ME547: Linear Systems
Modeling: Review of Laplace Transform

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From infinite series to Laplace

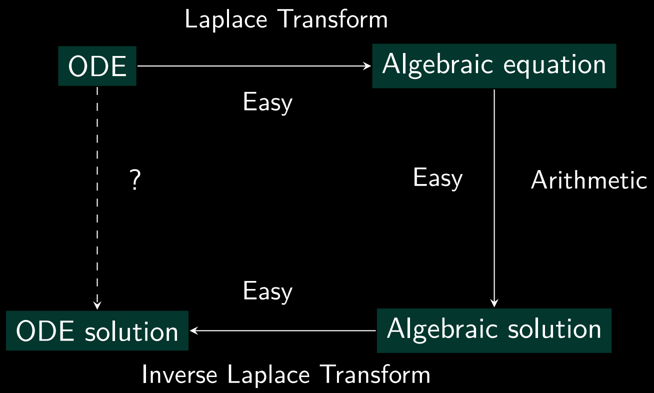
- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = ?$
- how does it relate to the Laplace transform?

Introduction

Pierre-Simon Laplace (1749-1827)

- "the French Newton" or "Newton of France"
- 13 years younger than Lagrange
- studied under Jean le Rond d'Alembert (co-discovered fundamental theorem of algebra, aka d'Alembert/Gauss theorem)

The Laplace approach to ODEs



Sets of numbers and the relevant domains

- *set*: a well-defined collection of distinct objects, e.g., $\{1, 2, 3\}$
- $\mathbb{R}$: the set of real numbers
- $\mathbb{C}$: the set of complex numbers
- $\in$: belong to, e.g., $1 \in \mathbb{R}$
- $\mathbb{R}_+$: the set of positive real numbers
- $\triangleq$: defined as, e.g., $y(t) \triangleq 3x(t) + 1$

Continuous-time functions

Formal notation:

$$f:\ \mathbb{R}_+ \to \mathbb{R}$$

where the domain of f is in $\mathbb{R}_+$, and the value of f is in $\mathbb{R}$

- we use $f(t)$ to denote a continuous-time function
- assume that $f(t) = 0$ for all $t < 0$

Laplace transform definition

For a continuous-time function

$$f: \mathbb{R}_+ \to \mathbb{R}$$

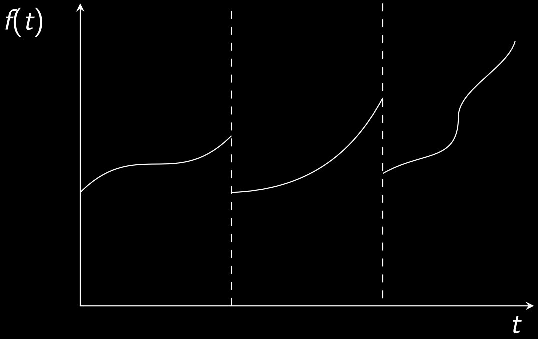
define Laplace Transform:

$$\boxed{F(s) = \mathcal{L}\{f(t)\} \triangleq \int_0^\infty f(t)e^{-st}dt}$$

$s \in \mathbb{C}$

Existence: Sufficient condition 1

- $f(t)$ is piecewise continuous

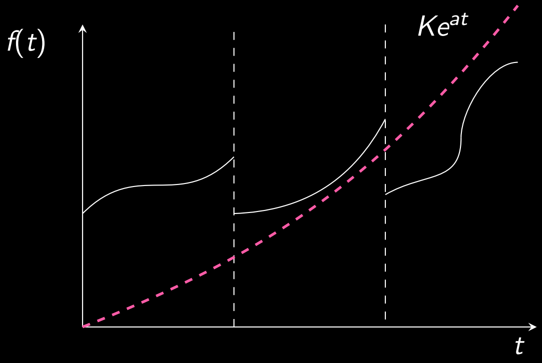


Existence: Sufficient condition 2

- $f(t)$ does not grow faster than an exponential as $t \to \infty$:

$$|f(t)| < ke^{\alpha t}, \text{ for all } t \geq t_0$$

for some constants: $k, \ \alpha, \ t_0 \in \mathbb{R}_+$.



