1 From Transfer Function to State Space: State-Space Canonical Forms

It is straightforward to derive the *unique* transfer function corresponding to a state-space model. The inverse problem, i.e., building internal descriptions from transfer functions, is less trivial and is the subject of *realization theory*.

A single transfer function has infinite amount of state-space representations. Consider, for example, the two models

$$\begin{cases}\dot{x} &= Ax + Bu\\ y &= Cx\end{cases}, \qquad \begin{cases}\dot{x} &= Ax + \frac{1}{2}Bu\\ y &= 2Cx\end{cases}$$

which share the same transfer function $C(sI-A)^{-1}B$.

We start with the most common realizations: controller canonical form, observable canonical form, and Jordan form, using the following unit problem:

$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}. \tag{1}$$

1.1 Controllable Canonical Form.

Consider first:

$$Y(s) = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} U(s). \tag{2}$$

Similar to choosing position and velocity in the spring-mass-damper example, we can choose

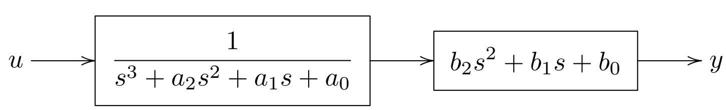
$$x_1 = y,\ x_2 = \dot{x}_1 = \dot{y},\ x_3 = \dot{x}_2 = \ddot{y}, \tag{3}$$

which gives

$$\frac{d}{dt}\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -a_0 & -a_1 & -a_2\end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix} 0\\ 0\\ 1\end{bmatrix} u \tag{4}$$

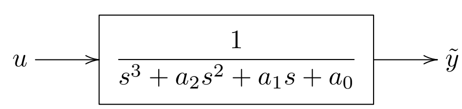
$$y = \begin{bmatrix} 1 & 0 & 0\end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix}$$

For the general case in (1), i.e., $\dddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0 y = b_2\ddot{u} + b_1\dot{u} + b_0 u$, there are terms with respect to the derivative of the input. Choosing simply (3) does not generate a proper state equation. However, we can decompose (1) as



(5)

The first part of the connection



(6)

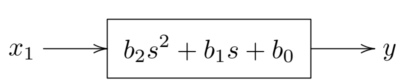
looks exactly like what we had in (2). Denote the output here as $\tilde{y}$. Then we have

$$\frac{d}{dt}\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -a_0 & -a_1 & -a_2\end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix} 0\\ 0\\ 1\end{bmatrix} u,$$

where

$$x_1 = \tilde{y},\ x_2 = \dot{x}_1,\ x_3 = \dot{x}_2. \tag{7}$$

Introducing the states in (7) also addresses the problem of the rising differentiations in u. Notice now, that the second part of (5) is nothing but



