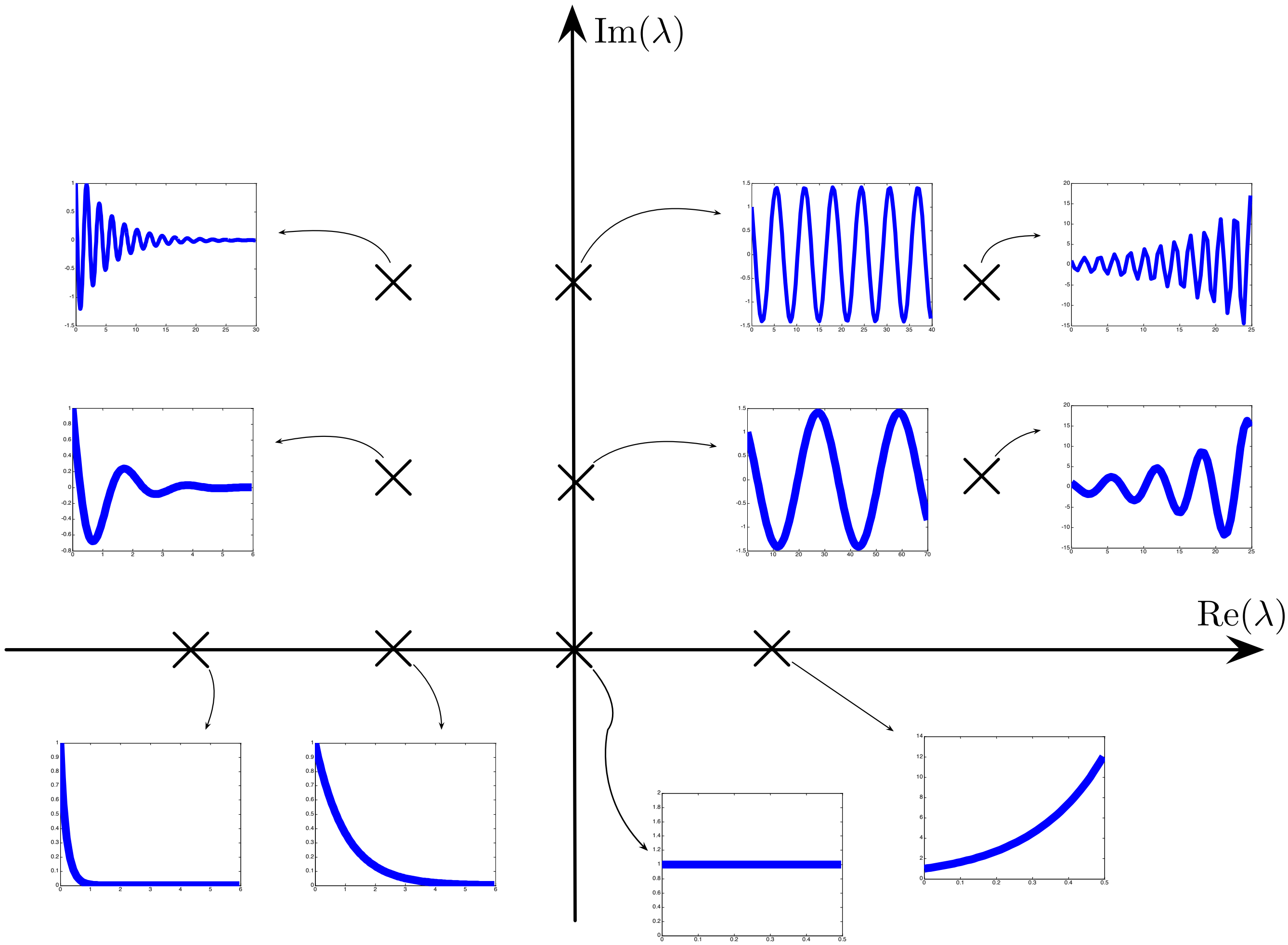
$$T = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}, \ T^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix}, \ e^{At} = Te^{\Lambda t}T^{-1} = T\begin{bmatrix} e^{jt} & 0 \\ 0 & e^{-jt} \end{bmatrix}T^{-1} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