Examples: Exponential

- $f(t) = e^{-at}, \; a \in \mathbb{C}$
- $F(s) = \frac{1}{s+a}$

Examples: Exponential

- $f(t) = 1(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$
- $F(s) = \frac{1}{s}$

Laplace transform and infinite series

- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = ?$

Examples: Sine

- $f(t) = \sin(\omega t)$
- $F(s) = \frac{\omega}{s^2+\omega^2}$
- Use: $\sin(\omega t) = \frac{e^{j\omega t}-e^{-j\omega t}}{2j}$, $\mathcal{L}\{e^{j\omega t}\} = \frac{1}{s-j\omega}$

Recall: Euler formula

$$e^{ja} = \cos a + j \sin a$$

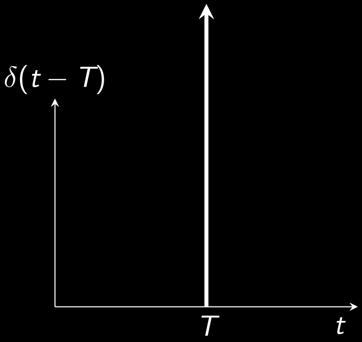
Leonhard Euler (04/15/1707 - 09/18/1783):

- Swiss mathematician, physicist, astronomer, geographer, logician and engineer
- studied under Johann Bernoulli
- teacher of Lagrange
- wrote 380 articles within 25 years at Berlin
- produced on average one paper per week at age 67, when almost blind!

Examples: Cosine

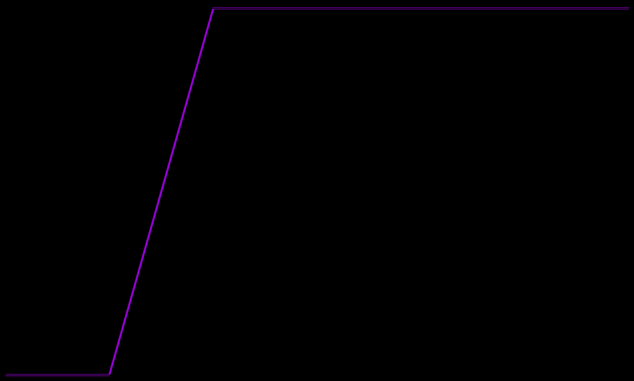
- $f(t) = \cos(\omega t)$
- $F(s) = \frac{s}{s^2+\omega^2}$

Examples: Dirac impulse



- a generalized function (formally, a distribution)
- e.g., consider $\dot{y} - ay = \dot{u} + bu$
  - if u is a unit step $1(t)$
  - $\dot{u}$ has a jump at 0
  - cannot directly evaluate $\dot{u}$!

Approximating the unit step



$$\mu_\epsilon(t) := \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{\epsilon}t & \text{for } 0 \le t < \epsilon \\ 1 & \text{for } \epsilon \le t \end{cases}$$

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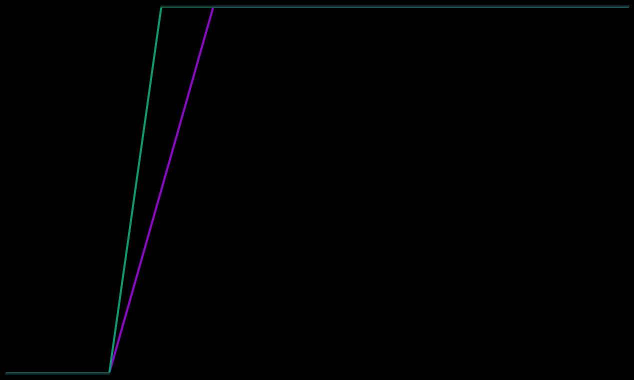