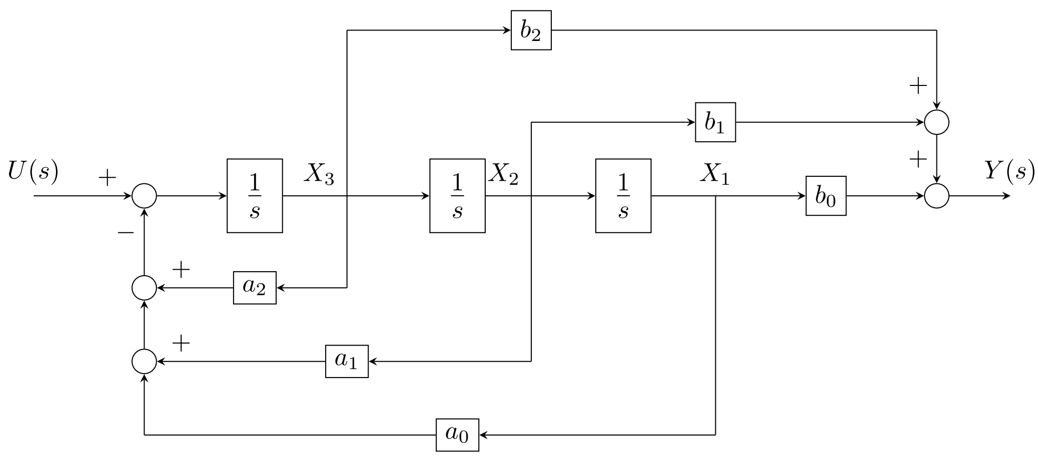
So

$$y = b_2\ddot{x}_1 + b_1\dot{x}_1 + b_0x_1 = b_2x_3 + b_1x_2 + b_0x_1 = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The above procedure constructs the *controllable canonical form* of the third-order transfer function (1):

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \tag{8}$$
$$y(t) = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

In a block diagram, the state-space system looks like



Example 1. Obtain the controllable canonical forms of the following systems

- $G(s) = \dfrac{s^2+1}{s^3+2s+10}$
- $G(s) = \dfrac{b_0s^2+b_1s+b_2}{s^3+a_0s^2+a_1s+a_2}$

General Case.

For a single-input single-output transfer function

$$G(s) = \frac{b_{n-1}s^{n-1}+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0} + d,$$

we can verify that

$$\Sigma_c = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{ccccc|c} 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} & 1 \\ \hline b_0 & b_1 & \cdots & b_{n-2} & b_{n-1} & d \end{array}\right] \tag{9}$$

realizes $G(s)$. This realization is called the *controllable canonical form.*

1.2 Observable Canonical Form.

Consider again

$$Y(s) = G(s)U(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} U(s).$$

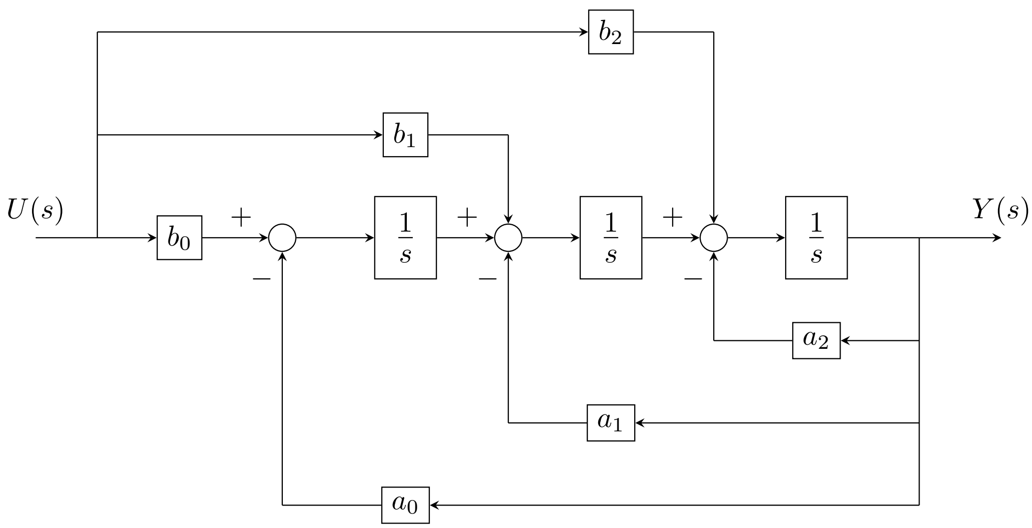
Expanding and dividing by s^3 yield

$$\left(1 + a_2\frac{1}{s} + a_1\frac{1}{s^2} + a_0\frac{1}{s^3}\right) Y(s) = \left(b_2\frac{1}{s} + b_1\frac{1}{s^2} + b_0\frac{1}{s^3}\right) U(s)$$

