Proofs of Bode's Integral Theorem

Xu Chen
chx AT uw.edu
Department of Mechanical Engineering
University of Washington

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In the literature, the Bode's Integral Theorem can be proven from different appraoches. We provide two in this set of notes.

Theorem. *(Bode's integral formula for continuous-time systems) Let $L(s)$ be a proper, scalar rational transfer function, of relative degree larger than 1. Let $S(s) = (1 + L(s))^{-1}$ and assume that $S(s)$ has no poles in the right half plane, and has $q \geq 0$ zeros in the closed right half plane, at locations p_1, p_2, ..., p_q. Then*

$$\int_0^{\infty} \ln |S(j\omega)| \, d\omega = \pi \sum_{k=1}^{q} p_k$$

Prove by direct construction

Proof. (key steps) Consider the simple case where we have a real unstable pole in $L(s)$. We construct the complex integral with s shown in the contour in Fig. 1. Here $R \to \infty$. Since $\ln S(s)$ is analytic within the contour, the whole contour integral sums up to zero. This is the result of Cauchy Integral Theorem. It is not difficult to show that the part of the integral along the arc with radius R is zero under the assumption of relative degree larger than 1.[1] Therefore the integral along the imaginary axis (which is the quantity that we want to compute) plus the integral along the contour C (consisting of the path $I \to II \to III$) in Fig. 2 is zero, namely, when $R \to \infty$

$$\int_{-j\infty}^{0} \ln S(s) \, ds + \int_0^{j\infty} \ln S(s) \, ds + \lim_{\epsilon \to 0} \int_C \ln S(s) \, ds = 0 \tag{1}$$

Now we focus on the contour C in Fig. 2. Decompose first

$$S(s) = (s - p) S^*(s)$$
$$\Rightarrow \int_C \ln S(s) \, ds = \int_C \ln (s - p) \, ds + \int_C \ln S^*(s) \, ds \tag{2}$$

so that we can separate the analytic part of $\ln S(s)$ as $\ln S^*(s)$. We will show that as $\epsilon \to 0$, $\int_C \ln S^*(s) \, ds \to 0$ and $\int_C \ln (s - p) \, ds$ approaches to some constant value that will show up in Bode's Integral Formula. For the first part, if we add a path IV to make a closed contour $I \to II \to III \to IV$, we have

$$\oint \ln S^*(s) \, ds = 0$$

[1] To see this, note that when $L(s)$ is small, a Taylor expansion for $\ln(1 + L(s))$ gives

$$\int_R \ln S(s) \, ds = -\int_R \ln (1 + L(s)) \, ds \approx -\int_R (\ln 1 + L(s)) \, ds \approx -\int_R L(s) ds$$

Since $L(s)$ decays to zero at a rate that is at least as fast as $1/s^2$ for large s, the above integral goes to zero when the radius of the circle goes to infinity.

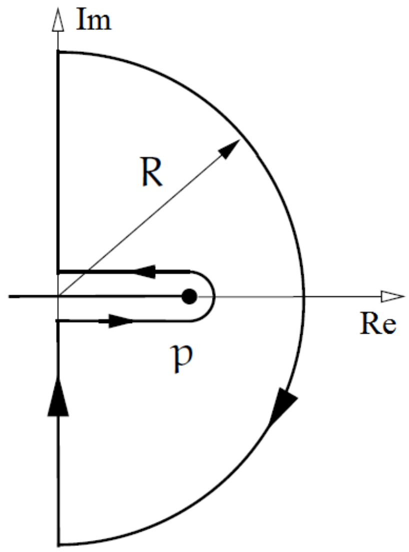

Figure 1: Contour of s for Bode's Integral

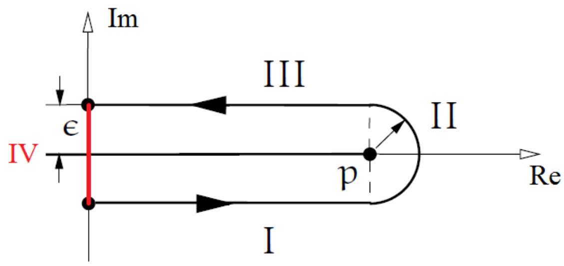

Figure 2: Partial contour for Bode's Integral, $\epsilon \to 0$.

due to the fact that $\ln S^*(s)$ is analytic on and within the contour. Hence

$$\begin{aligned}\int_C \ln S^*(s)\,ds + \int_{IV} \ln S^*(s)\,ds = 0 \\ \Rightarrow \int_C \ln S^*(s)\,ds = -\int_{IV} \ln S^*(s)\,ds = -\int_{\epsilon j}^{-\epsilon j} \ln S^*(s)\,ds \\ = \int_{-\epsilon j}^{\epsilon j} \ln S^*(s)\,ds\end{aligned}$$