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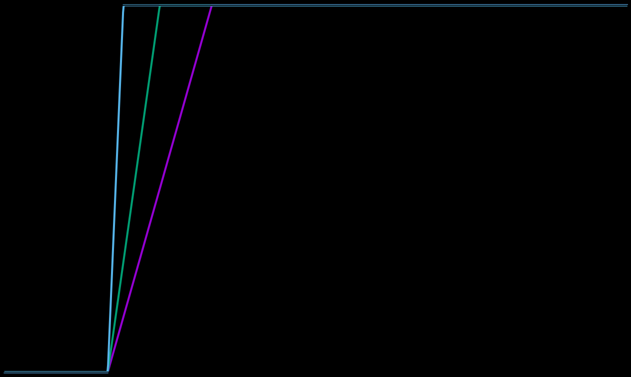

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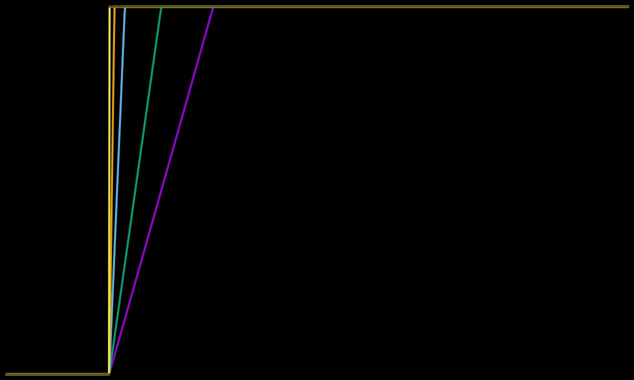

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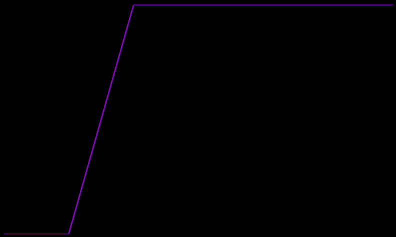
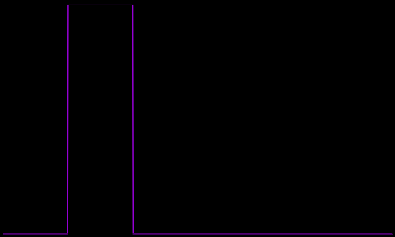
Approximating $\dot{1}(t)$





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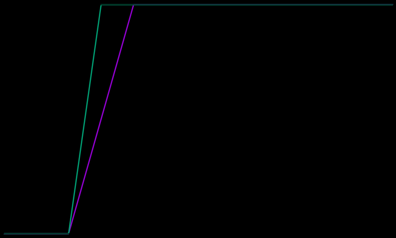

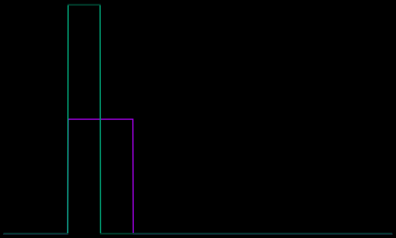

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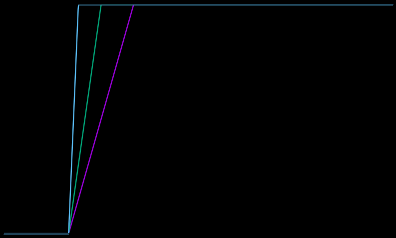

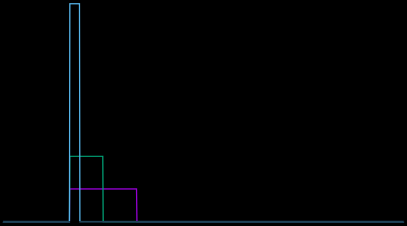

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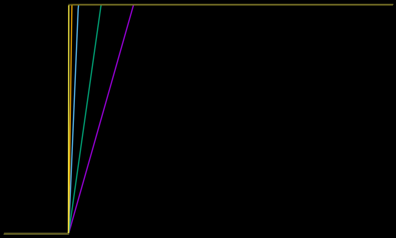

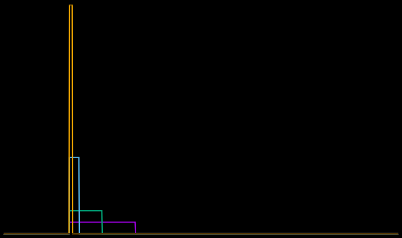

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Properties of the first-order approximation