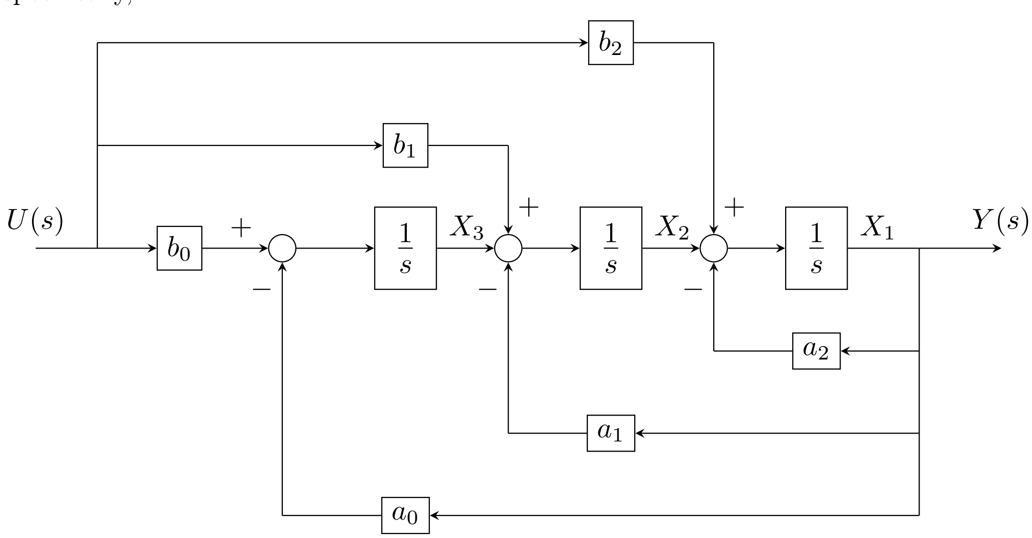
and therefore

$$\begin{aligned} Y(s) = &-a_2\frac{1}{s}Y(s) - a_1\frac{1}{s^2}Y(s) - a_0\frac{1}{s^3}Y(s) \\ &+ b_2\frac{1}{s}U(s) + b_1\frac{1}{s^2}U(s) + b_0\frac{1}{s^3}U(s). \end{aligned}$$

In a block diagram, the above looks like



or more specifically,



Here, the states are connected by

$$\begin{aligned} Y(s) &= X_1(s) & & & y(t) &= x_1(t) \\ sX_1(s) &= -a_2X_1(s) + X_2(s) + b_2U(s) & & & \dot{x}_1(t) &= -a_2x_1(t) + x_2(t) + b_2u(t) \\ sX_2(s) &= -a_1X_1(s) + X_3(s) + b_1U(s) & &\Rightarrow & \dot{x}_2(t) &= -a_1x_1(t) + x_3(t) + b_1u(t) \\ sX_3(s) &= -a_0X_1(s) + b_0U(s) & & & \dot{x}_3(t) &= -a_0x_1(t) + b_0u(t) \end{aligned}$$

or in matrix form:

$$\dot{x}(t) = \underbrace{\begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix}}_{A_o} x(t) + \underbrace{\begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix}}_{B_o} u(t) \tag{10}$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C_o} x(t)$$

The above is called the *observable canonical form* realization of $G(s)$.

Exercise 1. Verify that $C_o(sI - A_o)^{-1}B_o = G(s)$.

General Case.

In the general case, the *observable canonical form* of the transfer function

$$G(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} + d$$

is

$$\Sigma_o = \left[\begin{array}{c|c} A_o & B_o \\ \hline C_o & D_o \end{array}\right] = \left[\begin{array}{ccccc|c} -a_{n-1} & 1 & \cdots & 0 & 0 & b_{n-1} \\ -a_{n-2} & 0 & \cdots & 0 & 0 & b_{n-2}0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ -a_1 & 0 & \cdots & 0 & 1 & b_1 \\ -a_0 & & \cdots & & 0 & b_0 \\ \hline 1 & & \cdots & & & d \end{array}\right]. \tag{11}$$