As an exercise, for a general $A \in \mathbb{R}^{2\times 2}$ with complex eigenvalues $\sigma \pm j\omega$, you can show that by using $T = [t_R, t_I]$ where t_R and t_I are the real and the imaginary parts of t_1, an eigenvector associated with $\lambda_1 = \sigma + j\omega$, $x = Tx^*$ transforms $\dot{x} = Ax$ to

$$\dot{x}^*(t) = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} x^*(t)$$

and

$$e^{\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}t} = \begin{bmatrix} e^{\sigma t}\cos\omega t & e^{\sigma t}\sin\omega t \\ -e^{\sigma t}\sin\omega t & e^{\sigma t}\cos\omega t \end{bmatrix}.$$



The case with repeated eigenvalues, via generalized eigenvectors. Consider $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, which has two repeated eigenvalues $\lambda(A) = 2$ and

$$(A - \lambda I)\, t_1 = 0 \Rightarrow t_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

No other linearly independent eigenvectors exist. How do we go further? As A is already very similar to the Jordan form, we try instead

$$A \begin{bmatrix} t_1 & t_2 \end{bmatrix} = \begin{bmatrix} t_1 & t_2 \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},$$

which requires $At_2 = t_1 + \lambda t_2$, i.e.,

$$(A - \lambda I)\, t_2 = t_1 \Leftrightarrow \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} t_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\Rightarrow t_2 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

t_2 is linearly independent from t_1. Together, t_1 and t_2 span the 2-dimensional vector space. As such, t_2 is called a generalized eigenvector.

For general 3×3 matrices with $\det(\lambda I - A) = (\lambda - \lambda_m)^3$, i.e., $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_m$, we look for T such that

$$A = TJT^{-1}$$

where J has three canonical forms:

$$i), \begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix}, \; ii), \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix} \text{ or } \begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \lambda_m & 1 \\ 0 & 0 & \lambda_m \end{bmatrix}, iii), \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 1 \\ 0 & 0 & \lambda_m \end{bmatrix}.$$

- Case i): this happens when A has three linearly independent eigenvectors, i.e., $(A - \lambda_m I)t = 0$ yields t_1, t_2, and t_3 that span the 3-d vector space. This happens when nullity $(A - \lambda_m I) = 3$, namely, $\text{rank}(A - \lambda_m I) = 3 - \text{nullity}\,(A - \lambda_m I) = 0$.

- Case ii): this happens when $(A-\lambda_m I)t = 0$ yields two linearly independent solutions, i.e., when nullity $(A - \lambda_m I) = 2$. We then have, e.g.,

$$A[t_1, t_2, t_3] = [t_1, t_2, t_3] \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix} \Leftrightarrow [\lambda_m t_1, t_1 + \lambda_m t_2, \lambda_m t_3] = [At_1, At_2, At_3]$$

  t_1 and t_3 are the directly computed eigenvectors. For the generalized eigenvector t_2, the second column of the equality gives

$$(A - \lambda_m I)\, t_2 = t_1$$

- Case iii): this is for the case when $(A - \lambda_m I)t = 0$ yields only one linearly independent solution, i.e., when $\text{nullity}(A - \lambda_m I) = 1$. We then have,

$$A[t_1, t_2, t_3] = [t_1, t_2, t_3] \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 1 \\ 0 & 0 & \lambda_m \end{bmatrix} \Leftrightarrow [\lambda_m t_1, t_1 + \lambda_m t_2, t_2 + \lambda_m t_3] = [At_1, At_2, At_3]$$

  yielding

$$\begin{aligned} (A - \lambda_m I)\, t_1 &= 0 \\ (A - \lambda_m I)\, t_2 &= t_1 \\ (A - \lambda_m I)\, t_3 &= t_2 \end{aligned}$$

  where t_2 and t_3 are the generalized eigenvectors.