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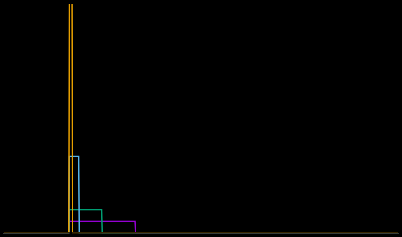
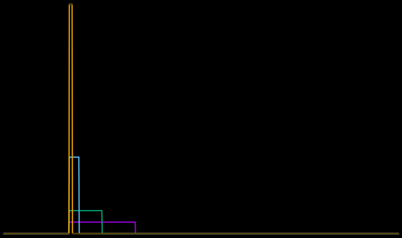


- $\int_{-\infty}^{\infty} \dot{\mu}_\epsilon(t)dt = 1$
- $\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} f(t)\dot{\mu}_\epsilon(t)dt = \lim_{\epsilon\to 0} \int_0^{\epsilon} f(t)\frac{1}{\epsilon}dt = f(0)$

General Dirac impulse properties



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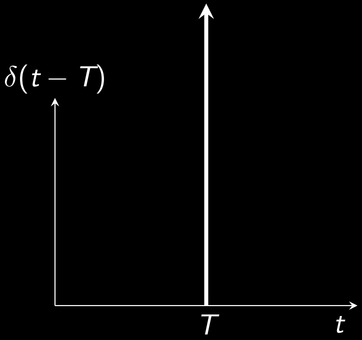
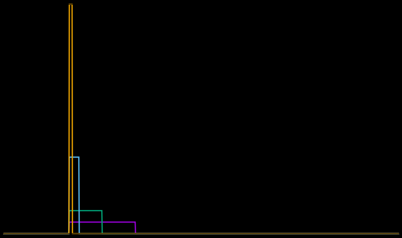


- $\int_0^{\infty} \delta(t-T)dt = 1$
- $\int_0^{\infty} \delta(t-T)f(t)dt = f(T)$

Challenges with the first-order approximation



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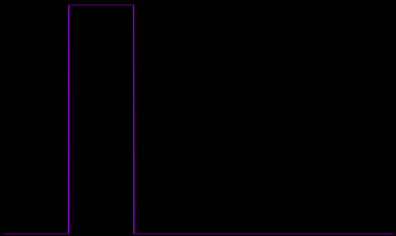
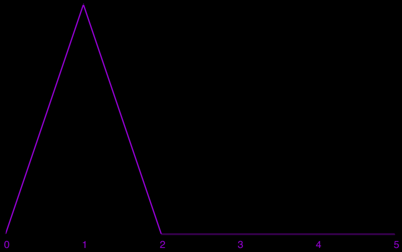
- $\dot{\mu}_\epsilon(t)$ is piecewise-continuous and not fully differentiable
- $\mu_\epsilon(t) \approx 1(t)$ is only first-order differentiable
- cannot handle, e.g., $\ddot{y} + 2\dot{y} - ay = \ddot{u} + 3\dot{u} + bu$

Second-order approximation of $\dot{1}(t)$





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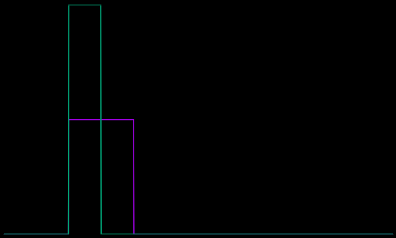

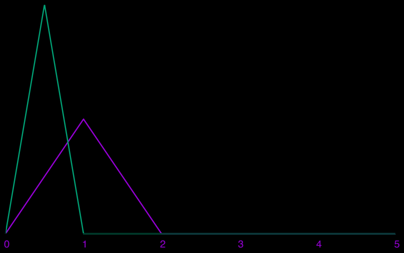

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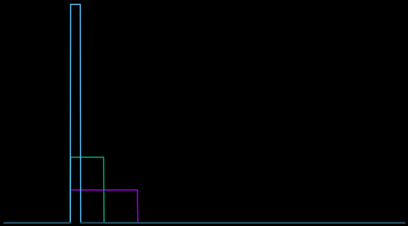