Exercise 2. Obtain the controllable and observable canonical forms of

$$G(s) = \frac{k_1}{s - p_1}.$$

1.3 Diagonal and Jordan canonical forms.

1.3.1 Diagonal form.

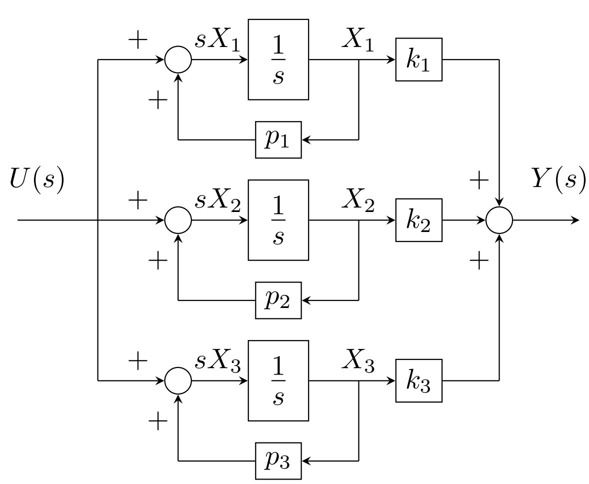
When

$$G(s) = \frac{B(s)}{A(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

and the poles of the transfer function $p_1 \neq p_2 \neq p_3$, we can write, using partial fractional expansion,

$$G(s) = \frac{k_1}{s - p_1} + \frac{k_2}{s - p_2} + \frac{k_3}{s - p_3},\ k_i = \lim_{p \to p_i} (s - p_i) \frac{B(s)}{A(s)},$$

namely



The state-space realization of the above is

$$A=\begin{bmatrix} p_1 & 0 & 0\\ 0 & p_2 & 0\\ 0 & 0 & p_3\end{bmatrix},\ B=\begin{bmatrix}1\\1\\1\end{bmatrix},\ C=\begin{bmatrix}k_1 & k_2 & k_3\end{bmatrix},\ D=0.$$

1.3.2 Jordan form.

If poles repeat, say,

$$G(s)=\frac{b_2s^2+b_1s+b_0}{s^3+a_2s^2+a_1s+a_0}=\frac{b_2s^2+b_1s+b_0}{(s-p_1)(s-p_m)^2},\ p_1\neq p_m\in\mathbb{R},$$

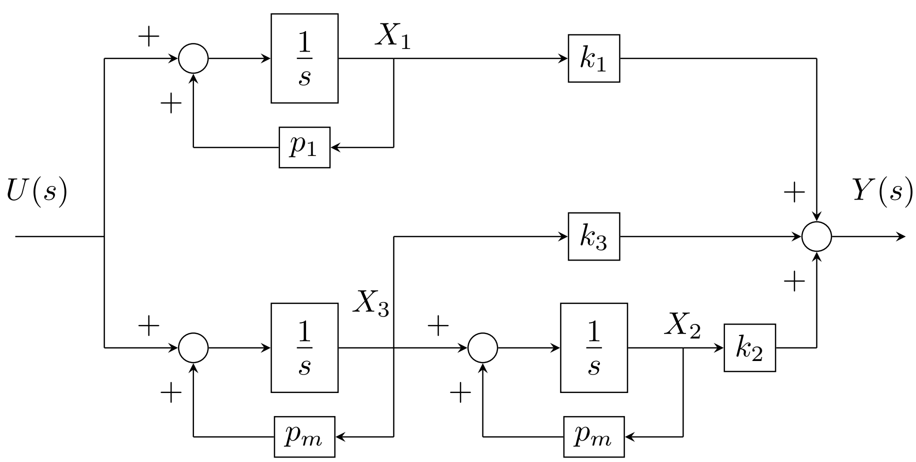
then partial fraction expansion gives

$$G(s)=\frac{k_1}{s-p_1}+\frac{k_2}{(s-p_m)^2}+\frac{k_3}{s-p_m},$$

where

$$\begin{aligned} k_1 &= \lim_{s\to p_1} G(s)(s-p_1)\\ k_2 &= \lim_{s\to p_m} G(s)(s-p_m)^2\\ k_3 &= \lim_{s\to p_m}\frac{d}{ds}\left\{G(s)(s-p_m)^2\right\}\end{aligned}$$