Example 2. Consider

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \; \det(A - \lambda I) = (\lambda + 1)(\lambda - 1) - 1 = \lambda^2 \Rightarrow \lambda_1 = \lambda_2 = 0.$$

Two repeated eigenvalues with $\text{rank}(A - 0I) = 1$ $\Rightarrow$only one linearly independent eigenvector:

$$(A - 0I)\, t_1 = 0 \Rightarrow t_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Generalized eigenvector:

$$(A - 0I)\, t_2 = t_1 \Rightarrow t_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Coordinate transform matrix:

$$T = [t_1, t_2] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \; T^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},$$

$$J = T^{-1}AT = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; e^{At} = Te^{Jt}T^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 - t & t \\ -t & 1 + t \end{bmatrix}.$$

The first eigenvector implies that if $x_1(0) = x_2(0)$ then the response is characterized by $e^{0t} = 1$, i.e., $x_1(t) = x_1(0) = x_2(0) = x_2(t)$. This makes sense because $\dot{x}_1 = -x_1 + x_2$ from the state equation.

Example 3 (Multiple eigenvectors)**.** Obtain the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}.$$

Analogous procedures give that

$$\lambda_1 = 5,\ \lambda_2 = \lambda_3 = -3.$$

So there are repeated eigenvalues. For $\lambda_1 = 5$, $(A - 5I)t_1 = 0$ gives

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} t_1 = 0 \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} t_1 = 0 \Rightarrow t_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

For $\lambda_2 = \lambda_3 = -3$, the characteristic matrix is

$$A + 3I = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}.$$

The second row is the first row multiplied by 2. The third row is the negative of the first row. So the characteristic matrix has only rank 1. The characteristic equation

$$(A - \lambda_2 I)\, t = 0$$

has two linearly independent solutions

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

Then

$$T = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \; J = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

Physical interpretation. When $\dot{x} = Ax$, $A = TJT^{-1}$ with $J = \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix}$, we have

$$x(t) = e^{At}x(0) = T \begin{bmatrix} e^{\lambda_m t} & te^{\lambda_m t} & 0 \\ 0 & e^{\lambda_m t} & 0 \\ 0 & 0 & e^{\lambda_m t} \end{bmatrix} T^{-1}x(0) = T \begin{bmatrix} e^{\lambda_m t} & te^{\lambda_m t} & 0 \\ 0 & e^{\lambda_m t} & 0 \\ 0 & 0 & e^{\lambda_m t} \end{bmatrix} \overset{I}{\cancel{T^{-1}T}} x^*(0)$$

If the initial condition is in the direction of t_1, i.e., $x^*(0) = [x_1^*(0), 0, 0]^T$ and $x_1^*(0) \neq 0$, the above equation yields $x(t) = x_1^*(0)t_1e^{\lambda_m t}$. If $x(0)$ starts in the direction of t_2, i.e., $x^*(0) = [0, x_2^*(0), 0]^T$, then $x(t) = x_2^*(0)(t_1te^{\lambda_m t} + t_2e^{\lambda_m t})$. In this case, the response does not remain in the direction of t_2 but is confined in the subspace spanned by t_1 and t_2.

Exercise 1. Obtain eigenvalues of J and e^{Jt} by inspection:

$$J = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}.$$

1.4 Explicit Computation of the State Transition Matrix A^k

Everything in computing the similarity transform $A = T\Lambda T^{-1}$ or $A = TJT^{-1}$ applies to the discrete-time case. The state transition matrix in this case is

$$\boxed{A^k = T\Lambda^k T^{-1} \text{ or } A^k = TJ^kT^{-1}.}$$

You should be able to derive these results:

J	J^k
$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$	$\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}$
$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$	$\begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix}$
$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$	$\begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2!}k\left(k-1\right)\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}$
$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$	$\begin{bmatrix} \lambda^k & k\lambda^{k-1} & 0 \\ 0 & \lambda^k & 0 \\ 0 & 0 & \lambda_3^k \end{bmatrix}$
$\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$	$r^k \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$, $r = \sqrt{\sigma^2+\omega^2}$, $\theta = \tan^{-1}\frac{\omega}{\sigma}$

Exercise 2. Write down J^k for $J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ and $J = \begin{bmatrix} -10 & 1 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -100 & 1 \\ 0 & 0 & 0 & -1 & -100 \end{bmatrix}$.