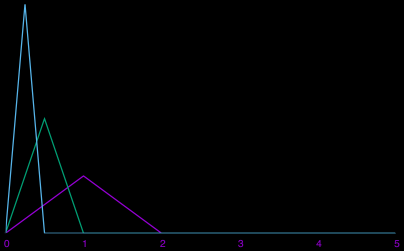

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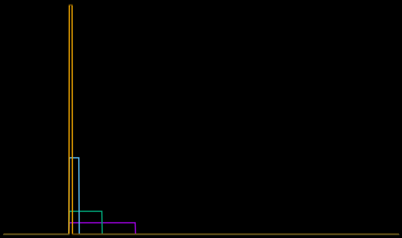

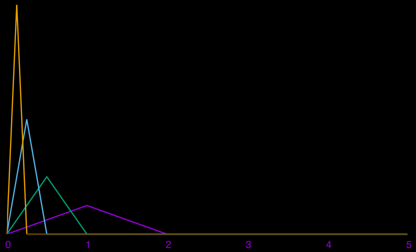

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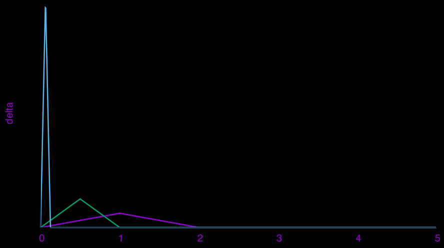

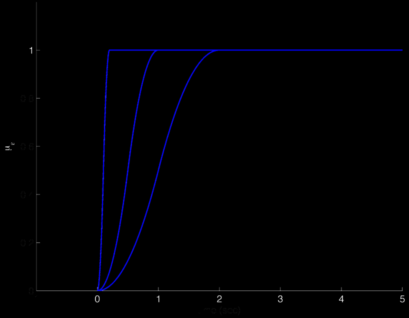

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- $\mu_\epsilon(t) = \int_0^t \delta_\epsilon(\tau) d\tau$: a smoother approximation of the unit step!
- is twice differentiable
- can keep on doing this to make δ_ϵ infinitely differentiable

Application of the concept

Transmission of Signal Nonsmoothness and Transient Improvement in Add-On Servo Control

Tianyu Jiang and Xu Chen

***Abstract*— Plug-in or add-on control is integral for high-performance control in modern precision systems. Despite the capability of greatly enhancing the steady-state performance, add-on compensation can introduce output discontinuity and significant transient response. Motivated by the vast application and the practical importance of add-on control designs, this paper identifies and investigates how general nonsmoothness in signals transmits through linear control systems. We explain the jump of system states in the presence of nonsmooth inputs in add-on servo enhancement, and derive formulas to mathematically characterize the transmission of the nonsmoothness. The results are then applied to devise fast transient responses over the traditional choice of add-on design at the input of the plant. Application examples to a manufacturing control system are conducted, with simulation and experimental results that validate the developed theoretical tools.**

***Index Terms*— Disturbance rejection, nonsmooth inputs, transient control.**

I. Introduction

PLUG-IN or add-on control design is central for servo enhancements in control engineering. In order to provide a storage capacity in the terabyte scale, a modern hard disk drive contains more than 900 000 data tracks within 1 in of the disk. Correspondingly, the width of each track, called track pitch (TP), can easily fall below 30 nm. During read/write operations, servo control must maintain a tracking error that is below 10% TP while strong external disturbances can induce tracking errors that are as large as 70% TP. Such large errors can only be attenuated by adding plug-in control commands. As another example, in high-speed wafer scanning for semiconductor manufacturing, [1] showed that 99.97% of the force commands in the positioning system are contributions of add-on feedforward control.

In feedback algorithms, add-on servo is central for a large class of design schemes that require a baseline feedback controller. Two examples are: disturbance observers [2] and Youla-parameterization-based loop shaping [3], [4]. Either for general low-frequency enhancement [5]–[7], or for the extensions to structured disturbance rejection [8]–[10], disturbance observers usually update the commands at the input side of the plant. Youla parameterization can be parameterized either as an add-on compensation at the plant input side [11], [12], or a combined compensation at the plant input and controller input [13], [14]. In feedforward-related control, adaptive or sensor-based feedforward compensation [15]–[17] can be configured as add-on algorithms either at the plant input or at the reference input (see more details in Section III).