In this way we need to just compute a line integral. $\ln S^*(s)$ is analytic so it is bounded by some finite value $f_m > 0$, therefore

$$\left|\int_{-\epsilon j}^{\epsilon j} \ln S^*(s)\,ds\right| \leq \int_{-\epsilon j}^{\epsilon j} |\ln S^*(s)|\,ds \leq \int_{-\epsilon j}^{\epsilon j} f_m ds = f_m 2\epsilon j \to 0 \tag{3}$$

Now switch to proving the second part. This needs just some small steps of algebra. From the fundamental result of

$$\ln x dx = d\left[x \ln x - x\right]$$

we have

$$\begin{aligned}\int_C \ln(s-p)\,ds &= \int_C d\left[(s-p)\ln(s-p)-(s-p)\right]\\ &= \left[(s-p)\ln(s-p)-(s-p)\right]\Big|_{-\epsilon j}^{\epsilon j}\\ &= \left[(s-p)\ln(s-p)\right]\Big|_{-\epsilon j}^{\epsilon j}+\left[-(s-p)\right]\Big|_{-\epsilon j}^{\epsilon j}\\ &= \left[s\ln(s-p)\right]\Big|_{-\epsilon j}^{\epsilon j}+\left[-p\ln(s-p)\right]\Big|_{-\epsilon j}^{\epsilon j}+\left[-(s-p)\right]\Big|_{-\epsilon j}^{\epsilon j}\end{aligned}$$

The terms $\left[s\ln(s-p)\right]\Big|_{-\epsilon j}^{\epsilon j}$ and $\left[-(s-p)\right]\Big|_{-\epsilon j}^{\epsilon j}$ all go to zero as $\epsilon\to 0$, for the remaining term we use the property of log functions:

$$\ln x = \ln\left(|x|\,e^{j\angle x}\right) = \ln|x| + \ln e^{j\angle x} = \ln|x| + j\angle x$$

and have

$$\begin{aligned}\lim_{\epsilon\to 0}\left[-p\ln(s-p)\right]\Big|_{-\epsilon j}^{\epsilon j} &= \lim_{\epsilon\to 0}\left[-p\ln|s-p| - pj\angle(s-p)\right]\Big|_{-\epsilon j}^{\epsilon j}\\ &= \lim_{\epsilon\to 0}\left[-pj\angle(s-p)\right]\Big|_{-\epsilon j}^{\epsilon j}\end{aligned}$$

Draw a picture of the vector $s-p$ in Fig. 2. We will see that as s goes along the contour starting at $-\epsilon j$ and ending at ϵj, the angular change of $\angle(s-p)$ is 2π as $\epsilon\to 0$. Hence

$$\int_C \ln(s-p)\,ds = \left[-pj\angle(s-p)\right]\Big|_{-\epsilon j}^{\epsilon j} \to -2\pi pj \qquad (4)$$

Combining (2) (3) and (4) we get

$$\int_C \ln S(s)\,ds \to -2\pi pj$$

as $\epsilon\to 0$. Using (1), we obtain

$$\int_{-j\infty}^{0} \ln S(s)\,ds + \int_0^{j\infty} \ln S(s)\,ds = 2\pi pj$$

When there are multiple unstable open-loop poles, the above analysis can be easily extended and we have

$$\int_{-j\infty}^{0} \ln S(s)\,ds + \int_0^{j\infty} \ln S(s)\,ds = 2j\pi\sum_k Re\,(p_k) = 2j\pi\sum_k p_k \qquad (5)$$

In control engineering we prefer using ω instead of s in the left half side of the above equation. To make this happen, we note that

$$\begin{aligned}\int_{-j\infty}^{0} \ln S(s)\,ds + \int_0^{j\infty} \ln S(s)\,ds &= j\int_{-\infty}^{0} \ln S(j\omega)\,d\omega + j\int_0^{\infty} \ln S(j\omega)\,d\omega\\ &= j\int_0^{\infty} \ln S(-j\omega)\,d\omega + j\int_0^{\infty} \ln S(j\omega)\,d\omega\\ &= j\int_0^{\infty} \left[\ln S(-j\omega) + \ln S(j\omega)\right] d\omega\\ &= j\int_0^{\infty} \ln\left[S(-j\omega)\,S(j\omega)\right] d\omega\\ &= 2j\int_0^{\infty} \ln|S(j\omega)|\,d\omega\end{aligned}$$