In state space, we have



The state-space realization of the above, called the Jordan canonical form,[1] is

$$A=\begin{bmatrix} p_1 & 0 & 0\\ 0 & p_m & 1\\ 0 & 0 & p_m\end{bmatrix},\ B=\begin{bmatrix}1\\0\\1\end{bmatrix},\ C=\begin{bmatrix}k_1 & k_2 & k_3\end{bmatrix},\ D=0.$$

1.4 Modified canonical form.

If the system has complex poles, say,

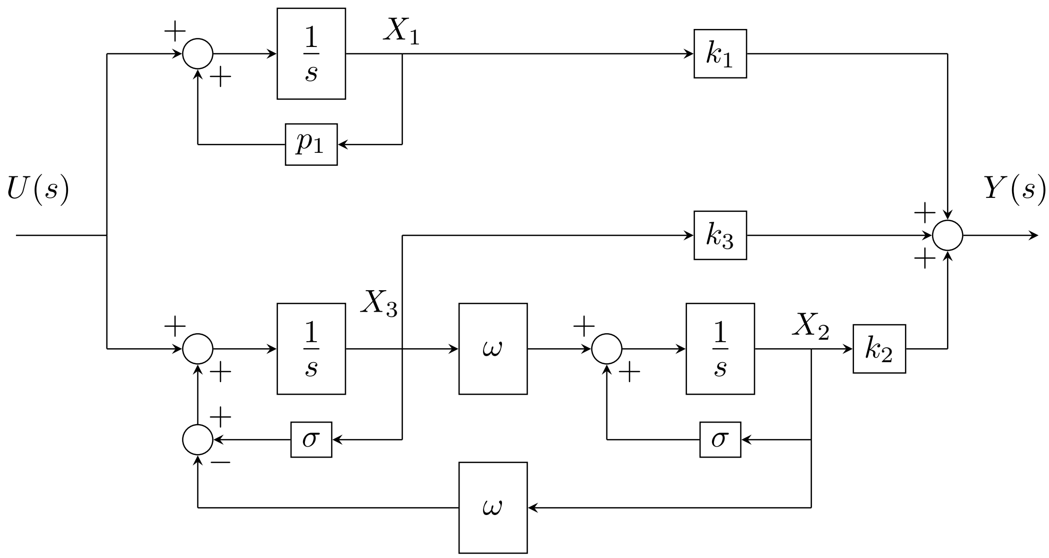
$$G(s)=\frac{b_2s^2+b_1s+b_0}{s^3+a_2s^2+a_1s+a_0}=\frac{b_2s^2+b_1s+b_0}{(s-p_1)\left[(s-\sigma)^2+\omega^2\right]},$$

then partial fraction expansion gives

$$G(s)=\frac{k_1}{s-p_1}+\frac{\alpha s+\beta}{(s-\sigma)^2+\omega^2},$$

which has the graphical representation as below:

[1] The A matrix is called a Jordan matrix.



Here $k_2 = (\beta + \alpha\sigma)/\omega$ and $k_3 = \alpha$.

You should be able to check that the block diagram matches with the transfer function realization.

The above can be realized by the modified Jordan form in state space:

$$A = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & \sigma & \omega \\ 0 & -\omega & \sigma \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix},\ D = 0.$$

1.5 Discrete-Time Transfer Functions and Their State-Space Canonical Forms

The procedures for finding state space realizations in discrete time is similar to the continuous time cases. The only difference is that we use

$$\mathcal{Z}\left\{x(k+n)\right\} = z^n X(z),$$

instead of

$$\mathcal{L}\left\{\frac{d^n}{dt^n}x(t)\right\} = s^n X(s),$$

assuming zero state initial conditions.