Fundamentally, add-on control brings servo enhancement by introducing new dynamic properties in closed-loop signals. Such a process induces certain degrees of nonsmoothness in the signals. For meeting future demands in high-precision systems, it is essential to understand what types of systems and add-on changes create large transient, and what are the mathematical relationships between the signal nonsmoothness and the induced transient. The importance of such considerations is verified in simulation and experiments in [18] and [19], which compared the transient performance in different feedforward control algorithms. Still, a full theoretical solution to the problem is intrinsically nontrivial, except for simple discontinuities, such as step and ramp signals. Despite the rich literature on designs to achieve the desired steady-state performance, sparse investigations on the transient in add-on compensation are available, and a full understanding of the theoretical add-on transient remains missing. This paper targets to bridge this gap. The focuses are twofold. First, we develop theoretical results about input-to-output discontinuity and reveal its practical importance for the transient performance in control design. Second, new investigations are made to examine the transient characteristics in different add-on control designs. We derive an exact mathematical formula for computing the changes in system outputs when the input and/or its derivatives have discontinuities, and provide computation of the associated transient response. One central result we obtain is that, the common choice of performing add-on control at the input side of the plant yields undesired long transients, if there are delays during turning ON the compensation. Solution of the problem is discussed in detail and verified on a precision motion control platform in semiconductor manufacturing.

The remainder of this paper is organized as follows. Section II describes the wafer scanner hardware on which

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Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

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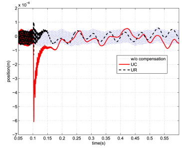

Fig. 10. Experimental comparison of add-on vibration compensations: compensation turned ON at 0.1 s, to attenuate a 500-Hz external vibration (the residual errors are from an internal 18-Hz motor force ripple).

rapidly with respect to time; and the obtained conclusions in the paper are increasingly important for avoiding large servo errors during controller implementation.

As a second example, we apply the developed tools to analyze a switched control scheme. Let $d = 0$ in Fig. 2. Consider the case of tracking a reference r as shown in the top plot of Fig. 11(a), which consists of a 10-Hz periodic signal and a 100-Hz signal that starts at around 0.6 s. r is designed to contain no discontinuities itself. To track the more aggressive 100-Hz reference signal, the feedback controller C switches to a more aggressive mode $C_2 = 40\,000 \times (1 + 3/s + 0.02\ s/(18\,000\ s + 1))$ at around 0.75 s, resulting in the improved tracking in Fig. 11(a). However, a detailed look at the control output indicates a significant increase of $|u(t)|$ as shown in Fig. 11(b). As the saturation limits of the control input are -10 and 10 V, such high-amplitude control inputs are extremely dangerous for application in practice, despite that the tracking error appears to be well controlled in simulation. Applying Theorem 2 to analyze the overlooked danger, one can find that due to the jump in the input to C_2, a significant discontinuity occurs in the output of C_2: $u(t_0^+) - u(t_0^-) = -991.2$ V; $\dot{u}(t_0^+) - \dot{u}(t_0^-) = 1.76255 \times 10^7$ V/s. The calculated -991.2 V jump in the control command can be seen to match well with the actual signal in Fig. 11(b). Furthermore, applying Proposition 5 gives the star-marked solid line in Fig. 11(c), which shows that the transient induced from the discontinuity in C_2 indeed is the main contributor of the abruptness in the overall control command.

With the prediction in Fig. 11(c), one can turn ON the input to C_2 first and slightly delay the engagement of the output of C_2, to avoid injecting the high-amplitude signals in the closed loop. For instance, a 20-step delay in turning ON the output of C_2 gives the servo results in Fig. 12, where in the top plot, the control command is seen to be maintained well under the saturation limits (actually no visual discontinuity or overshoot is observable from the new control command); and in the bottom plot, the error remains to be controlled with a slight 0.05 s longer transient compared with Fig. 11(a).[2]

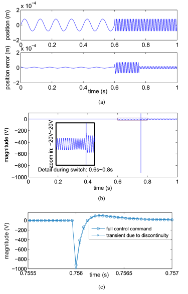

Fig. 11. Closed-loop signals with direct controller switching. (a) Reference and tracking error. (b) Corresponding control input. (c) Decomposition of control command: the transient due to discontinuity dominates in the postswitching transient control command.