Putting the above result to (5), we obtain the final conclusion

$$\boxed{\int_0^{\infty} \ln|S(j\omega)|\,d\omega = \pi\sum_k p_k}$$

□

Prove by integral formulas Review of some basic results:

- $\frac{d}{dx}\tan x = \sec^2 x$: this comes from $\frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$
- $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$: to see this, let $\tan\theta = x$, then

$$\begin{aligned}\frac{d}{dx}\arctan x &= \frac{d\theta}{dx} = \frac{1}{\frac{dx}{d\theta}} = \frac{1}{\sec^2\theta} = \cos^2\theta \\ &= \frac{\cos^2\theta}{\sin^2\theta + \cos^2\theta} = \frac{1}{1+\tan^2\theta} = \frac{1}{1+x^2}\end{aligned}$$

Fact 1. *If $a \in \mathcal{R}$ and $a \neq 0$, then*

$$\int_{-\sigma}^{\sigma} \ln\left(\omega^2 + a^2\right) d\omega = 2\sigma \ln\left(\sigma^2 + a^2\right) - 4\sigma + 4\,|a| \arctan\frac{\sigma}{|a|} \tag{6}$$

Proof.

$$\begin{aligned}\int_{-\sigma}^{\sigma} \ln\left(\omega^2 + a^2\right) d\omega &= \omega \ln\left(\omega^2 + a^2\right)\Big|_{-\sigma}^{\sigma} - \int_{-\sigma}^{\sigma} \omega \frac{2\omega}{\omega^2 + a^2} d\omega \\ &= 2\sigma \ln\left(\sigma^2 + a^2\right) - \int_{-\sigma}^{\sigma} \frac{2\omega^2 + 2a^2 - 2a^2}{\omega^2 + a^2} d\omega \\ &= 2\sigma \ln\left(\sigma^2 + a^2\right) - \int_{-\sigma}^{\sigma} \left(2 - 2\frac{a^2}{\omega^2 + a^2}\right) d\omega \\ &= 2\sigma \ln\left(\sigma^2 + a^2\right) - 4\sigma + \int_{-\sigma}^{\sigma} 2\frac{a^2}{\omega^2 + a^2} d\omega \\ &= 2\sigma \ln\left(\sigma^2 + a^2\right) - 4\sigma + 2\int_{-\sigma}^{\sigma} \frac{1}{\left(\frac{\omega}{|a|}\right)^2 + 1} d\omega \\ &= 2\sigma \ln\left(\sigma^2 + a^2\right) - 4\sigma + 2\,|a| \arctan\frac{\omega}{|a|}\Big|_{-\sigma}^{\sigma}\end{aligned}$$

where in the last equality we used the result that $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$. □

Lemma 2. *For any $a, b \in \mathcal{R}$*

$$\int_{-\infty}^{\infty} \ln\left|\frac{j\omega - a}{j\omega - b}\right|^2 d\omega = 2\pi\left(|a| - |b|\right)$$

Proof. We will do the case of $a \neq 0$ and $b \neq 0$ only.

$$\begin{aligned}\int_{-\infty}^{\infty} \ln\left|\frac{j\omega - a}{j\omega - b}\right|^2 d\omega &= \lim_{\sigma\to\infty} \int_{-\sigma}^{\sigma} \ln\frac{\omega^2 + a^2}{\omega^2 + b^2} d\omega \\ &= \lim_{\sigma\to\infty} \int_{-\sigma}^{\sigma} \left[\ln\left(\omega^2 + a^2\right) - \ln\left(\omega^2 + b^2\right)\right] d\omega\end{aligned}$$

Using (6) to the last term above yields

$$\begin{aligned}\int_{-\infty}^{\infty} \ln\left|\frac{j\omega - a}{j\omega - b}\right|^2 d\omega &= \lim_{\sigma\to\infty} \left\{\left[2\sigma \ln\left(\sigma^2 + a^2\right) - 4\sigma + 4\,|a| \arctan\frac{\sigma}{|a|}\right]\right. \\ &\qquad \left. - \left[2\sigma \ln\left(\sigma^2 + b^2\right) - 4\sigma + 4\,|b| \arctan\frac{\sigma}{|b|}\right]\right\} \\ &= 4\frac{\pi}{2}\left(|a| - |b|\right) \\ &= 2\pi\left(|a| - |b|\right)\end{aligned}$$

□

Fact 3. *Let $A(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = (s - \alpha_1)(s - \alpha_2)\ldots(s - \alpha_n)$. Then*

$$\sum_{i=1}^{n} \alpha_i = a_{n-1}$$

Main Steps of Proof

Setup:

Let

$$S(s) = \frac{D(s)}{D(s)+N(s)}$$

Partition the real and complex-conjugate roots such that

$$D(s) = \prod_{i=1}^{n-2n_c}(s-p_i)\prod_{i=1}^{n_c}(s-\alpha_i \pm j\beta_i)$$

$$D(s)+N(s) = \prod_{i=1}^{n-2m_c}(s-r_i)\prod_{i=1}^{m_c}(s-\gamma_i \pm j\eta_i)$$

where $\mathscr{R}(r_i) < 0 \ \forall i$ and $\mathscr{R}(\gamma_i) < 0 \ \forall i$, as the closed-loop roots are all strictly stable.