Exercise 3. Show that

$$J = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \Rightarrow J^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2!}k\left(k-1\right)\lambda^{k-2} & \frac{1}{3!}k\left(k-1\right)\left(k-2\right)\lambda^{k-3} \\ 0 & \lambda^k & k\lambda^{k-1} & \frac{1}{2!}k\left(k-1\right)\lambda^{k-2} \\ 0 & 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & 0 & \lambda^k \end{bmatrix}.$$

1.5 Transition Matrix via Inverse Transformation

We have now

	Continuous-time system	Discrete-time system
state equation	$\dot{x}(t) = Ax(t) + Bu(t)$, $x(0) = x_0$	$x(k+1) = Ax(k) + Bu(k)$, $x(0) = x_0$
solution	$x(t) = \underbrace{e^{At}x(0)}_{\text{free response}} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}_{\text{forced response}}$	$x(k) = \underbrace{A^kx(0)}_{\text{free response}} + \underbrace{\sum_{j=0}^{(k-1)} A^{(k-1-j)}Bu(j)}_{\text{forced response}}$
transition matrix	e^{At}	A^k

We also know from Laplace transform, that

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$X(s) = \underbrace{\left(sI-A\right)^{-1}x(0)}_{\text{free response}} + \underbrace{\left(sI-A\right)^{-1}BU(s)}_{\text{free response}}$$

Comparing $x(t)$ and $X(s)$ gives

$$\boxed{e^{At} = \mathcal{L}^{-1}\left\{\left(sI-A\right)^{-1}\right\}} \tag{16}$$

Example 4. Consider $A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$. We have

$$\begin{aligned} e^{At} &= \mathcal{L}^{-1} \begin{bmatrix} s-\sigma & -\omega \\ \omega & s-\sigma \end{bmatrix}^{-1} = \mathcal{L}^{-1} \left\{ \frac{1}{(s-\sigma)^2+\omega^2} \begin{bmatrix} s-\sigma & \omega \\ -\omega & s-\sigma \end{bmatrix} \right\} \\ &= e^{\sigma t} \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} \end{aligned}$$

Similarly, for the discrete time case, we have $X(z) = (zI-A)^{-1} zx(0) + (zI-A)^{-1} BU(s)$ and

$$\boxed{A^k = \mathcal{Z}^{-1}\left\{(zI-A)^{-1}z\right\}} \tag{17}$$

Example 5. Consider $A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$. We have

$$\begin{aligned} A^k &= \mathcal{Z}^{-1} \left\{ z \begin{bmatrix} z-\sigma & -\omega \\ \omega & z-\sigma \end{bmatrix}^{-1} \right\} = \mathcal{Z}^{-1} \left\{ \frac{z}{(z-\sigma)^2+\omega^2} \begin{bmatrix} z-\sigma & \omega \\ -\omega & z-\sigma \end{bmatrix} \right\} \\ &= \mathcal{Z}^{-1} \left\{ \frac{z}{z^2 - 2r\cos\theta z + r^2} \begin{bmatrix} z - r\cos\theta & r\sin\theta \\ -r\sin\theta & z-r\cos\theta \end{bmatrix} \right\},\ r = \sqrt{\sigma^2+\omega^2},\ \theta = \tan^{-1}\frac{\omega}{\sigma} \\ &= r^k \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix} \end{aligned}$$

Example 6. Consider $A = \begin{bmatrix} 0.7 & 0.3 \\ 0.1 & 0.5 \end{bmatrix}$. We have

$$\begin{aligned} (zI-A)^{-1} z &= \begin{bmatrix} \frac{z(z-0.5)}{(z-0.8)(z-0.4)} & \frac{0.3z}{(z-0.8)(z-0.4)} \\ \frac{0.1z}{(z-0.8)(z-0.4)} & \frac{z(z-0.7)}{(z-0.8)(z-0.4)} \end{bmatrix} = \begin{bmatrix} \frac{0.75z}{z-0.8} + \frac{0.25z}{z-0.4} & \frac{0.75z}{z-0.8} - \frac{0.75z}{z-0.4} \\ \frac{0.25z}{z-0.8} - \frac{0.25z}{z-0.4} & \frac{0.25z}{z-0.8} + \frac{0.75z}{z-0.4} \end{bmatrix} \\ \Rightarrow A^k &= \begin{bmatrix} 0.75(0.8)^k + 0.25(0.4)^k & 0.75(0.8)^k - 0.75(0.4)^k \\ 0.25(0.8)^k - 0.25(0.4)^k & 0.25(0.8)^k + 0.75(0.4)^k \end{bmatrix} \end{aligned}$$