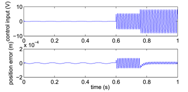

Fig. 12. Closed-loop signals with smoothened switching.

[2]Certainly, the transient can be further controlled using advanced switching mechanism. This paper focuses on providing the fundamental root causes and mathematical analysis tools.

Application of the concept

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Transmission of Signal Nonsmoothness and Transient Improvement in Add-On Servo Control

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***Abstract*— Plug-in or add-on control is integral for high-performance control in modern precision systems. Despite the capability of greatly enhancing the steady-state performance, add-on compensation can introduce output discontinuity and significant transient response. Motivated by the vast application and the practical importance of add-on control designs, this paper identifies and investigates how general nonsmoothness in signals transmits through linear control systems. We explain the jump of system states in the presence of nonsmooth inputs in add-on servo enhancement, and derive formulas to mathematically characterize the transmission of the nonsmoothness. The results are then applied to devise fast transient responses over the traditional choice of add-on design at the input of the plant. Application examples to a manufacturing control system are conducted, with simulation and experimental results that validate the developed theoretical tools.**

***Index Terms*— Disturbance rejection, nonsmooth inputs, transient control.**

I. Introduction

PLUG-IN or add-on control design is central for servo enhancements in control engineering. In order to provide a storage capacity in the terabyte scale, a modern hard disk drive contains more than 900 000 data tracks within 1 in of the disk. Correspondingly, the width of each track, called track pitch (TP), can easily fall below 30 nm. During read/write operations, servo control must maintain a tracking error that is below 10% TP while strong external disturbances can induce tracking errors that are as large as 70% TP. Such large errors can only be attenuated by adding plug-in control commands. As another example, in high-speed wafer scanning for semiconductor manufacturing, [1] showed that 99.97% of the force commands in the positioning system are contributions of add-on feedforward control.

In feedback algorithms, add-on servo is central for a large class of design schemes that require a baseline feedback controller. Two examples are: disturbance observers [2] and Youla-parameterization-based loop shaping [3], [4]. Either for general low-frequency enhancement [5]–[7], or for the extensions to structured disturbance rejection [8]–[10], disturbance observers usually update the commands at the input side of the plant. Youla parameterization can be parameterized either as an add-on compensation at the plant input side [11], [12], or a combined compensation at the plant input and controller input [13], [14]. In feedforward-related control, adaptive or sensor-based feedforward compensation [15]–[17] can be configured as add-on algorithms either at the plant input or at the reference input (see more details in Section III).

Fundamentally, add-on control brings servo enhancement by introducing new dynamic properties in closed-loop signals. Such a process induces certain degrees of nonsmoothness in the signals. For meeting future demands in high-precision systems, it is essential to understand what types of systems and add-on changes create large transient, and what are the mathematical relationships between the signal nonsmoothness and the induced transient. The importance of such considerations is verified in simulation and experiments in [18] and [19], which compared the transient performance in different feedforward control algorithms. Still, a full theoretical solution to the problem is intrinsically nontrivial, except for simple discontinuities, such as step and ramp signals. Despite the rich literature on designs to achieve the desired steady-state performance, sparse investigations on the transient in add-on compensation are available, and a full understanding of the theoretical add-on transient remains missing. This paper targets to bridge this gap. The focuses are twofold. First, we develop theoretical results about input-to-output discontinuity and reveal its practical importance for the transient performance in control design. Second, new investigations are made to examine the transient characteristics in different add-on control designs. We derive an exact mathematical formula for computing the changes in system outputs when the input and/or its derivatives have discontinuities, and provide computation of the associated transient response. One central result we obtain is that, the common choice of performing add-on control at the input side of the plant yields undesired long transients, if there are delays during turning ON the compensation. Solution of the problem is discussed in detail and verified on a precision motion control platform in semiconductor manufacturing.