If the relative degrees satisfy

$$\partial D(s) - 2 \geq \partial N(s)$$

then from Fact 3,

$$2\sum_{i=1}^{m_c}\gamma_i + \sum_{i=1}^{n-2m_c} r_i = 2\sum_{i=1}^{n_c}\alpha_i + \sum_{i=1}^{n-2n_c} p_i \tag{7}$$

Step 2:

Separate the roots on the left-half plane and the roots on the right-half plane such that:

$$\prod_{i=1}^{n-2n_c}(s-p_i) = \prod_{k=1}^{n_{sr}}(s-p_{sk})\prod_{h=1}^{n_{ur}}(s-p_{uh}),\ \ p_{sk} < 0, p_{uh} \geq 0$$

and

$$\prod_{i=1}^{n-2n_c}(s-\alpha_i \pm j\beta_i) = \prod_{k=1}^{n_{sc}}(s-\alpha_{sk} \pm j\beta_{sk})\prod_{h=1}^{n_{uc}}(s-\alpha_{uh} \pm j\beta_{uh}),\ \ \alpha_{sk} < 0, \alpha_{uh} \geq 0$$

Step 3:

Partitioning the integral

$$\begin{aligned} 2\int_0^\infty \ln|S(j\omega)|d\omega &= \int_0^\infty \ln|S(j\omega)|^2 d\omega \\ &= \int_0^\infty \ln\left|\frac{D(j\omega)}{D(j\omega)+N(j\omega)}\right|^2 d\omega \\ &= \int_0^\infty \ln\left|\frac{\prod_{i=1}^{n-2n_c}(j\omega-p_i)\prod_{i=1}^{n_c}(j\omega-\alpha_i \pm j\beta_i)}{\prod_{i=1}^{n-2m_c}(j\omega-r_i)\prod_{i=1}^{m_c}(j\omega-\gamma_i \pm j\eta_i)}\right|^2 d\omega \end{aligned}$$

and using Lemma 2 yield

$$2\int_0^\infty \ln|S(j\omega)|d\omega = \pi\left[2\sum_{i=1}^{n_c}|\alpha_i| + \sum_{i=1}^{n-2n_c}|p_i| - 2\sum_{i=1}^{m_c}|\gamma_i| - \sum_{i=1}^{n-2m_c}|r_i|\right]$$

Step 4:

As all closed-loop poles are on the left-half plane, we have

$$\begin{aligned} 2\sum_{i=1}^{m_c}|\gamma_i| + \sum_{i=1}^{n-2m_c}|r_i| &= -2\sum_{i=1}^{m_c}\gamma_i - \sum_{i=1}^{n-2m_c} r_i \\ &\overset{(7)}{=} -2\sum_{i=1}^{n_c}\alpha_i - \sum_{i=1}^{n-2n_c} p_i \end{aligned}$$

The roots of $D(s)$ are separeted in the left- and right-half planes. Hence

$$2\sum_{i=1}^{n_c}|\alpha_i| + \sum_{i=1}^{n-2n_c}|p_i| = 2\sum_{h=1}^{n_{uc}}\alpha_{uh} + \sum_{h=1}^{n_{ur}}p_{uh} - 2\sum_{k=1}^{n_{sc}}\alpha_{sk} - \sum_{k=1}^{n_{sr}}p_{sk}$$

Therefore

$$2\int_0^\infty \ln|S(j\omega)|d\omega = \pi\left[4\sum_{h=1}^{n_{uc}}\alpha_{uh} + 2\sum_{h=1}^{n_{ur}}p_{uh}\right]$$

In other words

$$\boxed{\int_0^\infty \ln|S(j\omega)|d\omega = \underbrace{2\pi\sum_{h=1}^{n_{uc}}\alpha_{uh} + \pi\sum_{h=1}^{n_{ur}}p_{uh}}_{\pi\times\text{sum of real parts of open-loop unstable poles}}}$$

Reference: B.F. Wu and E.A. Jonckheere, "A Simplified Approach to Bode's Theorem for Continuous-Time and Discrete-Time Systems," IEEE Transactions on Automatic Control, vol. 37, no. 11, November 1992, pp. 1797-1802.