We have the fundamental relationships:

$$x\left(k\right) \longrightarrow \boxed{z^{-1}} \longrightarrow x\left(k-1\right)$$

$$X\left(z\right) \longrightarrow \boxed{z^{-1}} \longrightarrow z^{-1}X\left(z\right)$$

$$x\left(k+n\right) \longrightarrow \boxed{z^{-1}} \longrightarrow x\left(k+n-1\right)$$

The discrete-time state-space description of a general transfer function $G(z)$ is

$$\begin{aligned} x\left(k+1\right) &= Ax\left(k\right) + Bu\left(k\right) \\ y\left(k\right) &= Cx\left(k\right) + Du\left(k\right) \end{aligned}$$

and satisfies $G\left(z\right) = C\left(zI - A\right)^{-1} B + D$.

Take again a third-order system as the example:

$$G\left(z\right) = \frac{b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0} = \frac{b_2 z^{-1} + b_1 z^{-2} + b_0 z^{-3}}{1 + a_2 z^{-1} + a_1 z^{-2} + a_0 z^{-3}}.$$

The A, B, C, D matrices of the canonical forms are exactly the same as those in continuous-time cases.

Controllable canonical form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Observable canonical form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Diagonal form (distinct poles):

$$G(z) = \frac{k_1}{z-p_1} + \frac{k_2}{z-p_2} + \frac{k_3}{z-p_3}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Jordan form (2 repeated poles):

$$G(z) = \frac{k_1}{z-p_1} + \frac{k_2}{(z-p_m)^2} + \frac{k_3}{z-p_m}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_m & 1 \\ 0 & 0 & p_m \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Jordan form (2 complex poles):

$$G(s) = \frac{k_1}{z-p_1} + \frac{\alpha z + \beta}{(z-\sigma)^2 + \omega^2}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & \sigma & \omega \\ 0 & -\omega & \sigma \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

where $k_2 = (\beta + \alpha\sigma)/\omega$, $k_3 = \alpha$.

Exercise: obtain the controllable canonical form for the following systems

- $G(s) = \frac{z^{-1} - z^{-3}}{1 + 2z^{-1} + z^{-2}}$
- $G(s) = \frac{b_0 z^2 + b_1 z + b_2}{z^3 + a_0 z^2 + a_1 z + a_2}$

1.6 Similar Realizations

Besides the canonical forms, other system realizations exist. Let us begin with the realization Σ of some transfer function $G(s)$. Let $T \in \mathbb{C}^{n\times n}$ be nonsingular. We can define *new* states by:

$$Tx^* = x.$$

We can rewrite the differential equations defining Σ in terms of these new states by plugging in $x = Tx^*$:

$$\frac{d}{dt}\left(Tx^*(t)\right) = ATx^*(t) + Bu(t),$$

to obtain

$$\Sigma^* : \begin{cases} \dot{x}^*(t) &= T^{-1}ATx^*(t) + T^{-1}Bu(t) \\ y(t) &= CTx^*(t) + Du(t) \end{cases}$$

This new realization

$$\Sigma^* = \left[\begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array}\right], \tag{12}$$

also realizes $G(s)$ and is said to be *similar* to Σ.

Similar realizations are fundamentally the same. Indeed, we arrived at Σ_{new} from Σ via nothing more than a change of variables.

Exercise 3 (Another observable canonical form.)**.** Verify that

$$\Sigma = \left[\begin{array}{ccc|c} -a_2 & 1 & 0 & b_2 \\ -a_1 & 0 & 1 & b_1 \\ -a_0 & 0 & 0 & b_0 \\ \hline 1 & 0 & 0 & d \end{array}\right]$$

is similar to

$$\Sigma^* = \left[\begin{array}{ccc|c} 0 & 0 & -a_0 & b_0 \\ 1 & 0 & -a_1 & b_1 \\ 0 & 1 & -a_2 & b_2 \\ \hline 0 & 0 & 1 & d \end{array}\right]$$