The remainder of this paper is organized as follows. Section II describes the wafer scanner hardware on which

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Laplace transform of the Dirac impulse

- $\mathcal{L}\{\delta(t)\} = \int_0^\infty e^{-st}\delta(t)dt = e^{-s0} = 1$
- because $\int_0^\infty \delta(t)f(t)dt = f(0)$

Properties of Laplace transform

Linearity

For any $\alpha, \beta \in \mathbb{C}$ and functions $f(t), g(t)$, let

$$F(s) = \mathcal{L}\{f(t)\}, \; G(s) = \mathcal{L}\{g(t)\}$$

then

$$\boxed{\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)}$$

Differentiation

Defining

$$\dot{f}(t) = \frac{df(t)}{dt}$$

$$F(s) = \mathcal{L}\{f(t)\}$$

- then

$$\boxed{\mathcal{L}\{\dot{f}(t)\} = sF(s) - f(0)}$$

- via integration by parts:

$$\begin{aligned}\mathcal{L}\{\dot{f}(t)\} &= \int_0^\infty e^{-st}\dot{f}(t)dt \\ &= -\int_0^\infty \frac{de^{-st}}{dt} f(t)dt + \left\{e^{-st}f(t)\right\}_{t=0}^{t\to\infty} \\ &= s\int_0^\infty e^{-st}f(t)dt - f(0) = sF(s) - f(0)\end{aligned}$$

Integration

Defining

$$F(s) = \mathcal{L}\{f(t)\}$$

then

$$\boxed{\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)}$$

Multiplication by e^{-at}

Defining

$$F(s) = \mathcal{L}\{f(t)\}$$

- then

$$\boxed{\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a)}$$

- Example:

$$\mathcal{L}\{1(t)\} = \frac{1}{s} \quad \mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2} \quad \mathcal{L}\{e^{-at}\sin(\omega t)\} = \frac{\omega}{(s+a)^2+\omega^2}$$

Multiplication by t

Defining

$$F(s) = \mathcal{L}\{f(t)\}$$

- then

$$\boxed{\mathcal{L}\left\{tf(t)\right\} = -\frac{dF(s)}{ds}}$$

- Example:

$$\mathcal{L}\{1(t)\} = \frac{1}{s} \quad \mathcal{L}\{t\} = \frac{1}{s^2}$$

Time delay τ

Defining

$$F(s) = \mathcal{L}\{f(t)\}$$

then

$$\boxed{\mathcal{L}\left\{f(t-\tau)\right\} = e^{-s\tau}F(s)}$$

Convolution

Given $f(t)$, $g(t)$, and

$$(f \star g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = (g \star f)(t)$$

- then

$$\boxed{\mathcal{L}\left\{(f \star g)(t)\right\} = F(s)G(s)}$$

- hence we have

$$\delta(t) \longrightarrow \boxed{G(s)} \longrightarrow g(t) = \mathcal{L}^{-1}\left\{G(s)\right\}$$

because

$$1 \longrightarrow \boxed{G(s)} \longrightarrow Y(s) = G(s) \times 1$$

Initial Value Theorem

If $f(0_+) = \lim_{t \to 0_+} f(t)$ exists, then

$$f(0_+) = \lim_{s \to \infty} sF(s)$$

Final Value Theorem

If $\lim_{t\to\infty} f(t)$ exists,

- then
$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$
- Example: find the final value of the system corresponding to:
$$Y_1(s) = \frac{3(s+2)}{s(s^2+2s+10)},\ \ Y_2(s) = \frac{3}{s-2}$$

Common Laplace transform pairs

$f(t)$	$F(s)$	$f(t)$	$F(s)$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$	e^{-at}	$\dfrac{1}{s+a}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$	t	$\dfrac{1}{s^2}$
$tx(t)$	$-\dfrac{dX(s)}{ds}$	t^2	$\dfrac{2}{s^3}$
$\frac{x(t)}{t}$	$\int_s^\infty X(s)\,ds$	te^{-at}	$\dfrac{1}{(s+a)^2}$
$\delta(t)$	1	$e^{-at}\sin(\omega t)$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
$1(t)$	$\dfrac{1}{s}$	$e^{-at}\cos(\omega t)$	$\dfrac{s+a}{(s+a)^2+\omega^2}$